**Introduction and Heuristic**

Let $G_\mathbb{Q}$ be the absolute Galois group of the rational numbers. A Galois representation is a continuous homomorphism

$$\rho : G_\mathbb{Q} \to G\ell_2(A),$$

for a topological ring $A$. In practice, Galois representations arise from the action of $G_\mathbb{Q}$ on the cohomology of varieties defined over $\mathbb{Q}$.

**Heuristic.** The image of a Galois representation should be as large as possible subject to the symmetries (cf. Definition 1) of the geometric object from which it arose.

**Definitions**

Fix a prime $p > 2$ and embeddings $\overline{Q} \hookrightarrow \mathbb{C}$ and $\overline{Q} \hookrightarrow \mathbb{Q}_p$.

**Definition 1.** [6] Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a cuspidal Hecke eigenform, and let $K$ be the number field generated by $(a_n : n \in \mathbb{Z}^+)$. An automorphism $\sigma$ of $K$ is a conjugate self-twist of $f$ if there is a nontrivial Dirichlet character $\eta_{\sigma}$ such that

$$a_{\sigma n} = \eta_{\sigma}(n) a_n$$

for almost all primes $\ell$. If the identity automorphism is a conjugate self-twist of $f$, then we say $f$ has complex multiplication (CM).

Let $K = \mathbb{Q}[[T]]$. For an integer $k \geq 2$ and a $p$-power root of unity $\zeta$, let $P_{k, \zeta} = (1 + T - \zeta(1 + p)T^p)\Lambda$. Such primes (and primes lying over them) are called arithmetic.

**Definition 2.** [2] Let $\mathbb{I}$ be an integral domain that is finite flat over $\Lambda$. A formal power series $F = \sum_{n=1}^{\infty} A_n q^n \in \mathbb{I}[[q]]$ is a Hida family if $A_p \in \mathbb{I}^+$ and, for every prime ideal $\mathfrak{p}$ of $\mathbb{I}$ lying over some $P_{k, \zeta}$, we have

- $A_p \mod \mathfrak{p} \in \mathbb{I}^+$ (rather than just $\overline{\mathbb{Q}}_p$)
- $f_{\mathfrak{p}} : = \sum_{n=1}^{\infty} (A_n \mod \mathfrak{p}) q^n$ is the $q$-expansion of a classical modular form of weight $k$ and the appropriate level and nebentypus.

**Main Theorem**

Let $F = \sum_{n=1}^{\infty} A_n q^n \in \mathbb{I}[[q]]$ be a Hida family. Hida showed [1] that there is a Galois representation $\rho_F : G_\mathbb{Q} \to G\ell_2(\mathbb{Q})$ that is unramified outside a finite set of primes and such that

$$\text{tr} \rho_F(\text{Frob}_\ell) = A_\ell$$

for all primes $\ell$ at which $\rho_F$ is unramified.

We can define conjugate self-twists of $F$ and the notion of CM following Definition 1 but replacing $K$ with the field of fractions of $\mathbb{I}$. For simplicity assume that $I$ is normal, and let $I_0$ be the subring of $I$ fixed by all conjugate self-twists of $F$.

**Theorem 1.** (L., [4]) Let $F$ be a non-CM Hida family. Assume that the residual representation $\bar{\rho}_F$ is absolutely irreducible and satisfies a minor $\mathbb{Z}_p$-regularity condition. Then there is a non-zero $\mathbb{I}_0$-ideal $a_0$ such that, in an appropriate basis, the image of $\bar{\rho}_F$ contains $\ker(\text{SL}_2(I_0) \to \text{SL}_2(I_0/a_0))$.

**Proof: Liftings Twists**

We keep the notation from Theorem 1 above. The following is a key input to the proof of Theorem 1.

**Theorem 2.** (L., [4]) Let $F$ be an arithmetic prime of $\mathbb{I}$ and $\bar{\sigma}$ a conjugate self-twist of $f_{\mathfrak{p}}$. If $\sigma$ preserves the local field generated by the Fourier coefficients of $f_{\mathfrak{p}}$, then $\sigma$ can be lifted to a conjugate self-twist $\bar{\sigma}$ of $F$.

**Proof: Reduction Steps**

**Theorem 3.** [7, 5] Let $F$ be a classical non-CM cuspidal eigenform. Let $O_0$ be the ring fixed by all conjugate self-twists of $f$. Then for any prime $\mathfrak{p}$, the image of $\rho_{f, \mathfrak{p}}$ contains an open subgroup of $O_{0, \mathfrak{p}}$. Let $G = \mathbb{I} / \mathfrak{p}P$ and $P$ be an arithmetic prime of $\Lambda$.

**Future Work**

- What is the largest $\mathbb{I}_0$-ideal $a_0$ that satisfies Theorem 1? We expect the answer is related to the congruence ideal of $F$, which in some cases can be related to $p$-adic $L$-functions.
- To what extent can we completely determine the image of $\bar{\rho}_F$?
- Is there an analogue of the Mumford-Tate Conjecture for $p$-adic families of Galois representations?

**References**


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**Contact Information**

Web: [www.math.ucla.edu/~jaclynlang](http://www.math.ucla.edu/~jaclynlang)
Email: jaclynlang@math.ucla.edu