A Discrete-Time LQG Compensator for Random Abstract Parabolic System

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Outline

Done Work

Alcohol Problem Formulation

Abstract Parabolic Systems

Abstract Parabolic Systems with Random Parameters

LQG Control and Compensator

Approximation and Convergence

Numerical Experiments

Done Work

- Extend the finite-dimensional discrete-time LQ control theory with LQG compensator (LQG control) to infinite-dimensional Hilbert space for parabolic systems.
- Establish abstract parabolic systems with random parameters in Bochner spaces wherein the random parameters are treated as additional spatial variables.
- Develop Galerkin-based finite-dimensional approximation and convergence theory for the established infinite-dimensional, discrete-time LQG control originated from the random abstract parabolic systems.
- Apply above-mentioned LQG control theory to intravenously-infused alcohol clamping studies based on population models and TAC sensing and implement numerical experiments.

LQG Flow

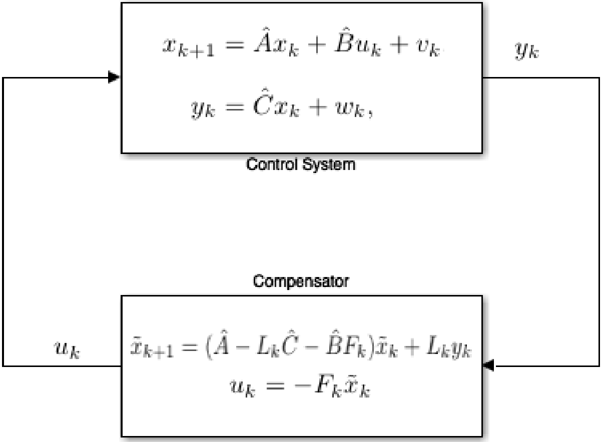

Figure: LQG Flow

Dynamical Model for the Transdermal Transport of Alcohol

$$\begin{aligned}
\frac{\partial \tilde{x}}{\partial t}(t,\eta) &= \alpha \frac{\partial^2 \tilde{x}}{\partial \eta^2}(t,\eta), \quad t>0, \quad \eta \in (0,1), \\
\frac{d\tilde{w}}{dt}(t) &= \beta \frac{\partial \tilde{x}}{\partial \eta}(t,0) - \gamma \tilde{w}(t) + \omega_1(t), \quad t>0, \\
\frac{d\tilde{v}}{dt}(t) &= -\delta \frac{\partial \tilde{x}}{\partial \eta}(t,1) - \frac{K\tilde{v}(t)}{M+\tilde{v}(t)} + b\tilde{u}(t) + \omega_2(t), \quad t>0,
\end{aligned} \tag{2.1}$$

with boundary conditions, controlled variable and observation:

$$\begin{aligned}
\tilde{x}(t,0) =& \tilde{w}(t), \quad \tilde{x}(t,1) = \tilde{v}(t), t>0, \\
\tilde{z}(t) =& \tilde{v}(t), \quad \tilde{y}(t) = \tilde{w}(t) + \zeta(t), \quad t>0,
\end{aligned} \tag{2.2}$$

respectively, and initial conditions:

$$\tilde{x}(0,\eta) = \varphi_0(\eta), \quad \eta \in (0,1), \quad \tilde{w}(0) = \theta_0, \quad \tilde{v}(0) = \xi_0. \tag{2.3}$$

- $\tilde{x}(t,\eta)$: the concentration of ethanol at time $t \geq 0$ and depth $\eta \in [0,1]$ in the epidermal layer.
- $\tilde{w}(t)$: the concentration of ethanol in the transdermal alcohol biosensor vapor collection chamber at time $t \geq 0$.
- $\tilde{v}(t)$: the concentration of ethanol in the blood at time $t \geq 0$.
- $\tilde{u}(t)$: the concentration of ethanol in the infused intravenous solution at time $t \geq 0$.
- ω_1, ω_2, and ζ: uncorrelated, zero-mean, stationary, Gaussian white noise processes with variances σ_1^2, σ_2^2, and σ^2, respectively.
- Normalized the thickness of the epidermal layer to be 1.

Equilibrium Solution

- Desired clamped blood alcohol level is $\tilde{v}(t) = \tilde{v}_0$.
- An equilibrium solution to the system (2.1) is given by

$$
\begin{aligned}
\tilde{x}(t,\eta) &= \tilde{x}_0(\eta) = \frac{\gamma \tilde{v}_0}{\gamma+\beta}\eta + \frac{\beta \tilde{v}_0}{\gamma+\beta} \\
\tilde{w}(t) &= \tilde{w}_0 = \frac{\beta \tilde{v}_0}{\gamma+\beta} \\
\tilde{v}(t) &= \tilde{v}_0 \\
\tilde{u}(t) &= \tilde{u}_0 = \frac{\delta\gamma\tilde{v}_0}{b(\gamma+\beta)} + \frac{K\tilde{v}_0}{b(M+\tilde{v}_0)}
\end{aligned}
$$

- Linearize about a clamped operating regime, $\tilde{x}_0$, $\tilde{w}_0$, $\tilde{v}_0$ and $\tilde{u}_0$, by writing $\tilde{x} = \tilde{x}_0 + x$, $\tilde{w} = \tilde{w}_0 + w$, $\tilde{v} = \tilde{v}_0 + v$, and $\tilde{u} = \tilde{u}_0 + u$.

