The Intricate Maze of Graph Complexes

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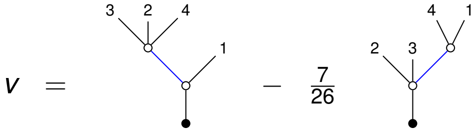
The cochain complex assembled from planar trees

The cochain complex $C^\bullet_{\mathrm{PT},n}$ consists of linear combinations[1] of rooted planar trees with n labeled leaves (and without bivalent vertices):

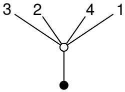


The degree of such tree $:= -$ number of internal (i.e. blue) edges.
The differential ∂ is defined as the sum over all possible contractions of internal edges (with appropriate sign factors).

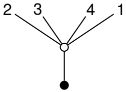
[1] The base field $\mathbb{K}$ has characteristic zero.

The above vector v has degree -1 and

$$\partial v \; = $$



$$- \; \frac{7}{26}$$



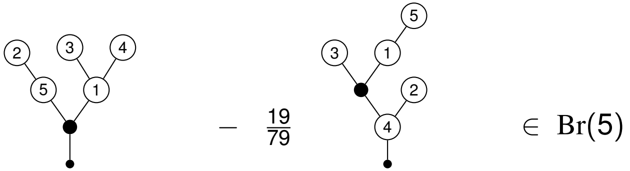
It is not hard to show that

$$H^i(C^{\bullet}_{\mathrm{PT},n}) = \begin{cases} \mathbb{K}\, S_n & \text{if } \; i = 2 - n\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

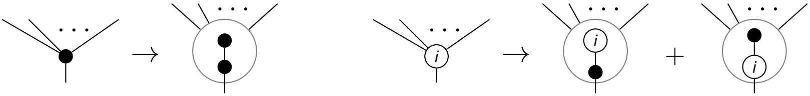
A neat topological proof of this fact is based on the use of Stasheff-Tamari polytopes (a.k.a. associahedra).

The cochain complex of brace trees $\mathrm{Br}(n)$

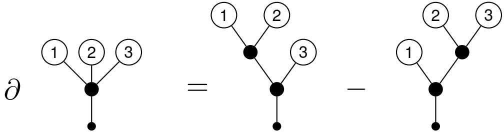
A vector in $\mathrm{Br}(n)$ is a linear combination of *brace trees*:

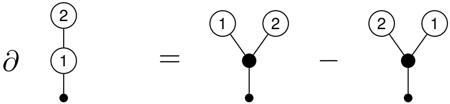


The degree of a brace tree $T = 2\times$ the number of neutral (black) vertices $-$ the number of non-root edges.
The differential $\partial(T)$ of T is the sum over all possible "splittings" of labeled and neutral vertices (with appropriate sign factors)



For example,. . .





The cohomology of $\mathrm{Br}(n)$ is . . .

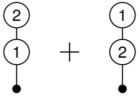
Theorem

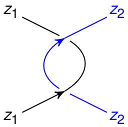
For every $n \geq 2$, we have

$$H^{\bullet}\big(\mathrm{Br}(n)\big) \;\cong\; H_{-\bullet}\big(\mathrm{Conf}(\mathbb{C}, n), \mathbb{K}\big),$$

where $\mathrm{Conf}(\mathbb{C}, n)$ *is the configuration space of n points on $\mathbb{C}$:*

$$\mathrm{Conf}(\mathbb{C}, n) := \big\{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \;\; \mathit{for} \;\; i \neq j\big\}.$$

For example, the class of  is represented by

the cycle  in $\mathrm{Conf}(\mathbb{C}, 2)$.

We need:

- **a set** $\mathrm{gra}_{n,k}$, $n \geq 1$, $k \geq 0$. An element of $\mathrm{gra}_{n,k}$ is a directed graph Γ with $V(\Gamma) = \{1, 2, \dots, n\}$ and $E(\Gamma) = \{1, 2, \dots, k\}$.

