A manifestation of the Grothendieck-Teichmueller group in geometry

Vasily Dolgushev

Temple University

- In 1990, V. Drinfeld introduced a (pro-nilpotent) infinite dimensional Lie algebra $\mathfrak{grt}$ (the Grothendieck-Teichmueller Lie algebra). This Lie algebra is graded by positive integers and, $\forall$ odd $n \ge 3$ $\exists$ a non-zero element $\sigma_n \in \mathfrak{grt}$ of degree n. We call elements $\{\sigma_n\}_{n \text{ odd}, \ge 3}$ *Deligne-Drinfeld elements*.

Kontsevich's conjecture on the action of $\mathfrak{grt}$

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- Let X be a smooth algebraic variety over an algebraically closed field $\Bbbk$ of characteristic 0. $\mathcal{O}_X$ is the structure sheaf, $\mathcal{T}_X$ is the tangent sheaf, and $\wedge_{\mathcal{O}_X} \mathcal{T}_X$ is the sheaf of polyvector fields on X. $\mathrm{Ch}(X)$ is the Chern character of $\mathcal{T}_X$.

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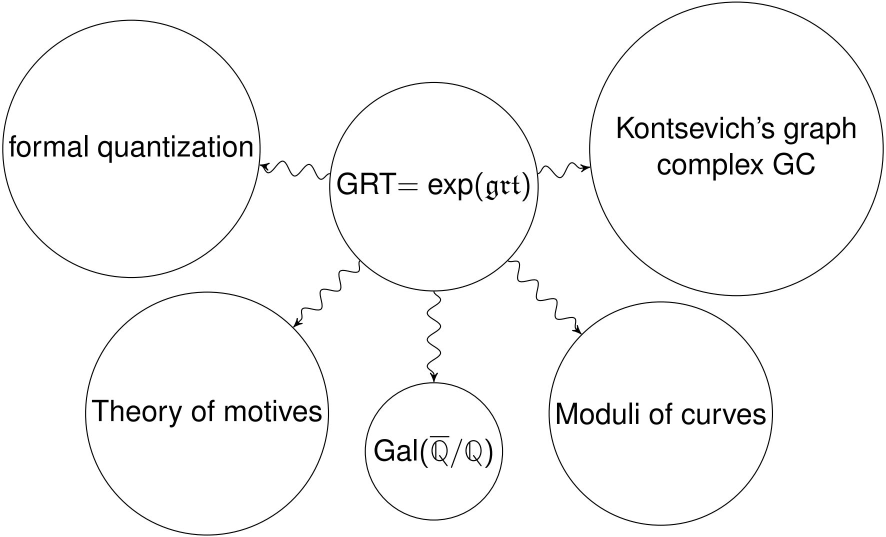
- In 1999, M. Kontsevich conjectured that $\mathfrak{grt}$ acts naturally on the cohomology $H^\bullet(X, \wedge_{\mathcal{O}_X} \mathcal{T}_X)$. The action of σ_n coincides (up to non-zero scalar) with the contraction with the n-th component $\mathrm{Ch}_n(X)$ of $\mathrm{Ch}(X)$.

Based on joint work arXiv:1211.4230 with Chris Rogers and Thomas Willwacher.

The Grothendieck-Teichmueller group GRT



- The family of *Drinfeld Kohno Lie algebras* $\mathfrak{t}_m$, $(m \ge 2)$. For every m, $\mathfrak{t}_m$ is generated by $\{t^{ij} = t^{ji}\}_{1 \le i \ne j \le m}$ subject to

$$[t^{ij}, t^{ik} + t^{jk}] = 0 \qquad \#\{i,j,k\} = 3\,,$$

$$[t^{ij}, t^{kl}] = 0 \qquad \#\{i,j,k,l\} = 4\,.$$

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- and the free Lie algebra $\mathfrak{lie}(x,y)$ in two symbols x, y.

