Lecture 10

Canonical free resolution of a DGCA (Part 2)

Let us construct the Chevalley-Eilenberg complex for a DGCA $(\mathcal{C}, d, [-])$ by suspension $sV$ of a graded vector space $V$ (or a cochain complex $V$) we mean $E \otimes V$ where $E$ is a 1-dim vector space placed in degree 1

$$(sV)^{\circ} = \#V^{\circ -1}$$

Similarly for desuspension:

$$(s^{-1}V)^{\circ} = \#V^{\circ +1}$$

As a graded vector space the CE complex is $C(s^{-1}E)$ (the free cocommutative coalgebra co-generated by $s^{-1}E$)
The differential $Q: C(S^1 \mathbb{C}) \to C(S^1 \mathbb{C})$ is defined by declaring that $Q$ is a coderivation and

$$p \circ Q(a) = -2a \quad \text{for all } a \in \mathbb{C}$$
$$p \circ Q(ab) = (-1)^{a+1} [a, b]$$
$$p \circ Q(a_1 \cdots a_k) = 0 \quad \text{for all } a_i \in \mathbb{C}, k \geq 2$$

We keep in mind that $a, b, a_i$ are elements of $S^1 \mathbb{C}$.

The sign $(-1)^{a+1} [a, b]$ has this explanation:

$$S^1 \mathbb{C} \xrightarrow{S} S^1 \mathbb{C} \xrightarrow{S} S^1 \mathbb{C}$$

$S^{-1}a = a + 1 \pmod{2}$

To show that $Q^2 = 0$, we notice that $|Q| = 1$.
\[ Q^2 = \frac{1}{2}(202 + 202) = \frac{1}{2} [Q, Q] \]

\[ \Rightarrow Q^2 \text{ is also a derivation.} \]

So we need to check that

\[ p \cdot Q^2 = 0 \]

\[ p \cdot Q^2(\alpha) = p \cdot Q(-2\alpha) = (-2)^2 \alpha = 0 \]

We already know that

\[ Q(\alpha \theta) = Q(\alpha, \theta) + Q(\theta) \alpha + (-1)^{\sigma+1} Q(\theta) \]

\[ = (-1)^{\sigma+1} [\theta, \alpha] - 3 \alpha \theta + (-1)^{\sigma} \alpha \theta \theta \]

Applying \( p \cdot Q \) to \( \alpha \theta \) we get

\[ p \cdot Q(\alpha \theta) = p \cdot Q((-1)^{\sigma+1} [\alpha, \theta] - (2) \alpha \theta + (-1)^{\sigma} \alpha \theta \theta) \]

\[ = (-1)^{\sigma} \{ [\alpha, \theta] - (-1)^{\sigma+1} \} \]

\[ + (-1)^{\sigma+1} \{ [\alpha, \alpha \theta] \} = 0 \]

by of Leibniz rule.
Exercise 6.7 Show that the Jacobi identity for $\mathfrak{f}$ implies that $p_{\mathfrak{f}}Q^3(a,b,c) = 0$ for $a,b,c \in \mathfrak{f}$.

For $a_1, \ldots, a_k$, we have $k \geq 3$.

$$p_{\mathfrak{f}}Q^2(a_1, \ldots, a_k) = p_{\mathfrak{f}}(Q_k(a_1, \ldots, a_k)) + \sum_{i < j} \pm Q_p(a_{i+p}, a_{j+p+1}, a_{i+p+k}) \sum_{p+k < k} \sum_{0 \leq r < k} = 0$$

Therefore $Q_k = 0$ for $k \geq 3$.

Remark A general codimension $Q$ on $C(\mathfrak{f}, \mathfrak{g})$ is called an $L_\infty$-algebra structure on $\mathfrak{g}$. 
For general L-algebra $Q_{1}, Q_{2}$, may be $\neq 0$.

We still have $(Q_{2})^{2} = 0$

$Q_{1}$ and $Q_{2}$ are compatible in the sense of the Leibniz rule but $[\beta, Q] = (-1)^{|\beta|} Q_{2}(\beta, \beta)$

does not satisfy the Jacobi identity.

