Lecture 5

$HH^0(A)$ for $A = \text{K} \langle x', x^2, \ldots, x^n \rangle$

(we assume that char K = 0)

It is known that $\text{Der}(A)$ are exactly vector fields on $\text{K}^n$.

We consider the following graded $\text{K}$-space

$\Lambda^0 \text{Der}(A) := \Lambda_A^0 \text{Der}(A)$

Every element of $\Lambda^m_A \text{Der}(A)$ can be written as

$\gamma = \gamma^{i_1 \ldots i_m}(x) \cdot \partial_{x^{i_1}} \Lambda \partial_{x^{i_2}} \Lambda \ldots \Lambda \partial_{x^{i_m}}$

where

$\partial_{x^{i_1}} \Lambda \partial_{x^{i_2}} \Lambda \ldots \Lambda \partial_{x^{i_m}} =$

$= \sum_{\sigma \in S_m} \frac{1}{m!} (-1)^{1_{16}} \partial_{x^{i_{\sigma(1)}}} \partial_{x^{i_{\sigma(2)}}} \ldots \partial_{x^{i_{\sigma(m)}}}$
\[ f_i \ldots f_n(x) \in \mathbb{k}(x_1, \ldots, x_n) \]

Thus  \( \Lambda^0 \text{Der}(A) = A \)

\( \Lambda^1 \text{Der}(A) = \text{Der}(A) \)...

\( \Lambda^0 \text{Der}(A) \) is the graded \( \mathbb{k} \)-space of polyvector fields (or polyderivations or multi-derivations)

\( \Lambda^0 \text{Der}(A) \) is a graded commutative algebra

\[ f, g \rightarrow f \Lambda g \]

\( \Lambda^{0+1} \text{Der}(A) \) is a graded Lie algebra

\( a, b \in A \)  \( \upsilon, \omega \in \text{Der}(A) \) we have

\[ [a, b]_S = 0 \quad [\upsilon, \omega]_S = \upsilon(\omega) \]

\[ [\upsilon, \omega]_S = \text{Lie bracket of \( \upsilon \) fields} \quad \text{and} \]

\[ -2 \]
\[ [\delta, 6, \Lambda 6_2]_5 = 6, \Lambda [\delta, 6_2]_5 \]

\[ +(-1)^{151(16_2+1)} [\delta, 6_1]_5 \Lambda 6_2 \]

is called the Schouten bracket.

Remark: Our definition of \( L, J_5 \)

\[ \hat{L} \]

is not conventional one.

The usual convention for the Leibnitz rule is

\[ [\delta, 6, \Lambda 6_2]_5 = [\delta, 6_1]_5 \Lambda 6_2 + \]

\[ +(-1)^{151(16_2+1)} 6_1 \Lambda [\delta, 6_2]_5 \]

These two different brackets are related by the rule

\[ [\delta, 6] = (-1)^{151}, 6_1 \Lambda [\delta, 6]_5 \]

The algebras \((\Lambda^{\infty+1}\text{Der}(A), L, J_5)\) and \((\Lambda^{\infty+1}\text{Der}(A), L, J_5)\) are isomorphic.
The isomorphism is given by
\[ f \rightarrow (-1)^{\frac{n}{2}} f \]

Define $f: K \rightarrow C$ as a map of (co)chain complexes. If $f$ is called a quasi-isomorphism if $f$ induces an isomorphism
\[ H^i(f): H^i(K) \cong H^i(C) \]

Every polyvector $\gamma \in \text{Der}^m(A)$ may be viewed as a Hochschild cochain in $C^m(A)$
\[ V(\gamma)(a_1, \ldots, a_m) = \gamma^{i_1 \ldots i_m}(a) \cdot a_{i_1} \cdot a_{i_2} \cdots a_{i_m} \]
Thus we get a natural map
\[ V: \text{Der}^1(A) \rightarrow C^1(A) \]

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Proposition: If we consider $\Lambda^\bullet \text{Der}(A)$ as a cochain complex with the zero differential, then

$$V: \Lambda^\bullet \text{Der}(A) \to (C^\bullet(A), \partial)$$

is a map of cochain complexes.

Proof: Let $f_i \cdots \mu_i(x)$ be a tensor that is not necessarily antisymmetric.

Exercise 4.1 in the notes.

Show that $a_1 \cdots a_m$,

$$f_i \cdots \mu_i(x) \partial_i a_1 \partial_\mu a_2 \cdots \partial_m a_m$$

is a cocycle in $C^m(A)$.

The exercise implies that

$$\partial V(f) = 0 = V(\text{difference in } \Lambda^\bullet \text{Der}(A))$$

The proposition is proved.
Theorem The map

\[ V : (\wedge^n \text{Der}(A), 0) \to (C^n(A), \partial) \]

is a quasi-isomorphism of cochain complexes.

The induced isomorphism

\[ H^i(V) : \wedge^n \text{Der}(A) \to H H^i(A) \]

is an isomorphism of graded commutative algebras and

\[ H^i(V) : \wedge^{n+1} \text{Der}(A) \to H H^{i+1}(A) \]

is an isomorphism of Lie algebras.