Linearized System

$$
\begin{aligned}
\frac{\partial x}{\partial t}(t,\eta) &= q_1 \frac{\partial^2 x}{\partial \eta^2}(t,\eta), \quad t>0, \quad \eta \in (0,1),\\
\frac{dw}{dt}(t) &= q_3 \frac{\partial x}{\partial \eta}(t,0) - q_4 w(t) + \omega_1(t), \quad t>0,\\
\frac{dv}{dt}(t) &= -q_5 \frac{\partial x}{\partial \eta}(t,1) - q_6 v(t) + q_2 u(t) + \omega_2(t), \quad t>0,
\end{aligned} \tag{2.4}
$$

with boundary conditions, controlled variable and observation:

$$
\begin{aligned}
x(t,0) =& w(t), \quad x(t,1) = v(t), \quad t>0,\\
z(t) =& v(t), \quad y(t) = w(t) + \zeta(t), \quad t>0,
\end{aligned} \tag{2.5}
$$

respectively, where in the equations (2.4), (2.5) the parameters $q_1 = \alpha$, $q_2 = b$, $q_3 = \beta$, $q_4 = \gamma$, $q_5 = \delta$, and $q_6 = \frac{KM}{(M+\tilde{v}_0)^2}$ are all positive.

Gelfand Triple

- Q: a compact subset of $\mathbb{R}^6$
- $H = \mathbb{R}^2 \times L^2(0,1)$ with the standard inner product and norm
- $H_q = \mathbb{R}^2 \times L^2(0,1)$ with the inner product

$$\left\langle (\theta, \xi, \varphi), (\bar{\theta}, \bar{\xi}, \bar{\varphi}) \right\rangle_q = \frac{q_1}{q_3}\theta\bar{\theta} + \frac{q_1}{q_5}\xi\bar{\xi} + \int_0^1 \varphi(\eta)\bar{\varphi}(\eta)d\eta.$$

- $V = \{(\theta, \xi, \varphi) \in \mathbb{R}^2 \times L^2(0,1) : \varphi \in H^1(0,1), \theta = \varphi(0), \xi = \varphi(1)\}$ with the inner product

$$\begin{aligned}\langle (\varphi(0), \varphi(1), \varphi), (\bar{\varphi}(0), \bar{\varphi}(1), \bar{\varphi}) \rangle_V \\ = \varphi(0)\bar{\varphi}(0) + \varphi(1)\bar{\varphi}(1) + \langle \varphi, \bar{\varphi} \rangle_{H^1(0,1)},\end{aligned}$$

  where $\langle \cdot, \cdot \rangle_{H^1(0,1)}$ denotes the standard inner product on $H^1(0,1)$.

- Gelfand triple $V \hookrightarrow H_q \hookrightarrow V^*$.

Bilinear Form $a(q;\cdot,\cdot)$ and $A(q)$

- Define the bilinear form $a(q;\cdot,\cdot)$: $V \times V \to \mathbb{R}$ by

$$a(q;(\varphi(0),\varphi(1),\varphi),(\bar{\varphi}(0),\bar{\varphi}(1),\bar{\varphi})) = \frac{q_1 q_4}{q_3}\varphi(0)\bar{\varphi}(0) + \frac{q_1 q_6}{q_5}\varphi(1)\bar{\varphi}(1) + q_1\int_0^1 \varphi'(\eta)\bar{\varphi}'(\eta)d\eta.$$

- $a(q;\cdot,\cdot)$ satisfies boundedness, V-coercivity, and continuity.
- $A(q) : Dom\,(A(q)) \subset H \to H$, $\langle A(q)\hat{\varphi},\hat{\psi}\rangle_{V^*,V} = -a(q;\hat{\varphi},\hat{\psi})$ for $\hat{\varphi} \in Dom\,(A(q))$, and $\hat{\psi} \in V$, where

$$Dom\,(A(q)) = \left\{\hat{\varphi} = (\varphi(0),\varphi(1),\varphi) \in V : \varphi \in H^2(0,1)\right\}.$$

- Operator $A(q)$ is densely defined on H_q, regularly dissipative, and self-adjoint. Therefore $A(q)$ is the infinitesimal generator of an exponentially stable, self-adjoint, analytic semigroup of bounded linear operators, $\{T(q;t) : t \geq 0\}$, on H_q.

