The Intricate Maze of Graph Complexes

Vasily Dolgushev

Temple University

*During the spring of 2016, V.D. is a visitor at the University of Pennsylvania.*
The cochain complex \( C_{PT,n}^\bullet \) consists of linear combinations\(^1\) of rooted planar trees with \( n \) labeled leaves (and without bivalent vertices):

The degree of such tree := − number of internal (i.e. blue) edges.

The differential \( \partial \) is defined as the sum over all possible contractions of internal edges (with appropriate sign factors).

\(^1\)The base field \( \mathbb{K} \) has characteristic zero.
For example,

The above vector $\nu$ has degree $-1$ and

$$\partial \nu = \begin{array}{c}
\begin{array}{c}
3 \\
2 \\
4 \\
1
\end{array}
\end{array} - \frac{7}{26} \begin{array}{c}
\begin{array}{c}
2 \\
3 \\
4 \\
1
\end{array}
\end{array}$$

It is not hard to show that

$$H^i(C_{PT,n}) = \begin{cases}
\mathbb{K} S_n & \text{if } i = 2 - n, \\
0 & \text{otherwise}.
\end{cases}$$

A neat topological proof of this fact is based on the use of Stasheff-Tamari polytopes (a.k.a. associahedra).
The cochain complex of brace trees $\text{Br}(n)$

A vector in $\text{Br}(n)$ is a linear combination of brace trees:

$$
\begin{align*}
2 & 3 & 4 \\
5 & 1 &
\end{align*}
\quad - \quad 
\begin{align*}
3 & 1 & 2 \\
5 & 4 &
\end{align*}
\quad \in \quad \text{Br}(5)
$$

The degree of a brace tree $T = 2 \times$ the number of neutral (black) vertices – the number of non-root edges.

The differential $\partial(T)$ of $T$ is the sum over all possible “splittings” of labeled and neutral vertices (with appropriate sign factors)
For example, 

\[ \partial \begin{array}{c}
1 \\
2 \\
3 
\end{array} = \begin{array}{c}
1 \\
2 \\
3 
\end{array} - \begin{array}{c}
2 \\
1 \\
3 
\end{array} \]

\[ \partial \begin{array}{c}
2 \\
1 
\end{array} = \begin{array}{c}
1 \\
2 
\end{array} - \begin{array}{c}
2 \\
1 
\end{array} \]
The cohomology of $\text{Br}(n)$ is...

**Theorem**

For every $n \geq 2$, we have

$$H^\bullet(\text{Br}(n)) \cong H_{-\bullet}(\text{Conf}(\mathbb{C}, n), \mathbb{K}),$$

where $\text{Conf}(\mathbb{C}, n)$ is the configuration space of $n$ points on $\mathbb{C}$:

$$\text{Conf}(\mathbb{C}, n) := \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}.$$ 

For example, the class of

$$\begin{array}{cc}
& 2 \\
1 & + \\
& 1
\end{array}$$

is represented by

the cycle

$$z_1 \rightarrow z_2 \leftarrow z_1$$

in $\text{Conf}(\mathbb{C}, 2)$. 

Vasily Dolgushev (Temple University)

Graph Complexes
The family of graph complexes $\{GC_d\}_{d \geq 1}$

We need:

- **a set** $\text{gra}_{n,k}$, $n \geq 1$, $k \geq 0$. An element of $\text{gra}_{n,k}$ is a directed graph $\Gamma$ with $V(\Gamma) = \{1, 2, \ldots, n\}$ and $E(\Gamma) = \{1, 2, \ldots, k\}$.

$\text{gra}_{n,k}$ is equipped with the action of $S_n \times (S_k \ltimes S_k^2)$;

- **the graded vector space**

$$fGC_d := \bigoplus_{n \geq 1, k \geq 0} \left( \mathcal{S}^{nd-d+(1-d)k} \otimes \text{K} \otimes \text{Or}^{n,k}_d \right) S_n \times (S_k \ltimes S_k^2),$$

where $\mathcal{S}$ is the operator which shifts the degree up by 1 and

$$\text{Or}^{n,k}_d := \begin{cases} 
\text{sgn}_{S_k} & \text{if } d \text{ is even}, \\
\text{sgn}_{S_n} \otimes (\text{sgn}_{S_2})^k & \text{if } d \text{ is odd}.
\end{cases}$$
For example, 

