The Goldman-Millson theorem revisited

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Based on joint work arXiv:1407.6735 with Christopher L. Rogers.

An L_∞-algebra is ...

Definition

An L_∞-algebra is a cochain complex (L, ∂) equipped with **symmetric** *multi-brackets of degree* 1 *($m \geq 2$)*

$$\{\ ,\ ,\dots,\ \}_m : S^m(L) \to L$$

which satisfy

$$\partial\{v_1, v_2, \dots, v_m\}_m + \sum_{i=1}^{m} \pm\{v_1, \dots, v_{i-1}, \partial v_i, v_{i+1}, \dots, v_m\}_m$$
$$+ \sum_{k=2}^{m-1} \sum_{\sigma \in \mathrm{Sh}_{k,m-k}} \pm\{\{v_{\sigma(1)}, \dots, v_{\sigma(k)}\}_k, v_{\sigma(k+1)}, \dots, v_{\sigma(m)}\}_{m-k+1} = 0\,.$$

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The base field $\Bbbk$ has characteristic zero.

The dg cocommutative coalgebra corresponding to L

Let (L, ∂) be a cochain complex and $S(L) = L \oplus \bigoplus_{m \ge 2} S^m(L)$ be the space of the truncated symmetric algebra. We view $S(L)$ as the **cocommutative coalgebra** with the standard comultiplication.

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An L_∞-structure on L is a degree 1 coderivation Q on the coalgebra $S(L)$ which satisfies the Maurer-Cartan (MC) equation

$$Q \circ Q = 0$$

and the condition $Q(v) = \partial(v) \ \ \forall \ v \in L$.

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The multi-brackets $\{\ ,\ ,\dots,\ \}_m$ are related to Q by the formula

$$\{v_1, v_2, \dots, v_m\}_m = p_L \circ Q(v_1 v_2 \dots v_m),$$

where p_L is the projection $S(L) \to L$.

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where p_L is the projection $S(L) \to L$.

To every L_∞-algebra L, we assign the *dg cocomm. coalgebra* $(S(L), Q)$.

An L_∞-morphism from L to $\tilde{L}$ is ...

Definition

An L_∞-morphism from an L_∞-algebra L to an L_∞-algebra $\tilde{L}$ is a homomorphism U of dg cocommutative coalgebras $(S(L), Q) \to (S(\tilde{L}), \tilde{Q})$.

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Recall that every homomorphism $U : S(L) \to S(\tilde{L})$ is uniquely determined by its composition $U' := p_{\tilde{L}} \circ U : S(L) \to \tilde{L}$ with the projection $p_{\tilde{L}} : S(\tilde{L}) \to \tilde{L}$ and **the linear term**

$$U_{(1)} := U'\big|_L : L \to \tilde{L}$$

is always a chain map from (L, ∂) to $(\tilde{L}, \tilde{\partial})$.

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An L_∞-morphism U is called a **quasi-isomorphism** if its linear term $U_{(1)}$ induces an isomorphism $H^\bullet(L, \partial) \to H^\bullet(\tilde{L}, \tilde{\partial})$.

Filtered L_∞-algebras

Definition

An L_∞-algebra L is **filtered** *if it is equipped with a descending filtration*

$$L = \mathcal{F}_1 L \supset \mathcal{F}_2 L \supset \mathcal{F}_3 L \supset \cdots , \quad \textit{such that} \quad L = \varprojlim_k L/\mathcal{F}_k L,$$

$$\partial(\mathcal{F}_i L) \subset \mathcal{F}_i L,$$

$$\{\mathcal{F}_{i_1} L, \mathcal{F}_{i_2} L, \dots, \mathcal{F}_{i_m} L\}_m \subset \mathcal{F}_{i_1+i_2+\cdots+i_m} L \quad \forall \;\; m \geq 2.$$