Operators Needed

- Input operator $B(q) \in \mathcal{L}(\mathbb{R}, H_q)$: $B(q)u = (0, q_2 u, 0)$.
- Noise-influenced operator $B_1 \in \mathcal{L}(\mathbb{R}^2, H_q)$: $B_1\omega = (\omega_1, \omega_2, 0)$.
- Observation operator $C \in \mathcal{L}(H_q, \mathbb{R})$: $C(\theta, \xi, \varphi) = \theta$.
- Controlled variable operator $D \in \mathcal{L}(H_q, \mathbb{R})$: $D(\theta, \xi, \varphi) = \xi$.
- Terminal penalty operator $G \in \mathcal{L}(H_q, H_q)$ is nonnegative. (e.g. $G = \rho D^* D$ with $\rho \geq 0$ in finite horizon problem).
- State covariance operator $\tilde{Q}(q) \in \mathcal{L}(H_q, H_q)$: $\tilde{Q}(q) = \hat{B}_1(q)\Sigma\hat{B}_1(q)^*$ where $\Sigma = diag(\sigma_1^2, \sigma_2^2) \in \mathbb{R}^{2\times 2}$.
- Output covariance matrix (operator) $\tilde{R} = \sigma^2 \in \mathbb{R}$.
- Consider zero-order hold input with sampling time $\tau > 0$.

Abstract Parabolic System

- Discrete-time dynamical system equivalent to linearized system (2.4)-(2.5)

$$\begin{aligned} x_{k+1} &= \hat{A}(q)x_k + \hat{B}(q)u_k + \hat{B}_1(q)\omega(k\tau), \\ x_0 &= (w(0), v(0), x(0,\cdot)), \\ y_k &= \hat{C}x_k + \zeta(k\tau). \end{aligned} \tag{3.1}$$

- $\omega(t) = [\omega_1(t), \omega_2(t)]^T$, $\hat{A}(q) = T(q;\tau) \in \mathcal{L}(H_q, H_q)$, $\hat{B}(q) = A(q)^{-1}(\hat{A}(q) - I)B(q) \in Dom(A(q)) \subset \mathcal{L}(\mathbb{R}, H_q) = H_q$, $\hat{B}_1(q) = A(q)^{-1}(\hat{A}(q) - I)B_1 \in Dom(A(q)) \times Dom(A(q)) \subset \mathcal{L}(\mathbb{R}^2, H_q) = H_q \times H_q$, $\hat{C} = C$.

Quadratic Performance Index

- Finite-time horizon problem

$$\hat{J}(u) = \mathbb{E}\Bigg\{ \sum_{k=k_0}^{k_1-1} \langle \hat{Q}x_k, x_k \rangle_{H_q} + \hat{r}u_k^2 + \langle \hat{G}x_{k_1}, x_{k_1} \rangle_{H_q} \Bigg\}.$$

- Infinite-time horizon problem

$$\hat{J}(u) = \mathbb{E}\Bigg\{ \sum_{k=k_0}^{\infty} \langle \hat{Q}x_k, x_k \rangle_{H_q} + \hat{r}u_k^2 \Bigg\}.$$

- $\hat{D} = D$, $\hat{Q} = \hat{D}^*\hat{D} \in \mathcal{L}(H_q, H_q)$, $\hat{G} = \rho\hat{D}^*\hat{D} \in \mathcal{L}(H_q, H_q)$, $\hat{R} = \hat{r}$.

Bochner Spaces and operator $\mathcal{A}$

- Treat q as a random parameter with known distribution described by the probability measure π.
- Bochner space $\mathcal{V} = L^2_\pi(Q;V)$, $\mathcal{H} = L^2_\pi(Q;H_q)$.
- Gelfand triple $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^*$.
- Define the bilinear form $\mathfrak{a}(\cdot,\cdot)$ on $\mathcal{V} \times \mathcal{V}$ and the operator $\mathcal{A} \in \mathcal{L}(\mathcal{V},\mathcal{V}^*)$ by:

$$\begin{aligned}\mathfrak{a}(\hat{\varphi},\hat{\psi}) =& \mathbb{E}_\pi\{a\left(q;\hat{\varphi}(q),\hat{\psi}(q)\right)\} \\ &= \int_Q a\left(q;\hat{\varphi}(q),\hat{\psi}(q)\right) d\pi(q) = -\langle \mathcal{A}\hat{\varphi},\hat{\psi}\rangle_{\mathcal{V}^*,\mathcal{V}},\end{aligned}$$

  for $\hat{\varphi},\hat{\psi} \in \mathcal{V}$, where $Dom(\mathcal{A}) = \{\varphi \in \mathcal{V} : \mathcal{A}\varphi \in \mathcal{H}\}$.
- Operator $\mathcal{A}$ is regularly dissipative and self-adjoint and can be restricted to $Dom(\mathcal{A})$ as the infinitesimal generator of an exponentially stable, analytic semigroup $\{\mathcal{T}(t) : t \geq 0\}$ on $\mathcal{H}$.

