The Goldman-Millson theorem revisited

Vasily Dolgushev

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Based on joint work arXiv:1407.6735 with Christopher L. Rogers.
An $L_\infty$-algebra is a cochain complex $(L, \partial)$ equipped with **symmetric** multi-brackets of degree 1 ($m \geq 2$)

$$\{ \ldots, \}^m : S^m(L) \to L$$

which satisfy

$$\partial \{ v_1, v_2, \ldots, v_m \}_m + \sum_{i=1}^{m} \pm \{ v_1, \ldots, v_{i-1}, \partial v_i, v_{i+1}, \ldots, v_m \}_m$$

$$+ \sum_{k=2}^{m-1} \sum_{\sigma \in \text{Sh}_{k,m-k}} \pm \{ \{ v_{\sigma(1)} , \ldots , v_{\sigma(k)} \}_{k} v_{\sigma(k+1)}, \ldots , v_{\sigma(m)} \}_{m-k+1} = 0.$$
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The base field $k$ has characteristic zero.
Let $(L, \partial)$ be a cochain complex and $S(L) = L \oplus \bigoplus_{m \geq 2} S^m(L)$ be the space of the truncated symmetric algebra. We view $S(L)$ as the **cocommutative coalgebra** with the standard comultiplication.
The dg cocommutative coalgebra corresponding to $L$

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An $L_\infty$-structure on $L$ is a degree 1 coderivation $Q$ on the coalgebra $S(L)$ which satisfies the Maurer-Cartan (MC) equation

$$Q \circ Q = 0$$

and the condition $Q(\nu) = \partial(\nu) \quad \forall \ \nu \in L.$
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The multi-brackets \(\{, , \ldots , \}\_m\) are related to \(Q\) by the formula

\[ \{v_1, v_2, \ldots , v_m\}_m = p_L \circ Q(v_1 v_2 \ldots v_m), \]

where \(p_L\) is the projection \(S(L) \to L.\)
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To every $L_\infty$-algebra $L$, we assign the $dg$ cocomm. coalgebra $(S(L), Q)$. 
An $L_\infty$-morphism from $L$ to $\tilde{L}$ is ...

**Definition**

An $L_\infty$-morphism from an $L_\infty$-algebra $L$ to an $L_\infty$-algebra $\tilde{L}$ is a homomorphism $U$ of dg cocommutative coalgebras $(S(L), Q) \rightarrow (S(\tilde{L}), \tilde{Q})$. 

Recall that every homomorphism $U: S(L) \rightarrow S(\tilde{L})$ is uniquely determined by its composition $U': p_{\tilde{L}} \circ U: S(L) \rightarrow \tilde{L}$ with the projection $p_{\tilde{L}}: S(\tilde{L}) \rightarrow \tilde{L}$ and the linear term $U(1): L \rightarrow \tilde{L}$ is always a chain map from $(L, \partial) \rightarrow (\tilde{L}, \tilde{\partial})$.

An $L_\infty$-morphism $U$ is called a quasi-isomorphism if its linear term $U(1)$ induces an isomorphism $H^\bullet(L, \partial) \rightarrow H^\bullet(\tilde{L}, \tilde{\partial})$. 

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An $L_\infty$-algebra $L$ is **filtered** if it is equipped with a descending filtration

$$L = \mathcal{F}_1L \supset \mathcal{F}_2L \supset \mathcal{F}_3L \supset \cdots,$$

such that

$$L = \varprojlim_{k} L/\mathcal{F}_kL,$$

$$\partial(\mathcal{F}_iL) \subset \mathcal{F}_iL,$$

$$\{\mathcal{F}_{i_1}L, \mathcal{F}_{i_2}L, \ldots, \mathcal{F}_{i_m}L\}_m \subset \mathcal{F}_{i_1+i_2+\ldots+i_m}L \quad \forall \ m \geq 2.$$
Filtered $L_\infty$-algebras

**Definition**

An $L_\infty$-algebra $L$ is **filtered** if it is equipped with a descending filtration

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such that

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**Assumption:** all $\infty$-morphisms $U$ in question are compatible with the filtrations in the sense that

$$U'(\mathcal{F}_{i_1} L \otimes \mathcal{F}_{i_2} L \otimes \cdots \otimes \mathcal{F}_{i_m} L) \subset \mathcal{F}_{i_1 + i_2 + \cdots + i_m} \tilde{L} \quad \forall \ m \geq 1.$$
Examples:

Let $A$ be an associative algebra over $\mathbb{k}$ and $\varepsilon$ be a formal deformation parameter. Then $L_A := \varepsilon C^\bullet(A, A)[[\varepsilon]]$ (with the Hochschild differential and the Gerstenhaber bracket) is a filtered dg Lie algebra with $F_k L_A := \varepsilon^k C^\bullet(A, A)[[\varepsilon]]$. 

