Construction of solutions of the critical SQG equation in bounded domains

Mihaela Ignatova

ABSTRACT. We consider the critical dissipative SQG equation in bounded domains, with the square root of the Dirichlet Laplacian dissipation. We construct global bounded interior Lipschitz solutions from arbitrary large bounded interior Lipschitz initial data.

1. Introduction

The surface quasi-geostrophic (SQG) equation is a nonlinear active scalar equation which can be derived from the full three dimensional equations of motion for rapidly rotating density stratified incompressible fluids. The equation was used as a model of frontogenesis in meteorology [**18**]. From a mathematical point of view, the SQG equation was initially proposed in [**9**] as a two dimensional model to study inviscid incompressible formation of singularities, and has attracted significant attention due its remarkable similarities with the three dimensional incompressible Euler equation. While the global regularity of solutions to SQG starting from smooth initial data is an outstanding open problem, the original blow-up scenario of [**9**] has been ruled out both analytically [**14**] and numerically [**8**], and nontrivial examples of global smooth solutions have been constructed [**3**]. Solutions of SQG and related equations without dissipation and with non-smooth initial data give rise to interface dynamics [**17**], [**2**] with potential finite time blow up [**16**].

The critical SQG equation is obtained from SQG by adding square root of the Laplacian dissipation, and has been the object of intensive study in the past decade. The solutions are transported by divergence-free velocities they create, and are smoothed out and decay due to the additional nonlocal dissipation. The critical space L^∞ remains invariant under the evolution [**20**]. Moreover, solutions with small initial L^∞ norm become smooth [**4**]. The global regularity of solutions initiating from large data was obtained independently in [**1**] and [**19**] by very different methods: using harmonic extension and the De Georgi methodology of zooming in, and passing from L^2 to L^∞ and from L^∞ to C^α in [**1**], and constructing a family of time invariant moduli of continuity in [**19**]. Several subsequent proofs were obtained (see [**12**] and references therein). All the proofs are dimension independent, but are in either the whole space $\mathbb{R}^d$ or the torus $\mathbb{T}^d$.

In a recent paper [**6**] we considered the critical SQG equation

$$\partial_t\theta + u\cdot\nabla\theta + \Lambda_D\theta = 0 \tag{1}$$

with

$$u = \nabla^\perp\Lambda_D^{-1}\theta = R_D^\perp\theta \tag{2}$$

in a bounded open domain $\Omega\subset\mathbb{R}^d$ with smooth boundary and smooth initial data. We denote by Δ the Laplacian operator with homogeneous Dirichlet boundary conditions and by Λ_D its square root defined in terms of eigenfunction expansions. In [**6**] we obtained global interior regularity results. More precisely, denoting

$$d(x) = dist(x,\partial\Omega), \tag{3}$$

we proved in [**6**] that smooth solutions obey a uniform bound

$$\sup_{x\in\Omega,\ 0\le t\le T} d(x)|\nabla\theta(x,t)| \le C \tag{4}$$

with C a time-independent constant depending only on initial data and the domain Ω.

Because no explicit kernel for the fractional Laplacian is available in general bounded domains, our approach, initiated in [**5**], was based on bounds on the heat kernel. In the present paper we provide a

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construction of solutions satisfying the above bound. Namely, we construct smooth up to the boundary solutions of an ϵ-approximate equation, and establish interior bounds of the type (4) which are uniform in ϵ. We then prove that in the limit of $\epsilon \to 0$ we obtain solutions of (1) which obey the bound (4).

The ϵ-approximate equations of the critical SQG are defined by

$$\partial_t \theta_\epsilon + u_\epsilon \cdot \nabla \theta_\epsilon + \Lambda_D \theta_\epsilon = 0, \tag{5}$$

with

$$u_\epsilon = \nabla^\perp \left(\Lambda_D^{-1}\right)_\epsilon \theta_\epsilon, \tag{6}$$

using a spectral regularization of Λ_D^{-1} depending on a parameter $\epsilon > 0$,

$$\left(\Lambda_D^{-1}\right)_\epsilon \theta = \int_\epsilon^\infty t^{-\frac{1}{2}} e^{t\Delta} \theta dt. \tag{7}$$

The main results of this paper are the following two theorems.