The Lie algebra $\mathfrak{grt}$ consists of ...

elements $\sigma(x,y) \in \mathfrak{lie}(x,y)$ satisfying

$$\sigma(y,x) = -\sigma(x,y)\,,$$

$$\sigma(x,y) + \sigma(y,-x-y) + \sigma(-x-y,x) = 0\,,$$

$$\sigma(t^{23},t^{34}) - \sigma(t^{13}+t^{23},t^{34}) + \sigma(t^{12}+t^{13},t^{24}+t^{34})$$

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The Lie bracket on $\mathfrak{grt}$ is the Ihara bracket:

$$[\sigma,\sigma']_{\mathrm{Ih}} := \delta_{\sigma}(\sigma') - \delta_{\sigma'}(\sigma) + [\sigma,\sigma']_{\mathfrak{lie}(x,y)}\,,$$

where $[\,,\,]_{\mathfrak{lie}(x,y)}$ is the usual bracket on $\mathfrak{lie}(x,y)$ and δ_σ is the derivation of $\mathfrak{lie}(x,y)$ defined by

$$\delta_\sigma(x) := 0\,, \qquad \delta_\sigma(y) := [y,\sigma(x,y)]\,.$$

Deligne-Drinfeld elements of $\mathfrak{grt}$

The above equations have neither linear nor quadratic solutions. The first non-trivial example of an element in $\mathfrak{grt}$ is

$$\sigma_3(x, y) = [x, [x, y]] - [y, [y, x]] \,.$$

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More generally,

Proposition (V. Drinfeld, 1990)

For every odd integer $n \geq 3$ there exists a non-zero vector $\sigma_n \in \mathfrak{grt}$ of degree n in symbols x and y such that

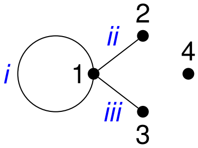
$$\sigma_n = \mathrm{ad}_x^{n-1}(y) + \dots$$

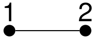
where $\dots$ is a sum of Lie words of degrees ≥ 2 in the symbol y.

$\{\sigma_n\}_{n \,\mathrm{odd}\, \geq 3}$ are called *Deligne-Drinfeld elements* of $\mathfrak{grt}$.

To define Kontsevich's graph complex GC we need...

- a set gra_n. An element of gra_n is a labelled graph Γ with n vertices and with total order on the set of its edges.





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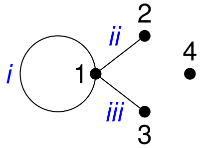
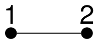
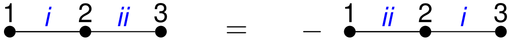




- and the graded vector space $\mathrm{Gra}(n)$ (for $n \geq 1$) spanned by elements of gra_n, modulo the relation $\Gamma^\tau = (-1)^{|\tau|}\Gamma$; Γ^τ and Γ correspond to the same labelled graph but differ only by permutation τ of edges. For example:



The full graph complex is ...

$$fGC := \prod_{n \ge 1} \mathfrak{s}^{2n-2} \Big(\mathrm{Gra}(n) \Big)^{S_n}$$

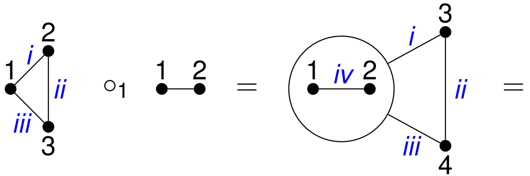
where $\mathfrak{s}$ shifts the degree up by 1.

$$[\gamma, \gamma'] := \gamma \bullet \gamma' - (-1)^{|\gamma||\gamma'|} \gamma' \bullet \gamma\,, \qquad \gamma \bullet \gamma' = \sum_{\tau \in \mathrm{Sh}_{k,n-1}} \tau(\gamma \circ_1 \gamma')\,,$$

$$\gamma \in \mathfrak{s}^{2n-2} \Big(\mathrm{Gra}(n) \Big)^{S_n}, \qquad \gamma' \in \mathfrak{s}^{2k-2} \Big(\mathrm{Gra}(k) \Big)^{S_k}.$$

The operation $\circ_1$ is ...