If $O$ is a dg operad and $P$ is a pseudo-cooperad with $P(0) = P(1) = 0$ then $\text{Conv}(P, O) := \prod_{n \geq 2} \text{Hom}_{S^n}(P(n), O(n))$ is a filtered dg Lie algebra with $F_k \text{Conv}(P, O) := \prod_{n \geq k+1} \text{Hom}_{S^n}(P(n), O(n))$. 

Let $X$ be a simply-connected space and $L_X := \bigoplus_{i \leq -2} \pi_{-i}(X) \otimes \mathbb{Q}$ be the minimal $L_\infty$-algebra representing the rational homotopy type of $X$. 

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The Goldman-Millson theorem
Recall that a MC element of a (filtered) $L_\infty$-algebra $L$ is a degree 0 element $\alpha \in L$ which satisfies
\[
\partial \alpha + \sum_{m=2}^{\infty} \frac{1}{m!} \{\alpha, \alpha, \ldots, \alpha\}_m = 0.
\]
Denote by $MC(L)$ the set of MC elements of an $L_\infty$-algebra $L$. 

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\text{MC elements and the de Rham-Sullivan algebra } \Omega_n
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Let $\Omega_n$ be the de Rham-Sullivan algebra of polynomial differential forms on the geometric simplex $\Delta^n$:

$$\Omega_n := \mathbb{k}[t_0, t_1, \ldots, t_n, dt_0, dt_1, \ldots, dt_n] / (t_0 + \cdots + t_n - 1, dt_0 + \cdots + dt_n)$$

Each $t_i$ has degree 0, each $dt_i$ has degree 1, $d(t_i) := dt_i$. 

MC elements and the de Rham-Sullivan algebra $\Omega_n$

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Each $t_i$ has degree 0, each $dt_i$ has degree 1, $d(t_i) := dt_i$.

To a filtered $L_\infty$-algebra $L$, we assign the simplicial set $\text{MC}_{\bullet}(L)$ with

$$\text{MC}_n(L) := \mathcal{MC}(L \hat{\otimes} \Omega_n).$$
A straightforward generalization of Prop 4.7 from E. Getzler, 2009 implies that

**Proposition**

*For every filtered $L\infty$-algebra $L$, the simplicial set $MC\bullet(L)$ is a Kan complex.*

We call $MC\bullet(L)$ the DGH $\infty$-groupoid corresponding to $L$. 
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**Proposition**

*For every filtered $L_\infty$-algebra $L$, the simplicial set $MC_\bullet(L)$ is a Kan complex.*

We call $MC_\bullet(L)$ the DGH $\infty$-groupoid corresponding to $L$.

**For example**, if $A$ is an associative algebra and $L_A := \varepsilon C^\bullet(A, A)[[\varepsilon]]$ then

$$\pi_0(MC_\bullet(L_A))$$

is the set of equivalence classes of 1-parameter formal deformations of $A$. 
More examples:

If $\mathcal{O}$ is a dg operad, $P$ is a pseudo-cooperad, and $L = \text{Conv}(P, \mathcal{O})$ then

$$
\pi_0\left( \text{MC}_\bullet \left( \text{Conv}(P, \mathcal{O}) \right) \right)
$$

is the set of homotopy classes of operad maps $\text{Cobar}(\mathcal{C}) \to \mathcal{O}$, where $\mathcal{C}$ is obtained from $P$ via adjoining the counit.
More examples:

If $\mathcal{O}$ is a dg operad, $P$ is a pseudo-cooperad, and $L = \text{Conv}(P, \mathcal{O})$ then

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is the set of homotopy classes of operad maps $\text{Cobar}(C) \to \mathcal{O}$, where $C$ is obtained from $P$ via adjoining the counit.

**Theorem (A. Berglund, 2011)**

For every nilpotent $L_\infty$-algebra $L$ and $\alpha \in MC(L)$, we have the isomorphism of groups

$$H^{-i}(L^\alpha) \cong \pi_i(\text{MC}_\bullet(L), \alpha), \quad \forall i \geq 1,$$

where $L^\alpha$ is obtained from $L$ via twisting the $L_\infty$-structure by $\alpha$ and the group structure on $H^{-1}(L^\alpha)$ is given the Campbell-Hausdorff formula.
More examples:

If $O$ is a dg operad, $P$ is a pseudo-cooperad, and $L = \text{Conv}(P, O)$ then

$$\pi_0\left( \text{MC}_\bullet\left( \text{Conv}(P, O) \right) \right)$$

is the set of homotopy classes of operad maps $\text{Cobar}(C) \to O$, where $C$ is obtained from $P$ via adjoining the counit.