$\mathrm{gra}_{n,k}$ is equipped with the action of $S_n \times (S_k \ltimes S_2^k)$;

- **the graded vector space**

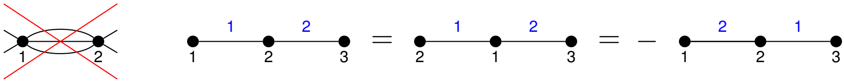
$$\mathrm{fGC}_d := \bigoplus_{n\geq 1,\, k\geq 0} \Big(\mathfrak{s}^{nd-d+(1-d)k}\, \mathbb{K}\mathrm{gra}_{n,k} \otimes \mathrm{Or}_d^{n,k} \Big)_{S_n \times (S_k \ltimes S_2^k)},$$

where $\mathfrak{s}$ is the operator which shifts the degree up by 1 and

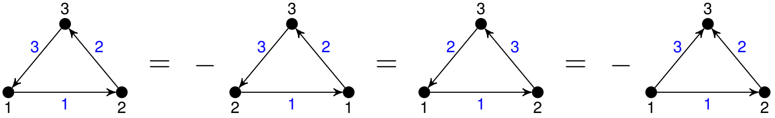
$$\mathrm{Or}_d^{n,k} := \begin{cases} \mathrm{sgn}_{S_k} & \text{if } d \text{ is even,} \\ \mathrm{sgn}_{S_n} \otimes (\mathrm{sgn}_{S_2})^{\otimes k} & \text{if } d \text{ is odd.} \end{cases}$$

For example,. . .

In fGC_2 (i.e. when *d* is even), we can forget about directions on edges, discard graphs with multiple edges, identify graphs which differ merely by the labels on vertices, and keep in mind that, the action of every *odd* permutation in S_k alters the sign factor in front of a graph:



In fGC_3 (i.e. when *d* is odd), we have[2]



[2]Multiple edges for fGC_3 are allowed.

fGC_d is a (graded) Lie algebra

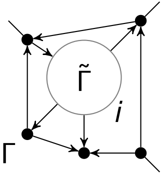
fGC_d has the Lie bracket:

$$[\Gamma, \tilde{\Gamma}] := \Gamma \bullet \tilde{\Gamma} - (-1)^{|\Gamma||\tilde{\Gamma}|} \tilde{\Gamma} \bullet \Gamma, \qquad \Gamma \bullet \tilde{\Gamma} := \sum_{i=1}^{n} \pm \Gamma \circ_i \tilde{\Gamma},$$

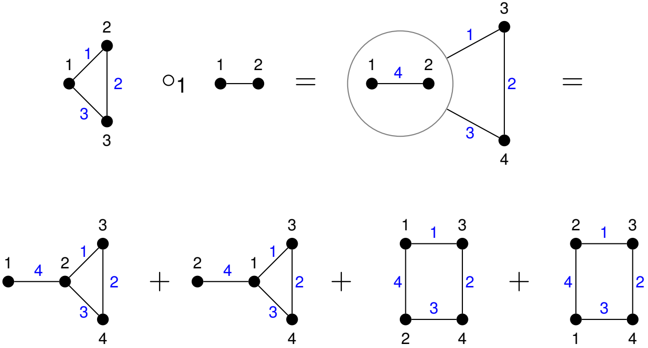
$$\Gamma \in \mathrm{gra}_{n,k}\,, \qquad \tilde{\Gamma} \in \mathrm{gra}_{m,q}\,,$$

where $\circ_i$ is this operation:

$$\Gamma \circ_i \tilde{\Gamma} \ := \ \sum$$



For example,. . .



If *d* is **odd**, we should also keep track of **directions** on edges and **sign factors.**

The graph complex GC_d

The graph $\Gamma_e = \overset{1}{\bullet}\!\!\longrightarrow\!\!\overset{2}{\bullet}$ satisfies the equation $[\Gamma_e, \Gamma_e] = 0$. So we introduce the differential on fGC_d

$$\partial := [\Gamma_e, \ \ ]\,.$$

Definition

The graph complex GC_d is the subcomplex of fGC_d which involves only graphs Γ satisfying these properties:

- *Γ is connected and*
- *each vertex of Γ has valency ≥ 3.*

The “hairy” graph complex $\mathrm{hGC}_{j,d}$

Let j be an integer. $\mathrm{hGC}_{j,d}$ is a modification of GC_d which involves graphs with a non-empty set of univalent vertices. As the graded vector space, $\mathrm{hGC}_{j,d}$ is

$$\bigoplus_{n\geq 0, m\geq 1, k} \left(\mathfrak{s}^{nd+mj+(1-d)k}\, \mathbb{K}\mathrm{hgra}(m,n,k) \otimes \mathrm{Or}^{m,n,k}_{j,d} \right)_{S_m\times S_n\times(S_k\ltimes S_2^k)},$$

where $\mathrm{hgra}_{m,n,k}$ is the set of connected graphs with the set of univalent vertices $\{1,2,\dots,m\}$, the set of vertices $\{1,2,\dots,n\}$ of valency ≥ 3 and the set of (directed) edges $\{1,2,\dots,k\}$. Finally, $\mathrm{Or}^{j,d}_{m,n,k}$ is a 1-dimensional representation of $S_m\times S_n\times(S_k\ltimes S_2^k)$ which “takes care” of identifying certain graph with appropriate sign factors.

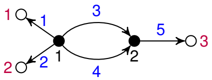
The differential is defined as the sum of expansions of vertices of valency ≥ 3 and keeping only the graphs satisfying our defining conditions.

The degree of a graph in $\mathrm{hGC}_{j,d}$ is

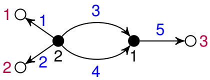
$$d \times \text{the number of vertices of val. } \geq 3$$

$$+ j \times \text{the number of univalent vertices} + (1-d) \times \text{the number of edges.}$$

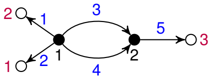
Examples of identifications in $\mathrm{hGC}_{j,d}$:



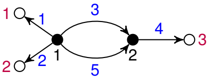
$= (-1)^d$



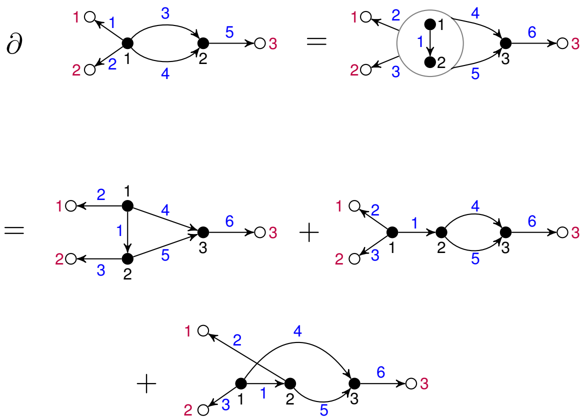
$= (-1)^j$



$= (-1)^{(1-d)}$



Example of computing the differential in $\mathrm{hGC}_{j,d}$



$\mathrm{hGC}_{1,3}$ shows up in knot theory!

A simple combinatorial argument $\Rightarrow \mathrm{hGC}_{1,3}$ lives in degrees ≤ 0.

Furthermore, due to D. Bar-Natan, *"On the Vassiliev knot invariants"*,

$$H^0(\mathrm{hGC}_{1,3}) := \mathrm{hGC}^0_{1,3} / \partial(\mathrm{hGC}^{-1}_{1,3})$$

is isomorphic to the space $\mathrm{CD}(S^1)$ of chord diagrams.

Every finite type invariant of framed knots factors through the map

$$LM_{\Phi} \;:\; \pi_0\big(\mathrm{Emb}_{fr}(S^1, S^3)\big) \longrightarrow \mathrm{CD}(S^1).$$

This map was constructed by M. Kontsevich, T.Q.T. Le and J. Murakami. It is often called *the universal Vassiliev invariant.*

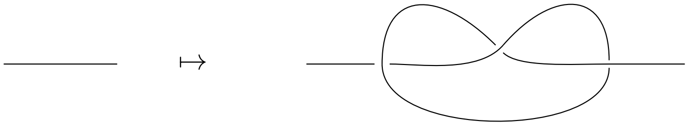
Topological meaning of $\mathrm{hGC}_{j,d}$

Due to results of G. Arone, T. Goodwillie, J. R. Klein, M. Kontsevich, P. Lambrechts, J. E. McClure, D. Sinha, J.H. Smith, V. Turchin, I. Volic, M. Weiss, and ..., the rational homotopy groups

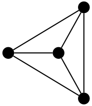
$$\pi_\bullet\big(\,\mathrm{Emb}_c(\mathbb{R}^j, \mathbb{R}^d)\big) \otimes \mathbb{Q}$$

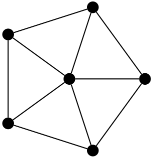
of the embedding space $\mathrm{Emb}_c(\mathbb{R}^j, \mathbb{R}^d)$ can be expressed in terms of $H^\bullet(\mathrm{hGC}_{j,d})$ if $d \geq 2j + 2$.