Instead we have

$$[\gamma, \epsilon] C + <p., = \pm 3 Q_{3}(\alpha, \beta, \gamma)$$

$$+ Q_{3}(\alpha, \epsilon, \gamma) + (-1)^{\alpha} Q_{3}(\alpha, \alpha, \gamma)$$

$$+ (-1)^{\gamma + \beta} Q_{3}(\alpha, \epsilon, \epsilon)$$

$Q_{0}Q = 0 \iff \text{10 quadratic relations of the form}$

$$Q_{p}(a_{6}(1), ..., a_{6}(p)) =$$

$$\sum_{i} \pm Q_{q+1} (Q_{p}(a_{6}(1), ..., a_{6}(p)), a_{6}(p+1), ..., a_{6}(p+q)) = 0$$
Let $B$ be a cocommutative coalgebra (without counit) possibly with a co-differential $\partial_B$.

We would like to describe the Harrison (co)chain complex of $B$.

As a vector space

$$\text{Harr}_n(B) = \text{Lie}_n(sB)$$

where $\text{Lie}_m(sB)$ is the $m$-th homogeneous component of the free Lie algebra generated by $sB$ (suspension of $B$).

The differential $\partial$ is defined by the requirement that $\partial$ is a derivation of $L, \partial$ on $\text{Lie}(sB)$ and
\[ 4 \phi \in \mathcal{B} \]

\[ \mathcal{D}(\phi) = -2^g \phi + \frac{1}{2} \sum (-1)^i \left[ \phi, \phi_i \right] \]

where \[ \Delta \phi = \sum_i \phi_i \otimes \phi_i \]

The identity \((\mathcal{D}^2) = 0\), the Leibniz rule for \(\mathcal{D}^2\) and \(\Delta\) and the associativity of \(\Delta \Rightarrow (\mathcal{D}^2) = 0\).

Let \(\mathcal{E}\) be a DGLA with differential and the bracket \([,]\).

The canonical free resolution of \(\mathcal{E}\) is

\[ (\text{Lie}(C(s^{-1}E)), \mathcal{D}, [ , ]) \]

(We use \([,]\) on \(\text{Lie}(...)\) to distinguish it from \([,]\) on \(E\))
We now want to construct a 9-isomorphism of DGLAs:

\[ \delta : \text{Lie}(sC(s^{-2}E)) \to \mathbb{E} \]

\( \delta \) is a map from a free Lie algebra.
So \( \delta \) is uniquely determined by the map

\[ \lambda = \text{Top} : sC(s^{-2}E) \to \mathbb{E} \]

where \( \rho : sC(s^{-2}E) \to \text{Lie}(sC(s^{-2}E)) \)

We define \( \lambda \) as

\[ \lambda(a + a_1a_2 + b_1b_2b_3 + \ldots) = a \quad (0 \leq i < p) \]

We need to show that \( \lambda \) is a map of cochain complexes. That is

\[ \delta(\partial x) = 2\delta(x) + x \in sC(s^{-2}E) \]
For $X = a \in \mathcal{L}$ we have
\[
\int (\theta X) = \int (\theta a) = \int (-Q a) = \int (-Q a^k)
\]
\[
= \int (\theta a) \subset \mathbb{D} = \mathbb{F}(a).
\]
For $X = a^k \in \mathcal{L}$ we have
\[
\int (\theta (a^k)) = -Q(a^k) + \frac{1}{2} (-1)^k \mathbb{F}(a^k) + \sum_{\lambda, \mu} \mathbb{F}(a^{k+1})
\]
\[
= (-1)^{a+1} [q, k] + (-1)^{a+1} [q, k]
\]
\[-Q(a)^k - \frac{1}{2} (-1)^a aQ(a).
\]
Thus
\[
\mathbb{F}(\theta (a^k)) = (-1)^a [q, k] + (1)^{a+1} [q, k] = 0.
\]
This is corrected by $\mathbb{F}(a^k) = 0$ for $k \geq 3$ we have
\[
\mathbb{F}(a_k, \ldots, a_k) = 0 \quad \mathbb{F}(a, \ldots, a) = 0
\]
\[ D(a_1, \ldots, a_n) = -Q(a_1, \ldots, a_n) \]

\[ + \sum \left[ \frac{\partial}{\partial a_i} \right] \]