Other Operators

- $\mathcal{B} \in \mathcal{L}(\mathbb{R}, \mathcal{H})$, $\mathcal{B}u = \mathbb{E}_\pi\{B(q)\}u$, $u \in \mathbb{R}$,
- $\mathcal{B}_1 \in \mathcal{L}(\mathbb{R}^2, \mathcal{H})$, $\mathcal{B}_1\omega = \mathbb{E}_\pi\{B_1\}\omega = B_1\omega$, $\omega \in \mathbb{R}^2$,
- $\mathcal{C} \in \mathcal{L}(\mathcal{H}, \mathbb{R})$, $\mathcal{C}\hat{\varphi} = \mathbb{E}_\pi\{C\hat{\varphi}\}$,
- $\mathcal{D} \in \mathcal{L}(\mathcal{H}, \mathbb{R})$, $\mathcal{D}\hat{\varphi} = \mathbb{E}_\pi\{D\hat{\varphi}\}$, $\hat{\varphi} \in \mathcal{H}$.

Abstract Parabolic System with Random Parameters

- Discrete-time dynamical system transferred from (3.1)

$$\begin{aligned} x_{k+1} =& \hat{\mathcal{A}} x_k + \hat{\mathcal{B}} u_k + \hat{\mathcal{B}}_1 \omega(k\tau), \quad x_0 = \hat{\varphi}_0, \\ y_k =& \hat{\mathcal{C}} x_k + \zeta(k\tau), \quad k = k_0, k_0+1, k_0+2, \ldots, \end{aligned} \tag{4.1}$$

- where $\hat{\mathcal{A}} = \mathcal{T}(\tau) \in \mathcal{L}(\mathcal{H}, \mathcal{H})$, $\hat{\mathcal{B}} = \mathcal{A}^{-1}(\hat{\mathcal{A}} - \mathcal{I})\mathcal{B} \in \mathcal{L}(\mathbb{R}, \mathcal{H})$, $\hat{\mathcal{B}}_1 = \mathcal{A}^{-1}(\hat{\mathcal{A}} - \mathcal{I})\mathcal{B}_1 \in \mathcal{L}(\mathbb{R}^2, \mathcal{H})$, $\hat{\mathcal{C}} = \mathcal{C} \in \mathcal{L}(\mathcal{H}, \mathbb{R})$, where $\mathcal{I}$ denotes the identity operator on $\mathcal{H}$.

Quadratic Performance Index

- Finite-time horizon problem

$$\hat{\mathcal{J}}(u) = \mathbb{E}\Big\{ \sum_{k=k_0}^{k_1-1} \Big\{ \langle \hat{\mathcal{Q}} x_k, x_k \rangle_{\mathcal{H}} + \hat{r} u_k^2 \Big\} + \langle \hat{\mathcal{G}} x_{k_1}, x_{k_1} \rangle_{\mathcal{H}} \Big\}, \tag{4.2}$$

- Infinite-time horizon problem

$$\hat{\mathcal{J}}(u) = \mathbb{E}\Big\{ \sum_{k=k_0}^{\infty} \Big\{ \langle \hat{\mathcal{Q}} x_k, x_k \rangle_{\mathcal{H}} + \hat{r} u_k^2 \Big\} \Big\}, \tag{4.3}$$

- $\hat{\mathcal{D}} = \mathcal{D} \in \mathcal{L}(\mathcal{H}, \mathbb{R})$, $\hat{\mathcal{Q}}, \hat{\mathcal{G}} \in \mathcal{L}(\mathcal{H}, \mathcal{H})$, $\hat{\mathcal{Q}} = \hat{\mathcal{D}}^* \hat{\mathcal{D}}$, and $\hat{\mathcal{G}} = \rho \hat{\mathcal{D}}^* \hat{\mathcal{D}}$.

LQG Control Problem

- Consider the discrete-time dynamical system (4.1) associated with quadratic performance indexes (4.2) and (4.3).
- The time-invariant finite-time horizon discrete-time LQG control is given by
  $(P1)$ Choose an input $\bar{u} \in L^2([0,T];\mathbb{R})$ for which the criterion (4.2) is minimized.
- The time-invariant infinite-time horizon discrete-time LQG control is given by
  $(P2)$ Choose an input $\bar{u} \in L^2([0,\infty);\mathbb{R})$ for which the criterion (4.3) is minimized.