THEOREM 1. *Let $\epsilon > 0$ and let $\theta_\epsilon(0) = \theta_0$ be an initial data, $\theta_0 \in W^{1,\infty}(\Omega)$. The ϵ-approximation (5)–(6) of the critical SQG has a unique, global, smooth up to the boundary solution. Let $T \geq 0$. The solution obeys the uniform in time bounds*

$$\sup_{t \in [0,T]} \|\theta_\epsilon(t)\|_{L^\infty(\Omega)} \leq \|\theta_0\|_{L^\infty(\Omega)}, \tag{8}$$

$$\sup_{t \in [0,T]} \|\theta_\epsilon(t)\|_{C^\alpha(\Omega)} \leq C \|\theta_0\|_{C^\alpha(\Omega)} \tag{9}$$

with C depending only on Ω but not on ϵ, and

$$\sup_{x \in \Omega, t \in [0,T]} d(x) |\nabla \theta_\epsilon(x,t)| \leq C \tag{10}$$

with C depending on Ω and $\|\theta_0\|_{W^{1,\infty}(\Omega)}$ but not on ϵ.

The sequence of approximations yields solutions of critical SQG.

THEOREM 2. *Let $\theta_0 \in L^\infty(\Omega)$ and let $T \geq 0$. Any sequence of solutions of the ϵ-approximations of SQG (5)–(6) with $\epsilon \to 0$ contains a subsequence θ_n converging strongly in $L^2(0,T;L^2(\Omega))$ to a weak solution $\theta \in L^\infty(0,T;L^\infty(\Omega)) \cap L^2(0,T;\mathcal{D}(\Lambda_D^{\frac{1}{2}}))$ of the critical SQG (1). If $\theta_0 \in W^{1,\infty}(\Omega)$, then θ obeys the uniform in time bound*

$$d(x)|\nabla \theta(x,t)| \leq C$$

with C depending only on Ω and $\|\theta_0\|_{W^{1,\infty}(\Omega)}$.

The Hölder space $C^\alpha(\Omega)$ for interior estimates used in the statement of Theorem 1 is defined below.

DEFINITION 1. *Let Ω be a bounded domain and let $0 < \alpha < 1$ be fixed. We say that $\theta \in C^\alpha(\Omega)$ if $\theta \in L^\infty(\Omega)$ and*

$$[f]_\alpha = \sup_{x \in \Omega} (d(x))^\alpha \left(\sup_{h \neq 0, |h| < d(x)} \frac{|f(x+h) - f(x)|}{|h|^\alpha} \right) < \infty. \tag{11}$$

The norm in $C^\alpha(\Omega)$ is

$$\|f\|_{C^\alpha} = \|f\|_{L^\infty(\Omega)} + [f]_\alpha. \tag{12}$$

The approximate equation (5)–(6) is devised so that it maintains the structure of transport by a divergence-free vector field. Because of this requirement, the advecting velocity does not have a sparse representation in terms of the eigenfunctions of the Dirichlet Laplacian. Instead, the stream function has a neat representation and is smooth (for fixed ϵ). This is used to prove that the solutions of the nonlinear ϵ-approximation are global and smooth up to the boundary. The strategy of proof of the ϵ-uniform bounds in Theorem 1 is similar to that of the proof of the a priori bounds of [**6**]. First, in order to obtain a priori L^∞ bounds we use the convex damping inequality (a Córdoba-Córdoba type inequality) proved in [**5**] for the Dirichlet