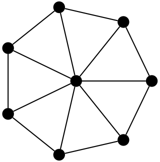
For example, $\mathbb{R} \to \mathbb{R}^4$:



Examples of degree zero cocycles in GC_2 are...

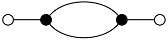
$\gamma_3 =$ 

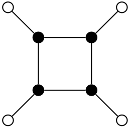
$\gamma_5 =$  $+ \ \ldots$

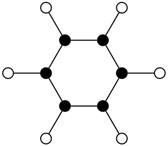
$\gamma_7 =$  $+ \ \ldots$ and so on.

Here $\ldots$ is a linear comb. of graphs with ≥ 2 vertices of valency ≥ 4.

Examples of degree zero cocycles in $\mathrm{hGC}_{1,3}$ are...

$w_2 =$ 

$w_4 =$ 

$w_6 =$ 

and so on.

A remark about the Euler characteristic

Let χ be an integer and $\mathrm{GC}_{d,\chi}$ (resp. $\mathrm{hGC}_{j,d,\chi}$) be the subcomplex of GC_d (resp. $\mathrm{hGC}_{j,d}$) spanned by graphs with the Euler characteristic χ.

It is easy to see that

$$\mathrm{GC}_d = \bigoplus_{\chi} \mathrm{GC}_{d,\chi}\,, \qquad \mathrm{hGC}_{j,d} = \bigoplus_{\chi} \mathrm{hGC}_{j,d,\chi}\,.$$

Moreover, $H^{\bullet}(\mathrm{GC}_{d,\chi})$ is isomorphic (up to a degree shift) to $H^{\bullet}(\mathrm{GC}_{2,\chi})$, if d is even, and isomorphic (up to a degree shift) to $H^{\bullet}(\mathrm{GC}_{3,\chi})$ if d is odd.

Similarly, $H^{\bullet}(\mathrm{hGC}_{j,d,\chi})$ “depends” (up to a degree shift) only on the parities of j and d. So, for the hairy case, it is sufficient to study only these four complexes $\mathrm{hGC}_{1,3}$, $\mathrm{hGC}_{2,3}$, $\mathrm{hGC}_{1,2}$, $\mathrm{hGC}_{2,2}$.

What do we know about GC_2?

- Due to T. Willwacher "*M. Kontsevich's graph complex and ...*", we know that

$$H^{<0}(\mathrm{GC}_2) = 0 \quad \text{and} \quad H^0(\mathrm{GC}_2) \cong \mathfrak{grt} \ \text{ as the Lie algebras,}$$

  where $\mathfrak{grt}$ is the Grothendieck-Teichmueller Lie algebra introduced by V. Drinfeld in 1990.

- It is conjectured that

$$H^1(\mathrm{GC}_2) \overset{?}{=} 0.$$

- Due to results of A. Khoroshkin, T. Willwacher, and M. Zivkovic "*Differentials on ...*" and a theorem of F. Brown, we know that "there are plenty of" cohomology classes in $H^{>1}(\mathrm{GC}_2)$.

To define the Lie algebra $\mathfrak{grt}$, we need...

- The family of *Drinfeld Kohno Lie algebras* $\mathfrak{t}_m$, $(m \ge 2)$. For every m, $\mathfrak{t}_m$ is generated by $\{t^{ij} = t^{ji}\}_{1 \le i \ne j \le m}$ subject to

$$[t^{ij}, t^{ik} + t^{jk}] = 0 \qquad \#\{i,j,k\} = 3\,,$$

$$[t^{ij}, t^{kl}] = 0 \qquad \#\{i,j,k,l\} = 4\,.$$

- and the free Lie algebra $\mathfrak{lie}(x,y)$ in two symbols x, y.