LQG Compensator

- Estimation system

$$\tilde{x}_{k+1} = \hat{\mathcal{A}}\tilde{x}_k + \hat{\mathcal{B}}u_k + \tilde{\mathcal{L}}(y(k\tau) - \hat{\mathcal{C}}\tilde{x}_k), \quad \tilde{x}_{k_0} = \tilde{\varphi}_0, \qquad (5.1)$$

where $\tilde{\varphi}_0 \in \mathcal{H}$ is arbitrary and the operator observer gain $\tilde{\mathcal{L}} \in \mathcal{L}(\mathbb{R}, \mathcal{H})$ is given by:

$$\tilde{\mathcal{L}} = \hat{\mathcal{A}}\tilde{\Pi}\hat{\mathcal{C}}^* \left\{\sigma^2 + \hat{\mathcal{C}}\tilde{\Pi}\hat{\mathcal{C}}^*\right\}^{-1},$$

with the operator $\tilde{\Pi}$ the unique positive semi-definite self-adjoint solution guaranteed to exist to the algebraic Riccati equation given by:

$$\tilde{\Pi} = \hat{\mathcal{A}}\left[\tilde{\Pi} - \tilde{\Pi}\hat{\mathcal{C}}^*\left(\sigma^2 + \hat{\mathcal{C}}\tilde{\Pi}\hat{\mathcal{C}}^*\right)^{-1}\hat{\mathcal{C}}\tilde{\Pi}\right]\hat{\mathcal{A}}^* + \tilde{\mathcal{Q}},$$

where $\tilde{\mathcal{Q}} = \hat{\mathcal{B}}_1\Sigma\hat{\mathcal{B}}_1^* \in \mathcal{L}(\mathcal{H}, \mathcal{H})$.

Solution to LQG Control $(P1)$

- By separation principle, the optimal control for $(P1)$ is given by $\bar{u}_k = -\hat{\mathcal{F}}_k \tilde{x}_k$.
- $\hat{\mathcal{F}}_k = \left\{\hat{r} + \hat{\mathcal{B}}^* \hat{\Pi}_k \hat{\mathcal{B}}\right\}^{-1} \hat{\mathcal{B}}^* \hat{\Pi}_k \hat{\mathcal{A}}$, where Π_k satisfies

$$\hat{\Pi}_k = \hat{\mathcal{A}}^* \left[\hat{\Pi}_{k+1} - \hat{\Pi}_{k+1} \hat{\mathcal{B}} \left(\hat{r} + \hat{\mathcal{B}}^* \hat{\Pi}_{k+1} \hat{\mathcal{B}}\right)^{-1} \hat{\mathcal{B}}^* \hat{\Pi}_{k+1}\right] \hat{\mathcal{A}} + \hat{\mathcal{Q}},$$

$$\hat{\Pi}_{k_1} = \hat{\mathcal{G}}.$$

- $\tilde{x}_k$ is generated by observation system (5.1).

Solution to LQG Control $(P2)$

- By separation principle, the optimal control for $(P2)$ is given by $\bar{u}_k = -\hat{\mathcal{F}}\tilde{x}_k$.
- $\hat{\mathcal{F}} = \left\{\hat{r} + \hat{\mathcal{B}}^*\hat{\Pi}\hat{\mathcal{B}}\right\}^{-1}\hat{\mathcal{B}}^*\hat{\Pi}\hat{\mathcal{A}}$, where Π satisfies the following algebraic Riccati equation

$$\hat{\Pi} = \hat{\mathcal{A}}^*\left[\hat{\Pi} - \hat{\Pi}\hat{\mathcal{B}}\left(\hat{r} + \hat{\mathcal{B}}^*\hat{\Pi}\hat{\mathcal{B}}\right)^{-1}\hat{\mathcal{B}}^*\hat{\Pi}\right]\hat{\mathcal{A}} + \hat{\mathcal{Q}}.$$

- $\tilde{x}_k$ is generated by observation system (5.1).

Finite-dimensional Subspace and Basis

- Let $\mathcal{V}^N$ be a finite-dimensional subspace of $\mathcal{V}$, satisfying $\mathcal{P}^N x \to x$, for $x \in \mathcal{V}$, where $\mathcal{P}^N$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{V}^N$.
- Parameter variable: random parameter q_i has bounded support $[a_i, b_i]$, $i = 1, 2, ..., 6$. Let Q be the compact subset of $\mathbb{R}^6$ given by $Q = \times_{i=1}^{6}[a_i, b_i]$. For $i = 1, 2, ..., 6$, partition $[a_i, b_i]$ into m_i equal subintervals, and let $\chi_j^{m_i}$ denote the characteristic function of the j-th subinterval, $j = 1, 2, ..., m_i$.
- Space variable: For $n = 1, 2, ...$ let $\{\varphi_j^n\}_{j=0}^n$ denote the standard linear B-splines on $[0, 1]$ with respect to the uniform mesh $\{0, \frac{1}{n}, \frac{2}{n}, ..., 1\}$ and set $\hat{\varphi}_j^n = (\varphi_j^n(0), \varphi_j^n(1), \varphi_j^n) \in V$.
- Basis on $\mathcal{V}^N$: $\Phi_J^N = \hat{\varphi}_{j_0}^n \Pi_{i=1}^6 \chi_{j_i}^{m_i}$, where $J = (j_0, j_1, ..., j_6)$, $j_0 \in \{0, 1, 2, ..., n\}$ and $j_i \in \{1, 2, ..., m_i\}$, $i = 1, 2, ..., 6$.