Proof By \( E[\theta^1, \theta^n] \) we denote the exterior algebra in variables \( \theta^1, \ldots, \theta^n \):

\[ E[\theta^1, \theta^n] = \frac{\text{free assot. alg}(\theta^1, \theta^n)}{(\theta^i \theta^j + \theta^j \theta^i)} \]
$E_m(\theta^i \theta^m)$ is the $k$-th component of $E(\theta^i \theta^m)$

$E_m(\theta^i \theta^m) = \text{span} \left\{ \theta^i \theta^m \right\}$

The Koszul complex is the graded $k$-space

$k = \bigotimes_k \left[ x^1, x^n, y^1, y^n \right] \otimes E_m(\theta^i \theta^m)$

with the differential $\delta$

$\delta = (x^i - y^i) \frac{\partial}{\partial \theta^i}$

$k(\delta, \delta) \otimes \text{E}_m(\delta) \overset{\delta}{\rightarrow} k(\delta, \delta) \overset{\delta}{\rightarrow} k(\delta, \delta)$

$\deg = 2 \quad \deg = 1 \quad \deg = 0$

$A = \bigotimes_k \left[ x^1, x^n \right] \quad A \otimes A^0 = \bigotimes_k \left[ x^1, x^n, y^1, y^n \right]$
Let us show that $K_n$ is a resolution of $A$ as a free $A \otimes \mathbb{A}^n$-module.

(Here $A = k[x_1, \ldots, x_n]$)

Change of variables

$$(x_1^{\circ}, y_1^{\circ}) \rightarrow (x_1^{\circ}, z_1^{\circ})$$

Then

$$z_1^{\circ} = x_1^{\circ} - y_1^{\circ}$$

We introduce the operator

$$\delta^* = \theta_i \frac{\partial}{\partial z_i}$$

It is easy to see that

$$\delta \delta^* + \delta^* \delta = \theta_i \frac{\partial}{\partial \theta_i} + z_i \frac{\partial}{\partial z_i}$$

counts

The degree in $\theta$

in $\theta$

in $z$
\[ \gamma : K_0 \to K_{0+1} \]

\[ \delta(\theta) = \begin{cases} 
\frac{\delta^*}{|\theta_1 + \theta_2|} \theta & \text{if} \quad \theta = \theta^{k \in \mathbb{Z}} x^3
\text{and} \quad k + l > 0 \\
0 & \text{if} \quad \theta = x^3
\end{cases} \]

\[ K = \bigoplus K_0 \]

let \( \lambda : K \to A \)

\[ \lambda(\theta) = \theta |_{z = 0} \theta = 0 \]

we have \( \forall \theta \in K \)

\[ \theta = \lambda(\theta) + \delta \tilde{\delta} \theta + \tilde{\delta} \delta \theta \]

\[ \begin{array}{ccc}
\rightarrow & 0 & \rightarrow 0 & \rightarrow A \\
\downarrow & \downarrow & \downarrow & \downarrow \lambda \\
\delta & K_2 & \delta & K_1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\delta & K_2 & \delta & K_0
\end{array} \]
\[
\lambda i = i \lambda \quad \text{and} \\
\lambda \partial = \partial \lambda \quad \text{is chain homotopic to} \quad \lambda \partial
\]

\[
\delta = \lambda \partial(\theta) = \delta \delta \theta + \delta \delta \theta
\]

\[
\Rightarrow \quad i \text{ and } \lambda \text{ give } \varphi \quad \text{as}
\]

\[
\varphi \text{-isomorphisms } \text{between}
\]

\[
\rightarrow 0 \rightarrow 0 \rightarrow A
\]

\[
\text{and} \quad K
\]

\[
i \text{ and } \lambda \text{ are compatible with}
\]

\[
A \text{-bi-module structure}
\]

Thus \( K \) is indeed a resolution of \( A \)

(as \( A \)-bi-module)

It is clear that

\[
\text{Hom}_{A \otimes A^{op}}(K^m, A) = \Lambda^m \text{Der}(A)
\]
It is also clear that
the differential on
\( \text{Hom}_A^{\text{op}}(K, A) \) is zero.
Consider the following map of
\( A \)-bimodules.

\[ \varphi: K \to B_\omega(A) \]

\[ \varphi(a(x), b(y)) \Theta_{\omega}^d \Theta_{\omega}^{\text{tr}} = \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^{\sigma(1)} (a(x), x^{1 \sigma(1)}, x^{1 \sigma(2)}, \ldots, x^{1 \sigma(m)}, b(y)) \]

**Exercise 4.2** Show that \( \varphi \) is

a map of cochain complexes

\( \varphi \) fits into the following commutative diagram

\[
\begin{array}{ccc}
K & \xrightarrow{\varphi} & B_\omega(A) \\
\downarrow \varphi & \Rightarrow & \downarrow \text{aug} = \mu \\
\text{aug} \circ \varphi & \Rightarrow & A
\end{array}
\]