Λ_D. The uniform L^∞ bounds are then used to bound finite difference quotients. These bounds employ nonlinear lower bounds for the fractional Laplacian with Dirichlet boundary conditions, in the spirit of [**11**]. A version for derivatives in bounded domains, proved in [**5**], was modified for finite differences. In order to make sense of finite differences near the boundary in a manner suitable for transport, we use a family of good cutoff functions depending on a scale ℓ (see Lemma 2). The bounds for finite difference quotients require information about the commutator between the fractional Laplacian and finite difference or gradient operators. The commutators are nontrivial and are the main reason why the final results are uniform only in the interior of the domain. The above considerations pertain to the dissipative operator Λ_D. The drift term involves a non-spectral operator, $R_D^\perp$. We use bounds for $R_D^\perp\theta$ which are based only on global a priori information on $\|\theta\|_{L^\infty}$ and the nonlinear lower bounds. Such an approach was initiated in [**11**] and [**12**]. In the bounded domain case the method of proof has to be quite different because the kernels are not explicit, and the operators are not spectral. Once global interior $C^\alpha(\Omega)$ bounds are obtained, in order to obtain global interior bounds for the gradient, we use a different nonlinear lower bound. This is a super-cubic bound which makes the gradient equation look subcritical in view of the acquired C^α bounds.

The proof of Theorem 2 is based on ideas of [**5**] where we proved the existence of global weak solutions of (1) in $L^2(\Omega)$. Using a similar approach together with weak continuity properties of the nonlinearity (as in [**20**]), the existence of L^2 weak solutions to the inviscid SQG in bounded domains was obtained in [**10**] (see also [**7**] for a proof by a vanishing viscosity approximation).

The paper is organized as follows: Section 2 is devoted to the proof of Theorem 1 concerning the ϵ-approximate equations. We first address nonuniform in ϵ bounds leading to the global existence and uniqueness of smooth up to the boundary solutions. We then prove uniform in ϵ bounds for the Riesz transforms of regularizations in a separate subsection. These together with nonlinear lower bounds and commutator estimates are used to establish global interior gradient bounds which are uniform in ϵ. In Section 3 we give the proof of the passage to the limit as $\epsilon \to 0$ and construction of solutions.

2. Proof of Theorem 1

2.1. Nonuniform bounds. We denote by w_j the $L^2(\Omega)$-normalized eigenfunctions of $-\Delta$ with corresponding eigenvalues λ_j, counted with their multiplicities,

$$-\Delta w_j = \lambda_j w_j \quad \text{in}\, \Omega, \tag{13}$$

$$w_{j|\partial\Omega} = 0, \tag{14}$$

for $j \in \mathbb{N}$. It is well known that $0 < \lambda_1 \leq ... \leq \lambda_j \to \infty$ and that $-\Delta$ is a positive selfadjoint operator in $L^2(\Omega)$ with domain $\mathcal{D}(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. Functional calculus can be defined using the eigenfunction expansion. In particular, for $s \geq 0$ and f in $\mathcal{D}(\Lambda_D^s) = \{f \in L^2(\Omega)\colon (\lambda_j^{\frac{s}{2}} f_j) \in \ell^2(\mathbb{N})\}$,

$$\Lambda_D^s f = (-\Delta)^{\frac{s}{2}} f = \sum_{j=1}^{\infty} \lambda_j^s f_j w_j \tag{15}$$

with coefficients

$$f_j = \int_\Omega f(y) w_j(y) dy.$$

We have the representation

$$\left(\Lambda_D^s f\right)(x) = c_s \int_0^\infty \left[f(x) - e^{t\Delta} f(x)\right] t^{-1-\frac{s}{2}} dt \tag{16}$$

for $f \in \mathcal{D}(\Lambda_D^s)$, where the heat operator $e^{t\Delta}$ is given by

$$(e^{t\Delta} f)(x) = \int_\Omega H_D(t,x,y) f(y) dy \tag{17}$$

with kernel $H_D(t,x,y)$.

We recall from [**6**] the lower bound that provides a strong boundary repulsive term.