The number of e.g. in either of monomials
is \( \geq 2 \)

\[ z - \sum_{k \leq 1} R_k(a_1, \ldots, a_n) \leq 1 \]
\[ + \sum \left[ \frac{\partial}{\partial a_i} \right] \]
\[ \# \geq 2 \quad \text{or} \quad \# \geq 2 \]

Thus \( A \) \( k \geq 3 \) \( \forall a_1, \ldots, a_n \in \mathbb{C} \)

\[ f(D(a_1, \ldots, a_n)) = 0 \]

Thus \( f \) is indeed a map of cochain complexes.

**Thm**

The map \( f \) is a quasi-isomorphism

**Proof**

We will prove this statement

for the case when \( f \) the derivative

\( \partial \in \mathfrak{L} \) is zero \( \partial = 0 \).
We introduce on $\text{Lie}(sC(s^{-E}))$ the following increasing filtration

$$Y = F_1 \text{Lie}(sC(s^{-E})) \subset F_2 \text{Lie}(sC(s^{-E})) \subset \ldots \subset F_m \text{Lie}(sC(s^{-E})) \subset \ldots$$

where $F_m = F_m \text{Lie}(sC(s^{-E}))$ is spanned by monomials

\[ [\ldots [\varphi_i, \varphi_k, \varphi_l, \ldots], \ldots, \varphi_2, \varphi_3] \]

with $k_1 + k_2 + \ldots + k_s \leq m$.

The differential $D$ consists of two parts. The first part does not change the degree $k_1 + k_2 + \ldots + k_s$. The second part corresponds to $\varphi_i \varphi_j \rightarrow [\varphi_i, \varphi_j]$. So, it lowers the sum $k_1 + k_2 + \ldots + k_s$ by 1.

Thus $D$ is compatible with the filtration.
We also introduce an $\mathcal{E}$-trivial filtration

$$\mathcal{E} = F_1 \mathcal{E} = F_2 \mathcal{E} = F_3 \mathcal{E} = \cdots$$

Then $\mathcal{E}$ is compatible with the filtration

$$\underbrace{\mathcal{E} \rightarrow \text{id}: F}$$

$$\phi_{|F_i} : F_i \rightarrow F_i \mathcal{E}$$

Thus $\phi_{|F_i}$ is a quasi-isomorphism from $F_i$ to $F_i \mathcal{E}$

For $m \geq 2$ we have the diagram

$$
\begin{array}{c}
0 \rightarrow F_{m-1} \mathcal{E} \rightarrow F_m \mathcal{E} \rightarrow 0 \rightarrow 0 \\
\uparrow \phi \quad \uparrow \phi \\
0 \rightarrow F_{m-1} \rightarrow F_m \rightarrow \frac{F_m}{F_{m-1}} \rightarrow 0
\end{array}
$$

If $\bigoplus_{m \geq 2} \frac{F_m}{F_{m-1}}$ is acyclic then

By induction $\phi_{|F_m} : F_m \rightarrow F_m \mathcal{E} = \mathcal{E}$ is a quasi-isomorphism.
The filtration $F_\cdot$ on $\text{Lie}(sC(s^{-2}))$ satisfies the following property:

$\forall X \in \text{Lie}(sC(s^{-2})) \exists m \in \mathbb{N}$ st.

$X \in F_m(\text{Lie}(sC(s^{-2})))$.

Thus, it is enough to show that

$\gamma : F_m \to \mathbb{L}$ is a $q$-isomorphism.

It remains to show that $\bigoplus_{m \geq 2} F_m / F_{m-1}$ is acyclic. In other words, we need to show that the cochain complex

$\text{Lie}(sC(s^{-2}))$ (where $E$ is a graded $\mathbb{C}$-space)

has non-trivial cocycles only in its first term.