**Theorem (A. Berglund, 2011)**

For every nilpotent $L_\infty$-algebra $L$ and $\alpha \in \mathcal{MC}(L)$, we have the isomorphism of groups

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where $L^\alpha$ is obtained from $L$ via twisting the $L_\infty$-structure by $\alpha$ and the group structure on $H^{-1}(L^\alpha)$ is given the Campbell-Hausdorff formula.

In particular, if $\text{MC}_\bullet(L)$ is simply-connected then $|\text{MC}_\bullet(L)|$ is a rational space.
The main result is ...

**Observation:** Every $L_\infty$-morphism $U : L \to \tilde{L}$ compatible with filtrations induces the map of simplicial sets

$$MC_\bullet(U) : MC_\bullet(L) \to MC_\bullet(\tilde{L}).$$

This way, the construction $MC_\bullet$ upgrades to a functor from the category of (filtered) $L_\infty$-algebras with $L_\infty$-morphisms to the category of simplicial sets.
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**Theorem (Christopher L. Rogers and V.D.)**

Let $L$ and $\tilde{L}$ be filtered $L_{\infty}$-algebras and $U$ be an $\infty$-morphism from $L$ to $\tilde{L}$ compatible with the filtrations. If the linear term $U_{(1)} : L \to \tilde{L}$ gives us a quasi-isomorphism

$$U_{(1)} \mid_{\mathcal{F}_mL} : \mathcal{F}_mL \to \mathcal{F}_m\tilde{L}$$

for every $m \geq 1$ then

$$\text{MC}_\bullet(U) : \text{MC}_\bullet(L) \to \text{MC}_\bullet(\tilde{L})$$

is a weak equivalence of simplicial sets.
Let \( \varphi : \mathcal{O} \to \tilde{\mathcal{O}} \) be a quasi-isomorphisms of dg operads and \( \mathcal{C} \) be a dg operad satisfying \( \mathcal{C}(0) = 0 \) and \( \mathcal{C}(1) = \mathbb{k} \). Then for every operad map \( \tilde{f} : \text{Cobar}(\mathcal{C}) \to \tilde{\mathcal{O}} \) there exists an operad map \( f : \text{Cobar}(\mathcal{C}) \to \mathcal{O} \) such that the diagram

\[
\begin{array}{ccc}
\text{Cobar}(\mathcal{C}) & \xrightarrow{\tilde{f}} & \tilde{\mathcal{O}} \\
\downarrow{f} & & \downarrow{\varphi} \\
\mathcal{O} & \xrightarrow{\varphi} & \tilde{\mathcal{O}}
\end{array}
\]

commutes up to homotopy. The homotopy class of \( f : \text{Cobar}(\mathcal{C}) \to \mathcal{O} \) is uniquely determined by the homotopy class of \( \tilde{f} \).
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Please, come to the next talk!
Does the homotopy type of MC\(_{\bullet}(L)\) depend on the choice of filtration?

Let \(G_{\bullet}L\) be another descending filtration on \(L\) such that \(L = G_1L\), \(L\) is complete with respect to this filtration, the multi-brackets are compatible with \(G_{\bullet}L\). Let \(J_{\bullet}L\) be the filtration on \(L\) obtained by intersecting

\[J_mL := F_mL \cap G_mL.\]

Then we have the DGH \(\infty\)-groupoids \(MC_{\bullet}^F(L)\), \(MC_{\bullet}^G(L)\), and \(MC_{\bullet}^J(L)\) of \(L\) constructed with the help of the filtrations \(F\), \(G\) and \(J\), respectively.
Does the homotopy type of $\text{MC}_\bullet(L)$ depend on the choice of filtration?

Let $G\cdot L$ be another descending filtration on $L$ such that $L = G_1 L$, $L$ is complete with respect to this filtration, the multi-brackets are compatible with $G\cdot L$. Let $J\cdot L$ be the filtration on $L$ obtained by intersecting

$$J_m L := F_m L \cap G_m L.$$ 

Then we have the DGH $\infty$-groupoids $\text{MC}_{\bullet}^{F}(L)$, $\text{MC}_{\bullet}^{G}(L)$, and $\text{MC}_{\bullet}^{J}(L)$ of $L$ constructed with the help of the filtrations $F$, $G$ and $J$, respectively.

Claim (based on a discussion with T. Willwacher)

The canonical maps of simplicial sets

$$\text{MC}_{\bullet}^{F}(L) \leftarrow \text{MC}_{\bullet}^{J}(L) \rightarrow \text{MC}_{\bullet}^{G}(L)$$

are weak homotopy equivalences.
References


THANK YOU!