PROPOSITION 1. *Let Ω be a bounded domain with smooth boundary. Let $0 \le s < 2$. There exists a constant $c > 0$ depending only on the domain Ω and on s, such that, for any Φ, a C^2 convex function satisfying $\Phi(0) = 0$, and any $f \in C_0^\infty(\Omega)$, the inequality*

$$\Phi'(f)\Lambda_D^s f - \Lambda_D^s(\Phi(f)) \ge \frac{c}{d(x)^s}\left(f\Phi'(f) - \Phi(f)\right) \tag{18}$$

holds pointwise in Ω.

Let us remark here that if in particular $s = 1$ and $\Phi(f) = \frac{f^2}{2}$, then the lower bound (18) is

$$\begin{aligned} D(f)(x) &= f(x)\Lambda_D f(x) - \frac{1}{2}\Lambda_D(f(x)^2) \\ &= \frac{c_1}{2}\int_0^\infty t^{-\frac{3}{2}}dt\int_\Omega H_D(t,x,y)(f(x)-f(y))^2 dy + \frac{c_1}{2}f(x)^2\Lambda_D 1 \\ &\ge \frac{c}{d(x)}f(x)^2 \end{aligned} \tag{19}$$

for $x \in \Omega$, where c is a constant depending only on Ω.

2.1.1. *Global existence of solutions to (5)–(6).* The construction of global smooth solutions up to the boundary to (5)–(6) is obtained by using Galerkin approximations, energy methods and the Aubin-Lions Lemma ([**5**, Theorem 4]). The Galerkin approximations are m-dimensional ODEs,

$$\partial_t\theta_m + P_m\left(\left(\nabla^\perp\left(\Lambda_D^{-1}\right)_\epsilon\theta_m\right)\cdot\nabla\theta_m\right) + \Lambda_D\theta_m = 0 \tag{20}$$

with initial data

$$\theta_m(0) = P_m\theta_0 \tag{21}$$

and where

$$P_m f = \sum_{j=1}^m f_j w_j. \tag{22}$$

Note that, while $\psi_m = \left(\Lambda_D^{-1}\right)_\epsilon\theta_m \in P_m L^2(\Omega)$, the velocity $u_m = \nabla^\perp\psi_m \notin P_m L^2(\Omega)$. The fact that u_m is divergence free and smooth follows from the smoothness of the eigenfunctions of the Dirichlet Laplacian. The Galerkin approximation is a priori bounded in $P_m L^2(\Omega)$, uniformly in m. The following lemma gives uniform bounds for derivatives at fixed $\epsilon > 0$.

LEMMA 1. *Let $\epsilon > 0$ be fixed, let $\theta \in L^2(\Omega)$ and let $\psi = \left(\Lambda_D^{-1}\right)_\epsilon\theta$. For any $M \ge 0$, we have $\psi \in \mathcal{D}(\Lambda_D^M)$ and there exists a constant $C_{M,\epsilon} > 0$ such that*

$$\|\Lambda_D^M\psi\|_{L^2(\Omega)} \le C_{M,\epsilon}\|\theta\|_{L^2(\Omega)}. \tag{23}$$

As a consequence, the velocities u_m are uniformly bounded in spaces of high regularity. The conclusion of the proof of existence of solutions to (5)–(6) is then done by passing to the limit $m \to \infty$ while $\epsilon > 0$ is fixed using the Aubin-Lions compactness lemma.