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Assumption: all ∞-morphisms U in question are compatible with the filtrations in the sense that

$$U'(\mathcal{F}_{i_1} L \otimes \mathcal{F}_{i_2} L \otimes \cdots \otimes \mathcal{F}_{i_m} L) \subset \mathcal{F}_{i_1+i_2+\dots+i_m} \tilde{L} \quad \forall \;\; m \geq 1.$$

Examples:

- Let A be an associative algebra over $\mathbb{k}$ and ε be a formal deformation parameter. Then $L_A := \varepsilon C^\bullet(A, A)[[\varepsilon]]$ (with the Hochschild differential and the Gerstenhaber bracket) is a filtered dg Lie algebra with $\mathcal{F}_k L_A := \varepsilon^k C^\bullet(A, A)[[\varepsilon]]$.

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- If $\mathcal{O}$ is a dg operad and P is a pseudo-cooperad with $P(0) = P(1) = \mathbf{0}$ then

$$\mathrm{Conv}(P, \mathcal{O}) := \prod_{n \ge 2} \mathrm{Hom}_{S_n}(P(n), \mathcal{O}(n))$$

is a filtered dg Lie algebra with

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$$\mathcal{F}_k \mathrm{Conv}(P, \mathcal{O}) := \prod_{n \ge k+1} \mathrm{Hom}_{S_n}(P(n), \mathcal{O}(n))$$
- Let X be a simply-connected space and
$$L_X := \bigoplus_{i \le -2} \pi_{-i}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$
be the minimal L_∞-algebra representing the rational homotopy type of X.

MC elements and the de Rham-Sullivan algebra Ω_n

Recall that a MC element of a (filtered) L_∞-algebra L is a degree 0 element $\alpha \in L$ which satisfies

$$\partial\alpha + \sum_{m=2}^{\infty} \frac{1}{m!}\{\alpha, \alpha, \dots, \alpha\}_m = 0.$$

Denote by $\mathcal{MC}(L)$ the set of MC elements of an L_∞-algebra L.

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Let Ω_n be the de Rham-Sullivan algebra of polynomial differential forms on the geometric simplex Δ^n:

$$\Omega_n := \Bbbk[t_0, t_1, \dots, t_n, dt_0, dt_1, \dots, dt_n] \big/ (t_0 + \dots + t_n - 1,\; dt_0 + \dots + dt_n)$$

Each t_i has degree 0, each dt_i has degree 1, $d(t_i) := dt_i$.

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Each t_i has degree 0, each dt_i has degree 1, $d(t_i) := dt_i$.

To a filtered L_∞-algebra L, we assign the simplicial set $\mathrm{MC}_\bullet(L)$ with

$$\mathrm{MC}_n(L) := \mathcal{MC}(L \,\hat{\otimes}\, \Omega_n).$$

Deligne-Getzler-Hinich (DGH) ∞-groupoid

A straightforward generalization of Prop 4.7 from *E. Getzler, 2009* implies that

Proposition

For every filtered L_∞-algebra L, the simplicial set $\mathrm{MC}_\bullet(L)$ is a Kan complex.

We call $\mathrm{MC}_\bullet(L)$ the DGH ∞-groupoid corresponding to L.

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We call $\mathrm{MC}_\bullet(L)$ the DGH ∞-groupoid corresponding to L.

For example, if A is an associative algebra and $L_A := \varepsilon C^\bullet(A, A)[[\varepsilon]]$ then

$$\pi_0\big(\mathrm{MC}_\bullet(L_A)\big)$$

is the set of equivalence classes of 1-parameter formal deformations of A.

More examples:

If $\mathcal{O}$ is a dg operad, P is a pseudo-cooperad, and $L = \text{Conv}(P, \mathcal{O})$ then

$$\pi_0\Big(\text{MC}_{\bullet}\left(\text{Conv}(P, \mathcal{O}) \right) \Big)$$

is the set of homotopy classes of operad maps $\text{Cobar}(\mathcal{C}) \to \mathcal{O}$, where $\mathcal{C}$ is obtained from P via adjoining the counit.

