Lecture 14  Construction of the quasi-isomorphism

\[ U : \text{Lie}(s \mathcal{C}(s^*P)) \rightarrow C^{*+}(A) \]

Let \( \{p_1, p_k, q_1, \ldots, q_m\} \) be a configuration representing a point in \( C_{k,m} \).

For every pair \( i,j \in \{1,2, \ldots, k\} \)

And pair \( i \in \{1,2, \ldots, k\} \) \( \ell \in \{1, \ldots, m\} \)

we have well-defined functions

\[ \psi(p_i, p_j) = \text{Arg} \left( \frac{p_j - p_i}{p_j - p_i} \right) = \frac{1}{2i} \ln \left( \frac{(p_j - p_i)(p_j - p_i)}{(p_i - p_i)(p_j - p_i)} \right) \]

\[ \psi(p_i, q_e) \]

\[ \psi(q_e, p_i) = \text{Arg} \left( \frac{q_e - p_i}{q_e - p_i} \right) \]

\[ \psi(q_e, q_e) = 0 \]
These functions are smooth and they extend smoothly to the boundary of $C_{k,m}$.

For example,

$$\phi(p_i, p_i) = \phi(p_i, p)$$

$$\phi(p, p_i) = \phi(p, p_i)$$

Then

$$\phi(p_i, p_i) = \phi(p_i, q)$$

$$\phi(p, p_i) = 0 (= \phi(q, p_i))$$

We call $\phi$ the angle function.
Def: An admissible graph $\Gamma$ is
an oriented graph with labels
such that

1) The set of vertices $V_\Gamma = \{1, 2, \ldots, k\} \cup \{\overline{1}, \overline{2}, \ldots, \overline{m}\}$
$k, m \in \mathbb{Z}_{\geq 0}$, $2k + m \geq 2$
The vertices from $\{1, 2, \ldots, k\}$ are called
the vertices of the first type
$\{\overline{1}, \overline{2}, \ldots, \overline{m}\}$ - vertices of the second type

2) every edge $(\nu_1, \nu_2) \in E_\Gamma$ starts
at a vertex of the 1st type

3) No top poles: \( \nu \)
\( (\) but we may have loops) \( )

we denote the set of edges originating
at $\nu \in \{1, 2, \ldots, k\}$ by $\text{Star}(\nu)$
We need to define

\[ U_k (\delta_1, \delta_2, \ldots, \delta_k) (a_1, \ldots, a_n) \]

\( \delta_i \in \Lambda^{d_i} \text{Der}(A) \) and \( a_i \in A \)

In other words \( |\delta_i| = d_i - 1 \)

Since \( |U_k| = 1 - k \) \( d_i \)‘s, \( k \), and \( n \) are related by the equation

\[ m - 1 = \sum_{i=1}^{k} (d_i - 1) + 1 - k \]

\[ m - 1 = \sum_{i=1}^{k} d_i - k + 1 - k \]

\[ 2k + m - 2 = \sum_{i=1}^{k} d_i \]

Let \( \Gamma \) be an admissible graph with \( k \) vertices of the 1st type
and in vertices of the second type
and there are exactly $d_i$ edges originating from the $i$-th vertex of the first type

To this graph $\Gamma$ we assign

$\Upsilon_\rho(\delta_1, \ldots, \delta_m)(\alpha_1, \ldots, \alpha_m)$ in the obvious way. Edges tell us how to form the differential expression in $\delta_1, \delta_2, \ldots, \delta_m, \alpha_1, \alpha_2, \ldots, \alpha_m$

For example, for $\lambda \in \Lambda^2 \, \text{Der}(A)$ and $\beta \in \Lambda^3 \, \text{Der}(A)$
\[ \mathcal{U}_\gamma (\beta) (x_1, x_2, x_3) = \lambda \delta (x) \beta (x) \prod_{i=1}^3 \delta_{x_i} a(x) \delta_{x_2} a_2 \delta_{x_3} a_3. \]