Proof of Lemma 1. For fixed $\epsilon > 0$, we write θ and $\psi = \left(\Lambda_D^{-1}\right)_\epsilon\theta$ in terms of the eigenfunctions expansions: $\theta = \sum_{j=1}^\infty \theta_j w_j$ and $\psi = \sum_{j=1}^\infty \psi_j w_j$ with coefficients $\theta_j = \int_\Omega \theta w_j$ and $\psi_j = \int_\Omega \psi w_j$, where

$$\psi_j = \int_\epsilon^\infty t^{-\frac{1}{2}}e^{-t\lambda_j}\theta_j dt.$$

We have

$$|\psi_j| \le e^{-\frac{\epsilon}{2}\lambda_j}\left(\int_\epsilon^\infty t^{-\frac{1}{2}}e^{-\frac{t}{2}\lambda_j}dt\right)|\theta_j|,$$

and changing the integration variable $t\lambda_j = s$,

$$\int_\epsilon^\infty t^{-\frac{1}{2}}e^{-\frac{t}{2}\lambda_j}dt = \left(\int_{\epsilon\lambda_j}^\infty s^{-\frac{1}{2}}e^{-\frac{s}{2}}ds\right)\lambda_j^{-\frac{1}{2}} \le \left(\int_0^\infty s^{-\frac{1}{2}}e^{-\frac{s}{2}}ds\right)\lambda_j^{-\frac{1}{2}} \le C\lambda_j^{-\frac{1}{2}},$$

leads to

$$|\psi_j| \le C\lambda_j^{-\frac{1}{2}} e^{-\frac{\epsilon}{2}\lambda_j} |\theta_j|.$$

Therefore, for any $M \ge 0$

$$\|\Lambda_D^M \psi\|_{L^2}^2 = \sum_{j=1}^{\infty} \lambda_j^M |\psi_j|^2 \le C \sum_{j=1}^{\infty} \lambda_j^{M-1} e^{-\epsilon\lambda_j} |\theta_j|^2 \le C_M \epsilon^{-M+1} \|\theta\|_{L^2}^2,$$

which concludes the proof of the lemma.

2.1.2. *Uniqueness of solutions to (5)–(6).* Let θ_i, $i = 1, 2$ be two solutions starting from the same initial data θ_0 in $L^2(\Omega)$ with $u_i = \nabla^\perp \left(\Lambda_D^{-1}\right)_\epsilon \theta_i$, $u_{i|\partial\Omega} \cdot n = 0$ and $\nabla \cdot u_i = 0$. Denote $\theta = \theta_1 - \theta_2$ and $u = u_1 - u_2$. Then θ obeys the equation

$$\partial_t \theta + u \cdot \nabla \theta_1 + u_2 \cdot \nabla \theta + \Lambda_D \theta = 0$$

with initial data $\theta_{|t=0} = 0$. Taking L^2-inner product with θ leads to

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 + \int_\Omega u \cdot \nabla\theta_1 \theta dx + \int_\Omega u_2 \cdot \nabla\theta\theta dx + \int_\Omega \theta \Lambda_D \theta dx = 0.$$

Using the Sobolev embedding and the continuity of the embedding $\mathcal{D}(\Lambda_D^s) \subset H^s(\Omega)$ for $s \ge 0$, we have

$$\|\nabla\theta_1\|_{L^\infty} \le C\|\Lambda_D \psi_1\|_{H^{2+\delta}} \le \|\Lambda_D^{3+\delta}\psi_1\|_{L^2} \le C\epsilon^{-1-\delta/2}\|\theta_1\|_{L^2} \tag{24}$$

holds for $\delta > 0$. By Proposition 1 with $s = 1$ and $\Phi(\theta_1) = \theta_1^2/2$, we get $\|\theta_1(t)\|_{L^2(\Omega)} \le \|\theta_0\|_{L^2(\Omega)}$ for $t \ge 0$, and so $\nabla\theta_1 \in L^\infty(\Omega)$ for $t \ge 0$. Now, observe that the second term is bounded

$$\left| \int_\Omega u \cdot \nabla\theta_1 \theta dx \right| \le \|u\|_{L^2} \|\nabla\theta_1\|_{L^\infty} \|\theta\|_{L^2} \le \|\nabla\theta_1\|_{L^\infty} \|\theta\|_{L^2}^2,$$

where

$$\|u\|_{L^2} = \|\nabla^\perp \psi\|_{L^2} = \|\Lambda_D \psi\|_{L^2} \le C_\epsilon \|\theta\|_{L^2}$$

by (23) with $M = 1$. The third term vanishes after integration by parts with u_2 divergence free and tangential to the boundary, and the last term is nonnegative. Thus

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_{L^2}^2 \le \|\nabla\theta_1\|_{L^\infty} \|\theta\|_{L^2}^2$$

and the uniqueness follows from Gronwall's inequality.