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Theorem (A. Berglund, 2011)

For every nilpotent L_∞-algebra L and $\alpha \in \mathcal{MC}(L)$, we have the isomorphism of groups

$$H^{-i}(L^\alpha) \cong \pi_i(\text{MC}_\bullet(L), \alpha), \qquad \forall i \geq 1,$$

where L^α is obtained from L via twisting the L_∞-structure by α and the group structure on $H^{-1}(L^\alpha)$ is given the Campbell-Hausdorff formula.

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In particular, if $\text{MC}_\bullet(L)$ is simply-connected then $|\text{MC}_\bullet(L)|$ is a rational space.

The main result is ...

Observation: Every L_∞-morphism $U : L \to \tilde{L}$ compatible with filtrations induces the map of simplicial sets

$$\mathrm{MC}_\bullet(U) : \mathrm{MC}_\bullet(L) \to \mathrm{MC}_\bullet(\tilde{L}).$$

This way, the construction $\mathrm{MC}_\bullet$ upgrades to a functor from the category of (filtered) L_∞-algebras with L_∞-morphisms to the category of simplicial sets.

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Theorem (Christopher L. Rogers and V.D.)

Let L and $\tilde{L}$ be filtered L_∞-algebras and U be an ∞-morphism from L to $\tilde{L}$ compatible with the filtrations. If the linear term $U_{(1)} : L \to \tilde{L}$ gives us a quasi-isomorphism

$$U_{(1)}\big|_{\mathcal{F}_m L} \;:\; \mathcal{F}_m L \;\to\; \mathcal{F}_m \tilde{L}$$

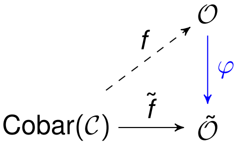
for every $m \geq 1$ then

$$\mathrm{MC}_\bullet(U) : \mathrm{MC}_\bullet(L) \to \mathrm{MC}_\bullet(\tilde{L})$$

is a weak equivalence of simplicial sets.

Applications

- Let $\varphi : \mathcal{O} \to \tilde{\mathcal{O}}$ be a quasi-isomorphisms of dg operads and $\mathcal{C}$ be a dg operad satisfying $\mathcal{C}(0) = \mathbf{0}$ and $\mathcal{C}(1) = \Bbbk$. Then for every operad map $\tilde{f} : \mathrm{Cobar}(\mathcal{C}) \to \tilde{\mathcal{O}}$ there exists an operad map $f : \mathrm{Cobar}(\mathcal{C}) \to \mathcal{O}$ such that the diagram



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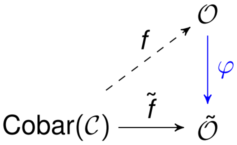

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- Please, come to the next talk!

Does the homotopy type of $\mathrm{MC}_\bullet(L)$ depend on the choice of filtration?

Let $\mathcal{G}_\bullet L$ be another descending filtration on L such that $L = \mathcal{G}_1 L$, L is complete with respect to this filtration, the multi-brackets are compatible with $\mathcal{G}_\bullet L$. Let $\mathcal{J}_\bullet L$ be the filtration on L obtained by intersecting

$$\mathcal{J}_m L := \mathcal{F}_m L \cap \mathcal{G}_m L.$$

Then we have the DGH ∞-groupoids $\mathrm{MC}^{\mathcal{F}}_\bullet(L)$, $\mathrm{MC}^{\mathcal{G}}_\bullet(L)$, and $\mathrm{MC}^{\mathcal{J}}_\bullet(L)$ of L constructed with the help of the filtrations $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{J}$, respectively.

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Claim (based on a discussion with T. Willwacher)

The canonical maps of simplicial sets

$$\mathrm{MC}^{\mathcal{F}}_\bullet(L) \longleftarrow \mathrm{MC}^{\mathcal{J}}_\bullet(L) \longrightarrow \mathrm{MC}^{\mathcal{G}}_\bullet(L)$$

are weak homotopy equivalences.

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THANK YOU!