The Lie algebra $\mathfrak{grt}$ consists of ...

elements $\sigma(x,y) \in \mathfrak{lie}(x,y)$ satisfying

$$\sigma(y,x) = -\sigma(x,y)\,,$$

$$\sigma(x,y) + \sigma(y,-x-y) + \sigma(-x-y,x) = 0\,,$$

$$\sigma(t^{23},t^{34}) - \sigma(t^{13}+t^{23},t^{34}) + \sigma(t^{12}+t^{13},t^{24}+t^{34})$$

$$-\sigma(t^{12},t^{23}+t^{24}) + \sigma(t^{12},t^{23}) = 0\,.$$

The Lie bracket on $\mathfrak{grt}$ is the Ihara bracket:

$$[\sigma,\sigma']_{\mathrm{Ih}} := \delta_\sigma(\sigma') - \delta_{\sigma'}(\sigma) + [\sigma,\sigma']_{\mathfrak{lie}(x,y)}\,,$$

where $[\ ,\ ]_{\mathfrak{lie}(x,y)}$ is the usual bracket on $\mathfrak{lie}(x,y)$ and δ_σ is the derivation of $\mathfrak{lie}(x,y)$ defined by

$$\delta_\sigma(x) := 0\,, \qquad \delta_\sigma(y) := [y,\sigma(x,y)]\,.$$

Deligne-Drinfeld elements of $\mathfrak{grt}$

The above equations have neither linear nor quadratic solutions. The first non-trivial example of an element in $\mathfrak{grt}$ is

$$\sigma_3(x, y) = [x, [x, y]] - [y, [y, x]] \,.$$

More generally,

Proposition (V. Drinfeld, 1990)

For every odd integer $n \ge 3$ there exists a non-zero vector $\sigma_n \in \mathfrak{grt}$ of degree n in symbols x and y such that

$$\sigma_n = \mathrm{ad}_x^{n-1}(y) + \dots$$

where $\dots$ is a sum of Lie words of degrees ≥ 2 in the symbol y .

$\{\sigma_n\}_{n \, \mathrm{odd} \, \ge 3}$ are called *Deligne-Drinfeld elements* of $\mathfrak{grt}$.

The Deligne-Drinfeld Conjecture states that ...

Conjecture

The Lie algebra $\mathfrak{grt}$ is freely generated by elements $\{\sigma_n\}_{n \text{ odd } \ge 3}$.

Numerical experiments performed by L. Albert, M. Espie, P. Harinck, J.-C. Novelli, G. Racinet, and C. Torossian show that this conjecture holds up to degree 16.

Remark

A result proved by F. Brown in 2011 implies that there exists a collection of Deligne-Drinfeld elements which generates a free Lie **subalgebra** *in $\mathfrak{grt}$.*

Remark (about $\mathfrak{grt} \cong H^0(\mathrm{GC}_2)$)

Willwacher's isomorphism $\mathfrak{grt} \cong H^0(\mathrm{GC}_2)$ sends the elements $\{\sigma_n\}_{n \text{ odd } \ge 3}$ to the cohomology classes of $\{\gamma_n\}_{n \text{ odd } \ge 3}$ introduced above.

The problem of deformation quantization

Recall that a star product on a smooth manifold M is an $\mathbb{R}[[\varepsilon]]$-bilinear associative multiplication

$$* : C^\infty(M)[[\varepsilon]] \otimes C^\infty(M)[[\varepsilon]] \to C^\infty(M)[[\varepsilon]]$$

of the form

$$f * g = f \cdot g + \sum_{k \geq 1} \varepsilon^k B_k(f, g),$$

where ε is a formal parameter and B_k are bidifferential operators.

The problem of deformation quantization is to describe equivalence classes of such star products on arbitrary manifold M.

Kontsevich's formality theorem

Let us denote by $\mathrm{PV}^\bullet(M)$ the algebra of polyvector fields on M. For example, $\mathrm{PV}^0(M) = C^\infty(M)$ and $\mathrm{PV}^1(M)$ is the space of vector fields. $\mathrm{PV}^\bullet(M)$ carries a natural Lie bracket $[\ ,\ ]_S$.

In 1997, M. Kontsevich proved that a theorem which implies that equivalence classes of star products on M are in bijection with (equivalence classes of)

$$\alpha = \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \cdots \in \varepsilon\, \mathrm{PV}^2(M)[[\varepsilon]], \qquad [\alpha, \alpha]_S = 0.$$

Such α may be thought of as a generalization of a Poisson structure on M and *Kontsevich's formality quasi-isomorphism* may be thought of a "bridge" connecting classical mechanics to quantum mechanics.