In fGC\(_2\) (i.e. when \(d\) is even), we can forget about directions on edges, discard graphs with multiple edges, identify graphs which differ merely by the labels on vertices, and keep in mind that, the action of every odd permutation in \(S_k\) alters the sign factor in front of a graph:

\[
\begin{align*}
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 3
\end{array}
\quad &=
\begin{array}{ccc}
1 & 2 & 3 \\
2 & 1 & 3
\end{array}
\quad =
\begin{array}{ccc}
2 & 1 & 3 \\
1 & 2 & 3
\end{array}
\quad =
\begin{array}{ccc}
1 & 2 & 3 \\
1 & 2 & 3
\end{array}
\end{align*}
\]

In fGC\(_3\) (i.e. when \(d\) is odd), we have\(^2\)

\[
\begin{align*}
\begin{array}{ccc}
3 & 2 & 1 \\
1 & 2 & 2
\end{array}
\quad &=
\begin{array}{ccc}
3 & 2 & 1 \\
2 & 1 & 1
\end{array}
\quad =
\begin{array}{ccc}
2 & 3 & 1 \\
1 & 1 & 2
\end{array}
\quad =
\begin{array}{ccc}
3 & 2 & 1 \\
1 & 2 & 2
\end{array}
\end{align*}
\]

\(^2\)Multiple edges for fGC\(_3\) are allowed.
$fGC_d$ is a (graded) Lie algebra

$fGC_d$ has the Lie bracket:

$$[\Gamma, \tilde{\Gamma}] := \Gamma \circ \tilde{\Gamma} - (-1)^{|\Gamma||\tilde{\Gamma}|} \tilde{\Gamma} \circ \Gamma, \quad \Gamma \circ \tilde{\Gamma} := \sum_{i=1}^{n} \pm \Gamma \circ_i \tilde{\Gamma},$$

where $\circ_i$ is this operation:

$$\Gamma \circ_i \tilde{\Gamma} := \sum \Gamma \tilde{\Gamma} i$$
For example, ...

If $d$ is odd, we should also keep track of directions on edges and sign factors.
The graph complex $\mathcal{GC}_d$

The graph $\Gamma_e = 1 \to 2$ satisfies the equation $[\Gamma_e, \Gamma_e] = 0$. So we introduce the differential on $\mathfrak{fGC}_d$

$$\partial := [\Gamma_e, ] .$$

**Definition**

The graph complex $\mathcal{GC}_d$ is the subcomplex of $\mathfrak{fGC}_d$ which involves only graphs $\Gamma$ satisfying these properties:

- $\Gamma$ is connected and
- each vertex of $\Gamma$ has valency $\geq 3$. 

Vasily Dolgushev (Temple University)
Let $j$ be an integer. $\mathsf{hGC}_{j,d}$ is a modification of $\mathsf{GC}_d$ which involves graphs with a non-empty set of univalent vertices. As the graded vector space, $\mathsf{hGC}_{j,d}$ is

\[
\bigoplus_{n \geq 0, m \geq 1, k} \left( s^{nd + mj + (1 - d)k} \mathsf{Khgra}(m, n, k) \otimes \mathsf{Or}_{j,d}^{m,n,k} \right) S_m \times S_n \times (S_k \ltimes S_k^2),
\]

where $\mathsf{hgra}_{m,n,k}$ is the set of connected graphs with the set of univalent vertices $\{1, 2, \ldots, m\}$, the set of vertices $\{1, 2, \ldots, n\}$ of valency $\geq 3$ and the set of (directed) edges $\{1, 2, \ldots, k\}$. Finally, $\mathsf{Or}_{j,d}^{m,n,k}$ is a 1-dimensional representation of $S_m \times S_n \times (S_k \ltimes S_k^2)$ which “takes care” of identifying certain graph with appropriate sign factors.

The differential is defined as the sum of expansions of vertices of valency $\geq 3$ and keeping only the graphs satisfying our defining conditions.
The degree of a graph in $hGC_{j,d}$ is

\[ d \times \text{the number of vertices of val. } \geq 3 \]

\[ + j \times \text{the number of univalent vertices} + (1 - d) \times \text{the number of edges}. \]

Examples of identifications in $hGC_{j,d}$:

\[ = (-1)^d \]

\[ = (-1)^j \]

\[ = (-1)^{(1-d)} \]
Example of computing the differential in $hGC_{j,d}$
$hGC_{1,3}$ shows up in knot theory!

A simple combinatorial argument $\Rightarrow hGC_{1,3}$ lives in degrees $\leq 0$.