2.2. Uniform bounds. We consider the ϵ-approximate equation (5)–(6) with fixed initial data $\theta_0 \in W^{1,\infty}(\Omega)$. By the convex damping inequality (18) with $s = 1$ we have the uniform in ϵ bound

$$\|\theta_\epsilon(t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty} \tag{25}$$

for $t \ge 0$.

Our approach is based on estimates on the finite differences $\delta_h \theta(x) = \theta(x + h) - \theta(x)$. These may not belong to the domain of the Dirichlet Λ_D, $H_0^1(\Omega)$. For this reason, as in [**6**], we use a family of good cutoff functions χ depending on a length scale ℓ:

LEMMA 2. *Let Ω be a bounded domain with C^2 boundary. For $\ell > 0$ small enough (depending on Ω) there exist cutoff functions χ with the properties: $0 \le \chi \le 1$, $\chi(y) = 0$ if $d(y) \le \frac{\ell}{4}$, $\chi(y) = 1$ for $d(y) \ge \frac{\ell}{2}$, $|\nabla^k \chi| \le C\ell^{-k}$ with C independent of ℓ and*

$$\int_\Omega \frac{(1 - \chi(y))}{|x - y|^{d+j}} dy \le \frac{C}{d(x)^j} \tag{26}$$

and

$$\int_\Omega \frac{|\nabla\chi(y)|}{|x - y|^{d-\alpha}} \le \frac{C}{d(x)^{1-\alpha}} \tag{27}$$

hold for $j > -d$, $\alpha < d$ and $d(x) \ge \ell$. We refer to such χ as a "good cutoff".

2.2.1. *Uniform bounds for Riesz transforms regularizations.* Let $\epsilon > 0$. We consider u given by

$$u = \nabla^{\perp}\left(\Lambda_D^{-1}\right)_{\epsilon}\theta \tag{28}$$

and establish estimates of finite differences of u in terms of θ, which are uniform in ϵ.

THEOREM 3. *Let χ be a good cutoff as in Lemma 2 above, and let u be defined by (28). Then*

$$|\delta_h u(x)| \le C\left(\sqrt{\rho D(f)(x)} + \|\theta\|_{L^\infty}\left(\frac{|h|}{d(x)} + \frac{|h|}{\rho}\right) + |\delta_h\theta(x)|\right) \tag{29}$$

holds for $d(x) \ge \ell$, $\rho \le cd(x)$, $f = \chi\delta_h\theta$, $D(f)$ defined in (19), and with C a constant depending only on Ω and not on ϵ.

Proof of Theorem 3. Let $\chi \in C_0^\infty(\Omega)$ be a good cutoff function with a fixed length scale $\ell > 0$ which satisfies $\chi(x) = 1$ if $d(x) \ge \frac{\ell}{2}$, $\chi(x) = 0$ if $d(x) \le \frac{\ell}{4}$, $|\nabla\chi(x)| \le C\ell^{-1}$, (26) and (27). We take $|h| \le \frac{\ell}{14}$. In view of the representation

$$\left(\Lambda_D^{-1}\right)_{\epsilon} = \int_{\epsilon}^{\infty} t^{-\frac{1}{2}} e^{t\Delta} dt \tag{30}$$

we have

$$\delta_h u(x) = \int_{\epsilon}^{\infty} t^{-\frac{1}{2}} dt \int_{\Omega} \delta_h^x \nabla_x^{\perp} H_D(x,y,t)\theta(y) dy \tag{31}$$