"Bridges" from classical to quantum mechanics

In my paper "*Stable formality quasi-isomorphisms for Hochschild cochains*", I proved a theorem which can be stated colloquially as

Theorem (a colloquial statement!)

The group

$$\exp\left(H^0(\mathrm{GC}_2)\right)$$

acts simply transitively on the set of bridges connecting classical mechanics to quantum mechanics.

Combining this result with Willwacher's theorem, we conclude that

Corollary (a colloquial statement!)

The Grothendieck-Teichmueller group $\exp(\mathfrak{grt})$ *acts simply transitively on the set of bridges connecting classical mechanics to quantum mechanics.*

Kontsevich's conjecture on $\mathfrak{grt}$ and $H^\bullet(X, \mathcal{PV}_X)$

Let X be a smooth algebraic variety over $\mathbb{K}$ and $\mathcal{PV}_X$ be the sheaf of polyvector fields on X. In 1999, M. Kontsevich conjectured that the Lie algebra $\mathfrak{grt}$ acts on

$$H^\bullet(X, \mathcal{PV}_X).$$

This action is compatible with the cup product. Moreover, the action of a Deligne-Drinfeld element σ_n coincides with the action of n-th component $\mathrm{Ch}_n(X)$ of the Chern character of X.

In paper C. Rogers, T. Willwacher, V.D. "*Kontsevich's graph complex, GRT, and* ...", we proved this conjecture. The graph complex GC_2 and the cocycles $\{\gamma_n\}_{n \geq 3,\, \mathrm{odd}}$ play an important role in the proof of this conjecture.

Open problems

Problem ($\sim$ to the Deligne-Drinfeld conjecture on $\mathfrak{grt}$)

Prove (or disprove) that the Lie algebra $H^0(\mathrm{GC}_2)$ is freely generated by the cohomology classes $\{[\gamma_n]\}_{n \ge 3,\, odd}$.

Problem (Drinfeld-Kontsevich)

Prove (or disprove) that $H^1(\mathrm{GC}_2) = 0$.

Problem

Does the universal Vassiliev invariant distinguish isotopy classes of (framed) knots?

Problem ("Does the universal Vassiliev invariant detect knot orientation?")

Prove (or disprove) that every cocycle in $\mathrm{hGC}_{1,3}$ with an odd number of hairs is trivial.

More open problems

Problem

According to T.Q.T. Le and J. Murakami, the universal Vassiliev invariant for knots does not depend on the choice of a Drinfeld associator. Is there an explicit construction of the map

$$\pi_0\big(\mathrm{Emb}_{fr}(S^1, S^3)\big) \to \mathrm{CD}(S^1)$$

which involves neither Drinfeld associator nor any kind of integration?

Problem

Is there a combinatorial proof of the fact that for every odd $n \geq 3$, there exists a cocycle $\gamma_n \in \mathrm{GC}_2$ which "starts with" the wheel with n spokes (as above)?

Yet two more open problems

Problem

Is there a precise relationship between GC_d *(or some version of this complex) for even* d *and* $\mathrm{GC}_{\tilde{d}}$ *(or some version of this complex) for odd* $\tilde{d}$*? Can a link between BFV and BV quantizations help us to establish this relationship?*

Problem (Goodwillie-Weiss approach versus the configuration space integral)

Establish a link between the "more direct" configuration space integral approach (a lá Bott-Taubes) to embedding spaces and the approach based on the functor calculus and formality for operads assembled from configuration spaces.

Selected references on graph complexes

[1] J. Conant and K. Vogtmann, On a theorem of Kontsevich, Algebr. Geom. Topol. 3 (2003) 1167–1224.

[2] V.A. Dolgushev and C.L. Rogers, Notes on algebraic operads, graph complexes, and Willwacher's construction, *Mathematical aspects of quantization*, 25–145, Contemp. Math., **583**, AMS, Providence, RI, 2012; arXiv:1202.2937.

[3] A. Khoroshkin, T. Willwacher, and M. Zivkovic, Differentials on graph complexes, arXiv:1411.2369

[4] A. Khoroshkin, T. Willwacher, and M. Zivkovic, Differentials on graph complexes II - hairy graphs, arXiv:1508.01281

[5] M. Kontsevich, Formal (non)commutative symplectic geometry. *The Gelfand Mathematical Seminars,* 1990–1992, 173–187, Birkhäuser Boston, Boston, MA, 1993.