$$\circ_1 : \mathrm{Gra}(n) \otimes \mathrm{Gra}(k) \to \mathrm{Gra}(k+n-1)\,.$$



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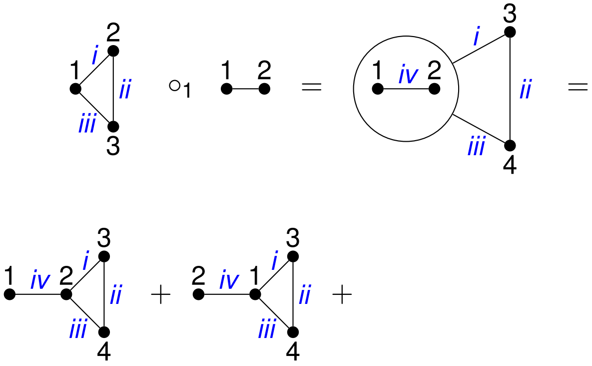

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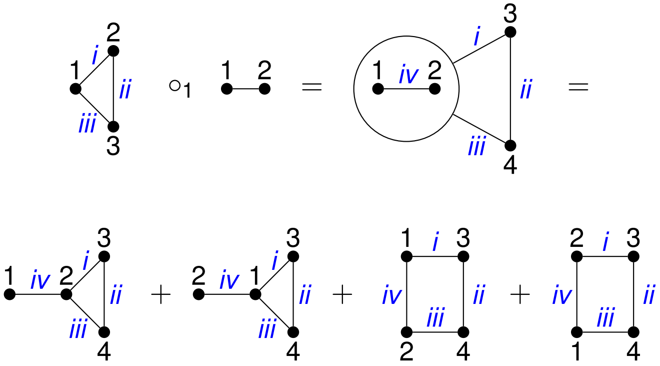
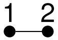


Kontsevich's graph complex *GC*

The graph $\Gamma_e =$ 

satisfies the equation $[\Gamma_e, \Gamma_e] = 0$. The differential on *fGC* is

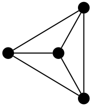
$$\partial := [\Gamma_e, \;\;].$$

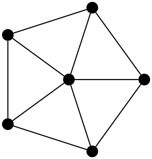
Definition

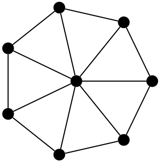
Kontsevich's graph complex GC is the subcomplex of fGC which involves only graphs Γ satisfying these properties:

- *Γ is connected and the complement of every vertex in Γ is also connected.*
- *Each vertex of Γ has valency ≥ 3.*

Examples of degree zero cocycles in *GC* are...

$\gamma_3 =$ 

$\gamma_5 =$  $+ \ \ldots$

$\gamma_7 =$  $+ \ \ldots$

$\ldots \qquad \ldots$

The Lie algebra $\mathfrak{grt}$ versus Kontsevich's graph complex

Theorem (Thomas Willwacher, 2010)

We have an isomorphism of Lie algebras

$$H^0(GC) \cong \mathfrak{grt}$$

Under this isomorphism Deligne-Drinfeld elements $\{\sigma_n\}_{n \; odd \; \ge 3}$ correspond to (non-zero scalar multiples of) classes of the above cycles $\{\gamma_n\}_{n \; odd \; \ge 3}$.

Main Results

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- proved that the action of the class $[\gamma_n] \in H^0(GC)$ (n odd ≥ 3) on $H^\bullet(X, \wedge_{\mathcal{O}_X} \mathcal{T}_X)$ coincides with the contractions with the n-th component $\mathrm{Ch}_n(X)$ of the Chern character $\mathrm{Ch}(X)$ of $\mathcal{T}_X$, and

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- showed that there are examples of smooth varieties X for which $\mathrm{Ch}_n(X)$ acts non-trivially on $H^\bullet(X, \wedge_{\mathcal{O}_X} \mathcal{T}_X)$.

THANK YOU!

[1] V.A. Dolgushev and C.L. Rogers, Notes on algebraic operads, graph complexes, and Willwacher's construction, *Mathematical aspects of quantization*, 25–145, Contemp. Math., **583**, AMS, Providence, RI, 2012; arXiv:1202.2937.

[2] V.A. Dolgushev, C.L. Rogers, and T.H. Willwacher, Kontsevich's graph complex, GRT, and the deformation complex of the sheaf of polyvector fields, arXiv:1211.4230.

[3] T. Willwacher, M. Kontsevich's graph complex and the Grothendieck-Teichmueller Lie algebra, arXiv:1009.1654.