Galerkin-based Approximation for $\mathcal{A}$

- For any $\varphi^N, \psi^N \in \mathcal{V}^N$, define $\mathcal{A}^N \in \mathcal{V}^N$ by

$$\begin{aligned} \langle \mathcal{A}^N \varphi^N, \psi^N \rangle_{\mathcal{V}^N, \mathcal{V}^N} &= -\mathfrak{a}(\varphi^N, \psi^N) \\ &= -\int_Q a(q; \varphi^N(q), \psi^N(q)) d\pi(q). \end{aligned}$$

- $\mathcal{A}^N$ is the infinitesimal generator of a uniformly continuous semigroup $\mathcal{T}^N(t) = e^{\mathcal{A}^N t}$ for $t \geq 0$.
- Trotter-Kato theorem: for each $x \in \mathcal{V}$, $\mathcal{T}^N(t)\mathcal{P}^N x \to \mathcal{T}(t)x$ in the $\mathcal{V}$ norm for $t > 0$, uniformly in t on compact sub intervals.
- $\hat{\mathcal{A}}^N \in \mathcal{L}(\mathcal{V}^N, \mathcal{V}^N)$: $\hat{\mathcal{A}}^N = \mathcal{T}^N(\tau) = e^{\mathcal{A}^N \tau}$.

Approximation for Other Operators

- $\hat{\mathcal{B}}^N = (\mathcal{A}^N)^{-1}(\hat{\mathcal{A}}^N - \mathcal{I}^N)\mathcal{P}^N\mathcal{B} \in \mathcal{L}(\mathbb{R}, \mathcal{V}^N)$.
- $\hat{\mathcal{B}}_1^N = (\mathcal{A}^N)^{-1}(\hat{\mathcal{A}}^N - \mathcal{I}^N)\mathcal{P}^N\mathcal{B}_1 \in \mathcal{L}(\mathbb{R}^2, \mathcal{V}^N)$.
- $\hat{\mathcal{C}}^N = \hat{\mathcal{C}}\mathcal{P}^N$.
- $\hat{\mathcal{Q}}^N = \mathcal{P}^N\hat{\mathcal{Q}}\mathcal{P}^N = (\hat{\mathcal{D}}^N)^*\hat{\mathcal{D}}^N$, where $\hat{\mathcal{D}}^N = \hat{\mathcal{D}}\mathcal{P}^N$.
- $\tilde{\mathcal{Q}}^N = \hat{\mathcal{B}}_1^N\Sigma(\hat{\mathcal{B}}_1^N)^* \in \mathcal{L}(\mathcal{V}^N, \mathcal{V}^N)$.

Finite-dimensional Approximation LQG Control on the infinite-time horizon

- Consider the following dynamical system

$$\begin{aligned} x_{k+1}^N =& \hat{\mathcal{A}}^N x_k^N + \hat{\mathcal{B}}^N u_k^N + \hat{\mathcal{B}}_1^N \omega(k\tau), \quad x_0^N = \mathcal{P}^N \hat{\varphi}_0. \\ y_k^N =& \hat{\mathcal{C}}^N x_k^N + \zeta(k\tau), \quad k = k_0, k_0+1, k_0+2, \ldots \end{aligned} \tag{6.1}$$

with the quadratic performance index

$$\hat{\mathcal{J}}^N(u^N) = \mathbb{E}\left\{ \sum_{k=k_0}^{\infty} \langle \hat{\mathcal{Q}}^N x_k^N, x_k^N \rangle_{\mathcal{H}} + \hat{r}(u_k^N)^2 \right\} \tag{6.2}$$

- Then the sequence of finite-dimensional approximating LQG control problems on the infinite-time horizon is
$(P2^N)$ Choose inputs $\bar{u} \in L^2([0,\infty);\mathbb{R})$ for which the criterion (6.2) is minimized.