Furthermore, due to D. Bar-Natan, "On the Vassiliev knot invariants",

$$H^0(hGC_{1,3}) := hGC^0_{1,3} / \partial(hGC^{-1}_{1,3})$$

is isomorphic to the space $CD(S^1)$ of chord diagrams.

Every finite type invariant of framed knots factors through the map

$$LM_\Phi : \pi_0(\text{Emb}_{fr}(S^1, S^3)) \longrightarrow CD(S^1).$$

This map was constructed by M. Kontsevich, T.Q.T. Le and J. Murakami. It is often called the universal Vassiliev invariant.
Topological meaning of $h\text{GC}_{j,d}$

Due to results of G. Arone, T. Goodwillie, J. R. Klein, M. Kontsevich, P. Lambrechts, J. E. McClure, D. Sinha, J.H. Smith, V. Turchin, I. Volic, M. Weiss, and ..., the rational homotopy groups

$$\pi_\bullet \left( \text{Emb}_c(R^j, R^d) \right) \otimes \mathbb{Q}$$

of the embedding space $\text{Emb}_c(R^j, R^d)$ can be expressed in terms of $H^\bullet(h\text{GC}_{j,d})$ if $d \geq 2j + 2$.

For example, $\mathbb{R} \to \mathbb{R}^4$:

\[
\begin{array}{c}
\text{---} \quad \rightarrow \\
\end{array} \quad \\
\begin{array}{c}
\hbox{non-planar graph}
\end{array}
\]
Examples of degree zero cocycles in $GC_2$ are...

$\gamma_3 = \begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (2,0);
\draw (0,0) -- (0,2);
\draw (0,0) -- (-2,0);
\end{tikzpicture}$

$\gamma_5 = \begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (2,0);
\draw (0,0) -- (0,2);
\draw (0,0) -- (-2,0);
\draw (0,0) -- (0,-2);
\draw (0,0) -- (2,-2);
\draw (0,0) -- (-2,-2);
\end{tikzpicture}$

$\gamma_7 = \begin{tikzpicture}[scale=0.5]
\draw (0,0) -- (2,0);
\draw (0,0) -- (0,2);
\draw (0,0) -- (-2,0);
\draw (0,0) -- (0,-2);
\draw (0,0) -- (2,-2);
\draw (0,0) -- (-2,-2);
\end{tikzpicture}$

Here $\ldots$ is a linear comb. of graphs with $\geq 2$ vertices of valency $\geq 4$. 

$+ \ldots$ and so on.
Examples of degree zero cocycles in $hGC_{1,3}$ are...

\[ w_2 = \begin{array}{c}
\text{Diagram 1}
\end{array} \quad w_4 = \begin{array}{c}
\text{Diagram 2}
\end{array} \]

\[ w_6 = \begin{array}{c}
\text{Diagram 3}
\end{array} \quad \text{and so on.} \]
A remark about the Euler characteristic

Let $\chi$ be an integer and $GC_d,\chi$ (resp. $hGC_{j,d,\chi}$) be the subcomplex of $GC_d$ (resp. $hGC_{j,d}$) spanned by graphs with the Euler characteristic $\chi$.

It is easy to see that

$$GC_d = \bigoplus_{\chi} GC_d,\chi, \quad hGC_{j,d} = \bigoplus_{\chi} hGC_{j,d,\chi}.$$

Moreover, $H^\bullet(GC_d,\chi)$ is isomorphic (up to a degree shift) to $H^\bullet(GC_{2,\chi})$, if $d$ is even, and isomorphic (up to a degree shift) to $H^\bullet(GC_{3,\chi})$ if $d$ is odd.

Similarly, $H^\bullet(hGC_{j,d,\chi})$ “depends” (up to a degree shift) only on the parities of $j$ and $d$. So, for the hairy case, it is sufficient to study only these four complexes $hGC_{1,3}$, $hGC_{2,3}$, $hGC_{1,2}$, $hGC_{2,2}$. 
What do we know about $GC_2$?

Due to T. Willwacher “M. Kontsevich’s graph complex and . . .”, we know that

$$H^<0(GC_2) = 0 \quad \text{and} \quad H^0(GC_2) \cong \mathfrak{grt} \quad \text{as the Lie algebras,}$$

where $\mathfrak{grt}$ is the Grothendieck-Teichmueller Lie algebra introduced by V. Drinfeld in 1990.

It is conjectured that

$$H^1(GC_2) \not\cong 0.$$

Due to results of A. Khoroshkin, T. Willwacher, and M. Zivkovic “Differentials on . . .” and a theorem of F. Brown, we know that “there are plenty of” cohomology classes in $H^{>1}(GC_2)$. 
To define the Lie algebra $\mathfrak{gr} t$, we need...