Further references related to graph complexes

[1] S. Barannikov, Modular operads and Batalin-Vilkovisky geometry, IMRN **19**, 2007.

[2] A. Hamilton and A. Lazarev, Graph cohomology classes in the Batalin-Vilkovisky formalism, J. Geom. Phys. **59**, 5 (2009) 555–575.

[3] E. Getzler, M.M. Kapranov, Modular operads, Compositio Math. **110**, 1 (1998) 65–126.

[4] K. Igusa, Graph cohomology and Kontsevich cycles, Topology **43**, 6 (2004) 1469–1510.

[5] M. Kapranov, Rozansky-Witten invariants via Atiyah classes, Compositio Math. **115**, 1 (1999) 71–113.

[6] M. Kontsevich, Rozansky-Witten invariants via formal geometry, Compositio Math. **115**, 1 (1999) 115–127.

Selected references on $\mathfrak{grt}$ and finite type knot invariants

[1] D. Bar-Natan, On the Vassiliev knot invariants, Topology **34**, 2 (1995) 423–472.

[2] F. Brown, Mixed Tate motives over $\mathbb{Z}$, Ann. of Math. (2) **175**, 2 (2012) 949–976.

[3] S. Chmutov, S. Duzhin, J. Mostovoy, Introduction to Vassiliev knot invariants, *Cambridge University Press*, 2012.

[4] V.A. Dolgushev, C.L. Rogers, and T.H. Willwacher, Kontsevich's graph complex, GRT, and the deformation complex of the sheaf of polyvector fields, Ann. of Math. (2) **182**, 3 (2015) 855–943.

[5] T. Willwacher, M. Kontsevich's graph complex and the Grothendieck-Teichmueller Lie algebra, Invent. Math. **200**, 3 (2015) 671–760.

Selected references on embedding spaces

[1] G. Arone and V. Turchin, On the rational homology of high-dimensional analogues of spaces of long knots, Geom. Topol. **18**, 3 (2014) 1261–1322.

[2] T.G. Goodwillie and M. Weiss, Embeddings from the point of view of immersion theory. II, Geom. Topol. **3** (1999)103–118.

[3] B. Munson and I. Volic, Cubical homotopy theory, Cambridge University Press, Cambridge, 2015. 625 pp.

[4] D.P. Sinha, Operads and knot spaces, JAMS **19**, 2 (2006) 461–486.

[5] M. Weiss, Embeddings from the point of view of immersion theory. I, Geom. Topol. **3** (1999) 67–101.

Further references on embedding spaces

[1] R. Bott and C. Taubes, On the self-linking of knots, J. Math. Phys. **35**, 10 (1994) 5247–5287.

[2] A.S. Cattaneo, P. Cotta-Ramusino, and R. Longoni, Configuration spaces and Vassiliev classes in any dimension, Algebr. Geom. Topol. **2** (2002), 949–1000.

[3] W. Dwyer and K. Hess, Long knots and maps between operads, Geom. Topol. **16**, 2 (2012) 919–955.

[4] J.E. McClure and J.H. Smith, A solution of Deligne's Hochschild cohomology conjecture, Contemp. Math., **293**, AMS, 2002.

[5] I. Volic, A survey of Bott-Taubes integration, J. Knot Theory Ramifications **16**, 1 (2007) 1–42.

Selected references on Kontsevich's formality and its generalizations

[1] V.A. Dolgushev, A Formality quasi-isomorphism for Hochschild cochains over rationals can be constructed recursively, *Replacement of* arXiv:1306.6733, *to appear*.

[2] V.A. Dolgushev, Stable formality quasi-isomorphisms for Hochschild cochains, arXiv:1109.6031.

[3] M. Kontsevich, Formality conjecture, *Deformation theory and symplectic geometry* (Ascona, 1996), 139–156, Math. Phys. Stud., 20, Kluwer Acad. Publ., Dordrecht, 1997.

[4] M. Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys., **66** (2003) 157–216.

[5] T. Willwacher, Deformation quantization and the Gerstenhaber structure on the homology of knot spaces, arXiv:1506.07078.

THANK YOU!