Solution to $(P2^N)$

- Optimal closed-loop state feedback form $\bar{u}_k^N = -\hat{\mathcal{F}}^N \tilde{x}_k^N$, $k = k_0, k_0 + 1, \cdots$, where

$$\hat{\mathcal{F}}^N = \left\{ \hat{r} + (\hat{\mathcal{B}}^N)^* \hat{\Pi}^N \hat{\mathcal{B}}^N \right\}^{-1} (\hat{\mathcal{B}}^N)^* \hat{\Pi}^N \hat{\mathcal{A}}^N, \tag{6.3}$$

and $\hat{\Pi}^N$ is the unique positive semi-definite, symmetric solution to the approximating ARE,

$$\hat{\Pi}^N = (\hat{\mathcal{A}}^N)^* \Big[\hat{\Pi}^N - \hat{\Pi}^N \hat{\mathcal{B}}^N \big(\hat{r} + (\hat{\mathcal{B}}^N)^* \hat{\Pi}^N \hat{\mathcal{B}}^N \big)^{-1} (\hat{\mathcal{B}}^N)^* \hat{\Pi}^N \Big] \hat{\mathcal{A}}^N + (\hat{\mathcal{D}}^N)^* \hat{\mathcal{D}}^N. \tag{6.4}$$

Solution to $(P2^N)$

- Approximation observer $\tilde{x}_k^N$ is given by

$$\begin{aligned}\tilde{x}_{k+1}^N =& \hat{\mathcal{A}}^N \tilde{x}_k^N + \hat{\mathcal{B}}^N u_k^N + \tilde{\mathcal{L}}^N (y(k\tau) - \hat{\mathcal{C}}^N \tilde{x}_k^N),\\ \tilde{x}_0^N =& \mathcal{P}^N \tilde{\varphi}_0,\end{aligned} \tag{6.5}$$

where $\tilde{\mathcal{L}}^N \in \mathcal{L}(\mathbb{R}, \mathcal{V}^N)$ is given by:

$$\tilde{\mathcal{L}}^N = \hat{\mathcal{A}}^N \tilde{\Pi}^N \hat{\mathcal{C}}^* \left\{ \sigma^2 + \hat{\mathcal{C}}^N \tilde{\Pi}^N (\hat{\mathcal{C}}^N)^* \right\}^{-1}, \tag{6.6}$$

with $\tilde{\Pi}^N$ the unique positive semi-definite symmetric solution to the ARE

$$\begin{aligned}\tilde{\Pi}^N =& \hat{\mathcal{A}}^N \Big[\tilde{\Pi}^N - \tilde{\Pi}^N (\hat{\mathcal{C}}^N)^* \left(\sigma^2 + \hat{\mathcal{C}}^N \tilde{\Pi}^N (\hat{\mathcal{C}}^N)^* \right)^{-1} \\ & \hat{\mathcal{C}}^N \tilde{\Pi}^N \Big] (\hat{\mathcal{A}}^N)^* + \hat{\mathcal{B}}_1^N \Sigma (\hat{\mathcal{B}}_1^N)^*.\end{aligned} \tag{6.7}$$

Parameter Setting

- $q_2 = 0.5$, $q_3 = 0.5$, $q_4 = 0.5$, $q_5 = 0.5$, $q_6 = 0.5$, $\sigma_1 = 0.05$, $\sigma_2 = 0.05$, $\sigma = 0.05$, $k_0 = 0$, $k_1 = \infty$, $\rho = 0$, and $\hat{r} = 0.1$.
- True $q_1 = 0.2$, random parameter $q_1 \sim \text{Beta}(\alpha, \beta)$ with $\alpha = 3$ and $\beta = 2$.
- Sampling interval $\tau = 0.1$.

Optimal Functional Control Gain

- Plot the functional control gains for $n = m = 4, 8, 16, 32$.

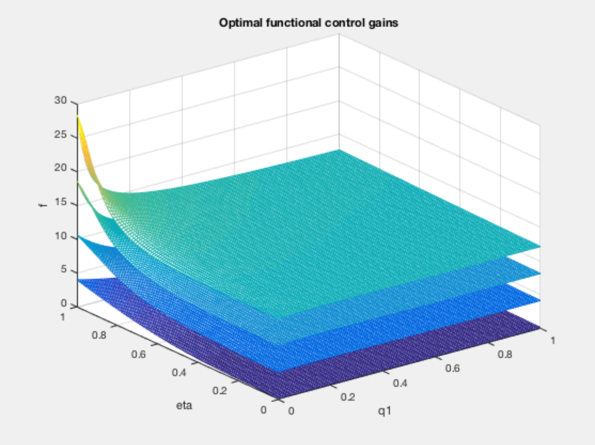

Figure: Optimal functional control gains for $n = m = 4, 8, 16, 32$.

Optimal Functional Observer Gain

- Plot the functional observer gains for $n = m = 4, 8, 16, 32$.