The family of Drinfeld Kohno Lie algebras $t_m$, $(m \geq 2)$. For every $m$, $t_m$ is generated by $\{t_{ij} = t_{ji}\}_{1 \leq i \neq j \leq m}$ subject to

$$[t_{ij}, t_{ik} + t_{jk}] = 0 \quad \#\{i, j, k\} = 3,$$

$$[t_{ij}, t_{kl}] = 0 \quad \#\{i, j, k, l\} = 4.$$ 

and the free Lie algebra $\mathfrak{lie}(x, y)$ in two symbols $x, y$. 
elements $\sigma(x, y) \in \text{lie}(x, y)$ satisfying

$$\sigma(y, x) = -\sigma(x, y),$$

$$\sigma(x, y) + \sigma(y, -x - y) + \sigma(-x - y, x) = 0,$$

$$\sigma(t^{23}, t^{34}) - \sigma(t^{13} + t^{23}, t^{34}) + \sigma(t^{12} + t^{13}, t^{24} + t^{34})$$

$$- \sigma(t^{12}, t^{23} + t^{24}) + \sigma(t^{12}, t^{23}) = 0.$$

The Lie bracket on $\mathfrak{grt}$ is the Ihara bracket:

$$[\sigma, \sigma']_{\text{Ih}} := \delta_\sigma(\sigma') - \delta_{\sigma'}(\sigma) + [\sigma, \sigma']_{\text{lie}(x, y)},$$

where $[\ , \ ]_{\text{lie}(x, y)}$ is the usual bracket on $\text{lie}(x, y)$ and $\delta_\sigma$ is the derivation of $\text{lie}(x, y)$ defined by

$$\delta_\sigma(x) := 0, \quad \delta_\sigma(y) := [y, \sigma(x, y)].$$
Deligne-Drinfeld elements of $\mathfrak{grt}$

The above equations have neither linear nor quadratic solutions. The first non-trivial example of an element in $\mathfrak{grt}$ is

$$\sigma_3(x, y) = [x, [x, y]] - [y, [y, x]].$$

More generally,

**Proposition (V. Drinfeld, 1990)**

For every odd integer $n \geq 3$ there exists a non-zero vector $\sigma_n \in \mathfrak{grt}$ of degree $n$ in symbols $x$ and $y$ such that

$$\sigma_n = \text{ad}^{n-1}_x(y) + \ldots$$

where $\ldots$ is a sum of Lie words of degrees $\geq 2$ in the symbol $y$.

$\{\sigma_n\}_{n \text{ odd } \geq 3}$ are called Deligne-Drinfeld elements of $\mathfrak{grt}$. 
The Deligne-Drinfeld Conjecture states that...

**Conjecture**

The Lie algebra $\mathfrak{g} \mathfrak{t}$ is freely generated by elements $\{\sigma_n\}_{n \text{ odd} \geq 3}$.

Numerical experiments performed by L. Albert, M. Espie, P. Harinck, J.-C. Novelli, G. Racinet, and C. Torossian show that this conjecture holds up to degree 16.

**Remark**

A result proved by F. Brown in 2011 implies that there exists a collection of Deligne-Drinfeld elements which generates a free Lie subalgebra in $\mathfrak{g} \mathfrak{t}$.

**Remark (about $\mathfrak{g} \mathfrak{t} \cong H^0(GC_2)$)**

Willwacher’s isomorphism $\mathfrak{g} \mathfrak{t} \cong H^0(GC_2)$ sends the elements $\{\sigma_n\}_{n \text{ odd} \geq 3}$ to the cohomology classes of $\{\gamma_n\}_{n \text{ odd} \geq 3}$ introduced above.
The problem of deformation quantization

Recall that a star product on a smooth manifold $M$ is an $\mathbb{R}[[\varepsilon]]$-bilinear associative multiplication

$$
* : C^\infty(M)[[\varepsilon]] \otimes C^\infty(M)[[\varepsilon]] \rightarrow C^\infty(M)[[\varepsilon]]
$$

of the form

$$
f \ast g = f \cdot g + \sum_{k \geq 1} \varepsilon^k B_k(f, g),
$$

where $\varepsilon$ is a formal parameter and $B_k$ are bidifferential operators.