Since \( \varphi \) and \( \mu \) are \( \mathbb{R} \)-isomorphisms
so is \( \varphi \).
Thus \( \varphi \) gives us a \( q \)-isomorphism

\[
\varphi = \text{Hom}(\varphi,)
\]

\[
\text{Hom}_{A \otimes A^\op} \left( B/(1), A \right) \rightarrow \text{Hom}_{A \otimes A^\op} \left( K_n(A), A \right)
\]

\[
C^i(A) \xrightarrow{\varphi} \Lambda^i \text{Der}(A)
\]

\[
\varphi(P) = \frac{1}{m!} \sum_{\sigma \in S_m} (-1)^{\sigma} \left( x^{\sigma(1)}, x^{\sigma(2)}, \ldots, x^{\sigma(m)} \right)
\]

\[
\times \sigma_i \wedge \sigma_i \wedge \ldots \wedge \sigma_i
\]

It is clear that \( \varphi \circ V = \text{id} \Lambda^i \text{Der}(A) \)

\[
\Lambda^i \text{Der}(A) \xrightarrow{\text{id}} \Lambda^i \text{Der}(A)
\]

\[
V \circ \varphi \sim \text{id}
\]

\[
C^i(A) \xrightarrow{\varphi} \Lambda^i \text{Der}(A)
\]

\[\Rightarrow \] \( \Lambda^i \text{Der}(A) \) induces a map of comm. alg. (Lie alg.s) on the level of cohomology.
Let us show that

\[ V(\delta_i) U V(\delta_2) - V(\delta_i \wedge \delta_2) = \mathcal{O}(\ldots) \]

when \( \delta_i \in A \) it is obvious.

Therefore it suffices to show that

\[ V(\delta_i \wedge \ldots \wedge \delta_i) U V(\delta_j \wedge \ldots \wedge \delta_j) \]

\[ - V(\delta_i \wedge \ldots \wedge \delta_i \wedge \delta_j \wedge \ldots \wedge \delta_j) \]

\[ = \mathcal{O}(\ldots) \]

For \( k = 1 \) we have

\[ \delta_i \otimes \delta_j \wedge \ldots \wedge \delta_j = \delta_i \wedge \delta_j \wedge \ldots \wedge \delta_j = \]

\[ \sum_{\sigma \in S_k} \pm \delta_j^{(\sigma(1))} \otimes \delta_j^{(\sigma(2))} \otimes \ldots \otimes (\delta_i \otimes \delta_j^{(\sigma(1))} + \delta_j^{(\sigma(1))} \otimes \delta_i) \]

\[ \otimes \delta_j^{(\sigma(3))} \otimes \ldots \otimes \delta_j^{(\sigma(m))} \]
Exercise: Show that

\[
\partial_j \wedge \partial_{j_2} \wedge \ldots \wedge (\partial_i \wedge \partial_{j_3} + \partial_{j_3} \wedge \partial_i) \wedge \partial_{j_{i+1}} \wedge \ldots \wedge \partial_{j_{m}} = \\
= \mathcal{H}och \left[ \sum (-1)^{s} \partial_j \wedge \partial_{j_2} \wedge \ldots \wedge \partial_{j_{s}} \wedge \partial_{j_{s+1}} \wedge \ldots \wedge \partial_{j_{m}} \right]
\]

Thus, for a vector field $v$ we have

\[
V(v) \mathcal{V}(\mathcal{V}) - \mathcal{V}(v \mathcal{V}) = \mathcal{D}(\ldots)
\]

Then we proceed by induction on the degree of a polyvector:

\[
V(\mathcal{V}_1 \wedge \mathcal{V}_2) \mathcal{V}(\mathcal{V}) - V(\mathcal{V}_1 \wedge \mathcal{V}_2 \mathcal{V}) = \\
= V(\mathcal{V}_1) V(\mathcal{V}_2) \mathcal{V}(\mathcal{V}) - \mathcal{D}(\ldots) V(\mathcal{V}) \\
- V(\mathcal{V}_1 \wedge \mathcal{V}_2 \mathcal{V}) = V(\mathcal{V}_1) V(V(\mathcal{V}_2 \mathcal{V})) \\
- V(\mathcal{V}_1 \wedge \mathcal{D}(\ldots) - \mathcal{D}(\ldots) V(\mathcal{V}) - V(\mathcal{V}_1 \wedge \mathcal{V}_2 \mathcal{V}) \\
= \mathcal{D}(\ldots) \text{ by inductive assumption}
\]
As a graded commutative algebra $HH^*(A)$ is generated by $HH^0(A)$ and $HH^1(A)$

Due to Leibniz rule it suffices to check

$[V(\delta_1), V(\delta_2)]_g - V([\delta_1, \delta_2]) = 0$\quad (\ldots)

for $\delta_1, \delta_2 \in A \oplus \text{Der}_2(A)$

But for $\delta_1, \delta_2 \in A \oplus \text{Der}_2(A)$

$[V(\delta_1), V(\delta_2)]_g - V([\delta_1, \delta_2]) = 0$

The Hochschild-Kostant-Rosenberg Thom is proved.

Remark The Hochschild-Kostant-Rosenberg Thom holds for any regular commutative algebra.