Assignment 4

Exercise 1. (4 pts.) We consider the polynomial algebra $A = \mathbb{K}[x^1, x^2, \ldots, x^d]$. Set $\partial_i := \partial_{x^i}$ and prove that for every $1 \leq k \leq m$ the sum
\[
\partial_i \cup \partial_{i_1} \cup \partial_{i_2} \cup \cdots \cup \partial_{i_m} + \partial_{i_k} \cup \partial_{i_1} \cup \partial_{i_2} \cup \partial_{i_{k-1}} \cup \partial_{i_k} \cup \partial_{i_{k+1}} \cup \cdots \cup \partial_{i_m}
\]
is exact in $C^\bullet(A)$.

Exercise 2. (3 pts.) Consider the Gerstenhaber algebra
\[
V_A = \mathbb{K}[x^1, \ldots, x^d, \theta_1, \ldots, \theta_d]
\]
of polyvector fields on $\mathbb{K}^d$. Let $F$ be a (linear) map
\[
F : V_A \to C^\bullet(A)
\]
defined by the formula
\[
F(\gamma)(a_1, a_2, \ldots, a_m) = \gamma^{i_m \cdots m-1 \cdot i_1}(x)(\partial_{x^{i_1}} a_1)(\partial_{x^{i_2}} a_2) \cdots (\partial_{x^{i_m}} a_m),
\]
where $\gamma = \gamma^{i_1 \cdots i_m}(x)\theta_{i_1} \theta_{i_2} \cdots \theta_{i_m}$.

Prove that
\[
F(\gamma, \sigma) = \{F(\gamma), F(\sigma)\}
\]
whenever $\gamma$ and $\sigma$ are generators of $V_A$, i.e. $\gamma = x^i$ and $\sigma = x^j$, or $\gamma = \theta_i$ and $\sigma = x^j$, or $\gamma = x^i$ and $\sigma = \theta_j$, or $\gamma = \theta_i$ and $\sigma = \theta_j$.

Exercise 3. (3 pts.) During our lectures we proved that the map $F$ (defined in (2)) is compatible with the “Lie brackets” $[,]_S$ and $\{,\}$ up to homotopy. Namely, for every pair $\gamma, \sigma \in V_A$ the difference
\[
F(\gamma, \sigma)_S - \{F(\gamma), F(\sigma)\}
\]
is exact in $C^\bullet(A)$. Consider the case when both $\gamma$ and $\sigma$ are bivectors, i.e.
\[
\gamma = \gamma^{ij}(x)\theta_i \theta_j, \quad \sigma = \sigma^{kl}(x)\theta_k \theta_l.
\]
Show that, in this case,
\[
F(\gamma, \sigma)_S - \{F(\gamma), F(\sigma)\} \neq 0.
\]