The problem of deformation quantization is to describe equivalence classes of such star products on arbitrary manifold $M$. 
Kontsevich’s formality theorem

Let us denote by $PV^\bullet(M)$ the algebra of polyvector fields on $M$. For example, $PV^0(M) = C^\infty(M)$ and $PV^1(M)$ is the space of vector fields. $PV^\bullet(M)$ carries a natural Lie bracket $[\ ,\ ]_S$.

In 1997, M. Kontsevich proved that a theorem which implies that equivalence classes of star products on $M$ are in bijection with (equivalence classes of)

$$\alpha = \varepsilon \alpha_1 + \varepsilon^2 \alpha_2 + \cdots \in \varepsilon \ PV^2(M)[[\varepsilon]], \quad [\alpha, \alpha]_S = 0.$$

Such $\alpha$ may be thought of as a generalization of a Poisson structure on $M$ and Kontsevich’s formality quasi-isomorphism may be thought of a “bridge” connecting classical mechanics to quantum mechanics.
In my paper “Stable formality quasi-isomorphisms for Hochschild cochains”, I proved a theorem which can be stated colloquially as:

**Theorem (a colloquial statement!)**

The group \( \exp \left( H^0(GC_2) \right) \)
acts simply transitively on the set of bridges connecting classical mechanics to quantum mechanics.

Combining this result with Willwacher’s theorem, we conclude that:

**Corollary (a colloquial statement!)**

The Grothendieck-Teichmueller group \( \exp(\mathfrak{grt}) \) acts simply transitively on the set of bridges connecting classical mechanics to quantum mechanics.
Let $X$ be a smooth algebraic variety over $\mathbb{K}$ and $\mathcal{PV}_X$ be the sheaf of polyvector fields on $X$. In 1999, M. Kontsevich conjectured that the Lie algebra $\mathfrak{grt}$ acts on

$$H^\bullet(X, \mathcal{PV}_X).$$

This action is compatible with the cup product. Moreover, the action of a Deligne-Drinfeld element $\sigma_n$ coincides with the action of $n$-th component $Ch_n(X)$ of the Chern character of $X$.

In paper C. Rogers, T. Willwacher, V.D. “Kontsevich’s graph complex, GRT, and ...”, we proved this conjecture. The graph complex $GC_2$ and the cocycles $\{\gamma_n\}_{n \geq 3, \text{odd}}$ play an important role in the proof of this conjecture.
### Open problems

<table>
<thead>
<tr>
<th>Problem (∼ to the Deligne-Drinfeld conjecture on $\mathfrak{g}_\mathfrak{t}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prove (or disprove) that the Lie algebra $H^0(GC_2)$ is freely generated by the cohomology classes ${[\gamma n]}_{n \geq 3, \text{odd}}$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem (Drinfeld-Kontsevich)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prove (or disprove) that $H^1(GC_2) = 0$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Does the universal Vassiliev invariant distinguish isotopy classes of (framed) knots?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem (“Does the universal Vassiliev invariant detect knot orientation?”)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prove (or disprove) that every cocycle in $hGC_{1,3}$ with an odd number of hairs is trivial.</td>
</tr>
</tbody>
</table>
More open problems

Problem

According to T.Q.T. Le and J. Murakami, the universal Vassiliev invariant for knots does not depend on the choice of a Drinfeld associator. Is there an explicit construction of the map

\[ \pi_0(\text{Emb}_{fr}(S^1, S^3)) \to \text{CD}(S^1) \]

which involves neither Drinfeld associator nor any kind of integration?

Problem

Is there a combinatorial proof of the fact that for every odd \( n \geq 3 \), there exists a cocycle \( \gamma_n \in \text{GC}_2 \) which “starts with” the wheel with \( n \) spokes (as above)?
Yet two more open problems

**Problem**

Is there a precise relationship between $GC_d$ (or some version of this complex) for even $d$ and $GC_{\tilde{d}}$ (or some version of this complex) for odd $\tilde{d}$? Can a link between BFV and BV quantizations help us to establish this relationship?

**Problem (Goodwillie-Weiss approach versus the configuration space integral)**

Establish a link between the “more direct” configuration space integral approach (a lâ Bott-Taubes) to embedding spaces and the approach based on the functor calculus and formality for operads assembled from configuration spaces.
Selected references on graph complexes


Further references related to graph complexes


Selected references on $\mathfrak{grt}$ and finite type knot invariants


Selected references on embedding spaces


Further references on embedding spaces


Selected references on Kontsevich’s formality and its generalizations

[1] V.A. Dolgushev, A Formality quasi-isomorphism for Hochschild cochains over rationals can be constructed recursively, 


THANK YOU!