We define
\[ \mathcal{U} (\sigma_1, \ldots, \sigma_k) (x_1, \ldots, x_m) = \sum_{\text{perms} \gamma} W_{\gamma} \mathcal{U}_\gamma (\sigma_1, \ldots, \sigma_k | x_1, \ldots, x_m) \]

where (the weight of the graph)
\[ W_{\gamma} = \frac{1}{(2\pi)^{2k+m-2}} \int \frac{\Lambda \delta \psi}{\mathcal{C}_{k,m}} \]

Since \( \frac{k}{2} \sum_{i=1}^3 \delta_{x_i} = 2k+m-2 \) a top degree form on \( C_{k,m} \).
For example, for $k=1$ we have $z_1 + m - 2 = d_1$ so

$$m = d = d_1.$$ We have exactly 1 graph

From this picture we see that

$$\text{Arg}(q - p) = 2\pi - \alpha$$
$$\text{Arg}(q - \overline{p}) = \alpha$$

Thus

$$\text{Arg} \left( \frac{q - p}{q - \overline{p}} \right) = \pi + \varphi - (\pi - \varphi) = 2\varphi.$$

and

$$0 \leq \varphi \leq \pi$$
Thus for the graph $Γ$, we have

$$W_1 = \frac{1}{(2\pi)^m} \int 2^m \, d\xi_1 \, dx_2 \ldots \, dx_m = \int_{0 < \xi_1, \xi_2, \ldots, \xi_m \leq \pi} \ldots$$

"simplex"

$$= \frac{2^m}{(2\pi)^m} \int_0^\pi dx_1 \int_0^{\xi_1} dx_2 \int_0^{\xi_2} \ldots \int_0^{\xi_{m-1}} dx_m = \frac{1}{m!}$$

Thus $U_1(\mathbf{r}) = \frac{1}{m!} V(\mathbf{r})$

Where $V$ is from the Hochschild-Kostant-Rosenberg Theorem.

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How to prove that \( \mathcal{U} \) is a quasi-isomorphism?

We have the diagram:

\[
\begin{align*}
\mathcal{U} : & \quad \text{Lie}(s \mathfrak{C}(s^{\ast} \mathcal{P})^\ast)) \\
\end{align*}
\]

\[
\begin{align*}
P^* & \xrightarrow{\mathcal{U}} C^{\ast+1}(\mathcal{A}) \\
\end{align*}
\]

from the Hochschild-Kostant-Rosenberg theorem.

This diagram is NOT commutative.

\[
\begin{align*}
\delta_1, \delta_2 \in P^* \\
\end{align*}
\]

\[
\begin{align*}
\mathcal{U} : & \quad \delta_1, \delta_2 \\
\end{align*}
\]

\[
\begin{align*}
[\delta_1, \delta_2] & \xrightarrow{\delta_1, \delta_2} \left( V(\delta_1), V(\delta_2) \right) \mathcal{G} \\
\text{Example} & \quad \left( V(\delta_1, \delta_2) \right) \neq \left( V(\delta_1), V(\delta_2) \right) \mathcal{G}
\end{align*}
\]
Indeed the cohomology $H^i(\text{Lie}(\text{...}))$ is spanned by classes of $\mathcal{f} \in P$.

and $\mathcal{U} \mathcal{f} \in P^c \subseteq \text{Lie}(sC(s^pP))$

we have

$V(\mathcal{f}(\delta)) = \mathcal{U}(\delta)$

Since $H^i(\mathcal{f})$ and $H^i(\mathcal{U})$ are isomorphisms,

then so is $H^i(\mathcal{U})$. 
Remark about signs:

M. Kontsevich uses a convention for which

\[ U_k(... \sigma_1, \sigma_2...) = -(-1)^{|\sigma_1| |\sigma_2|} U_k(... \sigma_2, \sigma_1...) \]

We use a convention convention for which

\[ U_k(... \sigma_1, \sigma_2...) = (-1)^{(1+|\sigma_1|)(1+|\sigma_2|)} U_k(... \sigma_2, \sigma_1...) \]

To get from one sign convention to another we need to switch from

\[ s^1 \sigma_1 \otimes s^2 \sigma_2 \otimes ... \otimes s^k \]

to

\[ (s^1 \otimes ... \otimes s^{-1}) \otimes (\sigma_1 \otimes \sigma_2 \otimes ... \otimes \sigma_k) \]