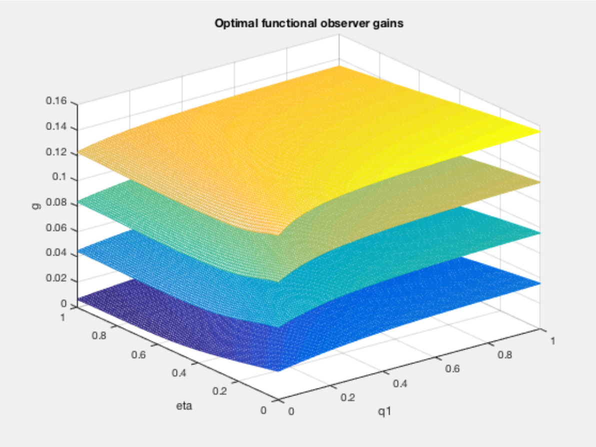

Figure: Optimal functional observer gains for $n = m = 4, 8, 16, 32$.

Optimal Functional Control/Observer Gain Convergence

- L^2 norm of the difference between the approximating optimal functional control gains and the *infinite dimensional* $(n = m = 32)$ optimal functional control gains.

$m = n$	4	8	12	16	20	24	28
Norm $(\times 10^{-4})$	18.00	10.00	5.18	2.61	1.25	0.54	0.17

- L^2 norm of the difference between the approximating optimal functional observer gains and the *infinite dimensional* $(n = m = 32)$ optimal functional observer gains.

$m = n$	4	8	12	16	20	24	28
Norm $(\times 10^{-5})$	12.62	7.39	4.81	3.21	2.09	1.24	0.56

Optimal Functional Control Gains for Different Values of q_1

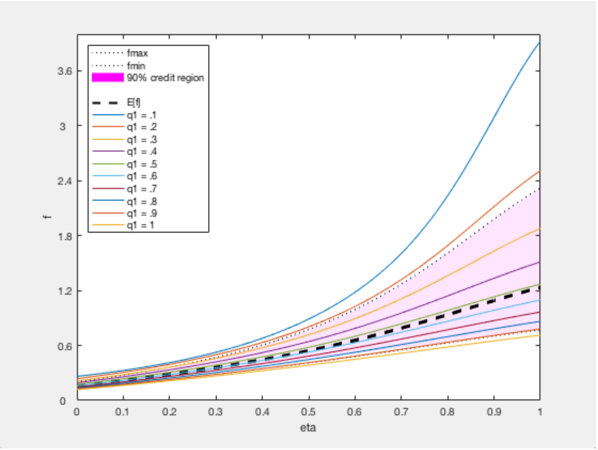

Figure: Optimal functional control gains $\hat{f}_3$ for various values of q_1 and the expected values of these gains when $q_1 \sim \text{Beta}(\alpha, \beta)$ with $\alpha = 3$ and $\beta = 2$.

BAC Comparison

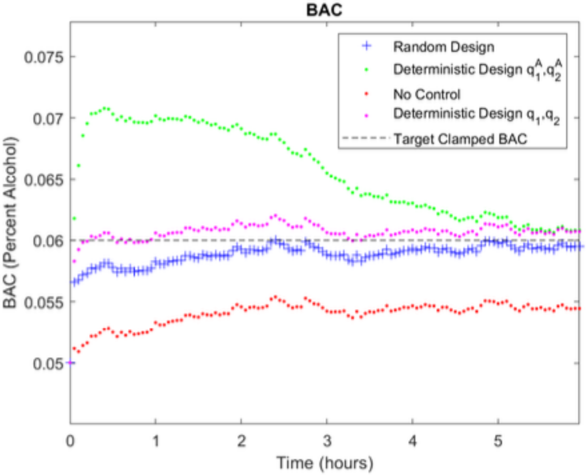

Figure: BAC comparison

Ethanol Infusion Rate (Input) Comparison

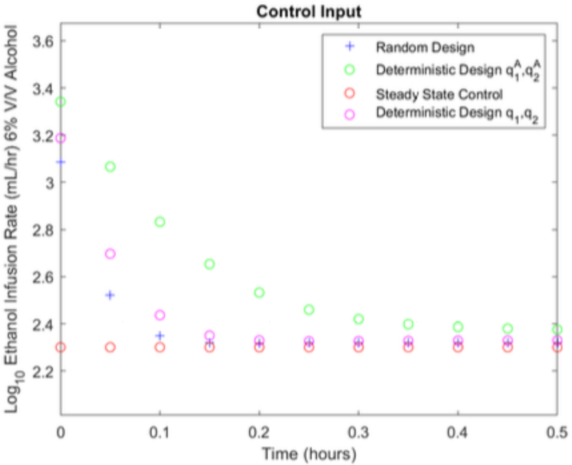

Figure: Ethanol infusion rate (input) comparison

Thank you!