for $x \in \Omega$ with $d(x) \ge \frac{\ell}{4}$. Let $\rho = \rho(x,h) > 0$ be a length scale satisfying

$$\rho \le cd(x), \tag{32}$$

to be determined later. We split

$$\delta_h u = \delta_h u^{in} + \delta_h u^{out} \tag{33}$$

where

$$\delta_h u(x)^{in} = \int_{\epsilon}^{\rho^2} t^{-\frac{1}{2}} dt \int_{\Omega} \delta_h^x \nabla_x^{\perp} H_D(x,y,t)\theta(y) dy \tag{34}$$

and

$$\delta_h u(x)^{out} = \int_{\rho^2}^{\infty} t^{-\frac{1}{2}} dt \int_{\Omega} \delta_h^x \nabla_x^{\perp} H_D(x,y,t)\theta(y) dy. \tag{35}$$

We recall important bounds on the Dirichlet heat kernel [**6**, Appendix 1]:

$$\nabla_x\nabla_x H_D(x,y,t) \le Ct^{-1-\frac{d}{2}} e^{-\frac{|x-y|^2}{\bar{K}t}}, \tag{36}$$

$$\int_{\Omega} |(\nabla_x + \nabla_y) H_D(x,y,t)|\, dy \le Ct^{-\frac{1}{2}} e^{-\frac{d(x)^2}{\bar{K}t}} \tag{37}$$

and

$$\int_{\Omega} |\nabla_x(\nabla_x + \nabla_y) H_D(x,y,t)|\, dy \le Ct^{-1} e^{-\frac{d(x)^2}{\bar{K}t}} \tag{38}$$

valid for $t \le cd(x)^2$ and $0 < t \le T$.

Using (36), we have

$$|\delta_h u^{out}(x)| \le C\|\theta\|_{L^\infty}\frac{|h|}{\rho}. \tag{39}$$

In order to bound $\delta_h u^{in}$, we write

$$\delta_h u^{in}(x) = u_h(x) + v_h(x) \tag{40}$$

where

$$u_h(x) = \int_{\epsilon}^{\rho^2} t^{-\frac{1}{2}} dt \int_{\Omega} \nabla_x^{\perp} H(x,y,t)(\chi(y)\delta_h\theta(y) - \chi(x)\delta_h\theta(x)) dy \tag{41}$$

and

$$v_h(x) = v_1(x) + v_2(x) + v_3(x) + \chi(x)\delta_h\theta(x)v_4(x) \tag{42}$$

with

$$v_1(x) = \int_\epsilon^{\rho^2} t^{-\frac{1}{2}} dt \int_\Omega \nabla_x^\perp (H_D(x+h,y,t) - H_D(x,y,t))(1-\chi(y))\theta(y)dy, \tag{43}$$

$$v_2(x) = \int_\epsilon^{\rho^2} t^{-\frac{1}{2}} dt \int_\Omega \nabla_x^\perp (H_D(x+h,y,t) - H_D(x,y-h,t))\chi(y)\theta(y)dy, \tag{44}$$

$$v_3(x) = \int_\epsilon^{\rho^2} t^{-\frac{1}{2}} dt \int_\Omega \nabla_x^\perp H_D(x,y,t)(\chi(y+h) - \chi(y))\theta(y+h)dy, \tag{45}$$

and

$$v_4(x) = \int_\epsilon^{\rho^2} t^{-\frac{1}{2}} dt \int_\Omega \nabla_x^\perp H_D(x,y,t)dy. \tag{46}$$

We first estimate u_h using the bound on the heat kernel

$$|\nabla H_D(x,y,t)| \le C\frac{1}{\sqrt{t}}\left(1 + \frac{|x-y|}{\sqrt{t}}\right) H_D(x,y,t) \tag{47}$$

for $t \le cd(x)^2$ and the Cauchy-Schwarz inequality

$$\begin{aligned} |u_h(x)| &\le C\int_\epsilon^{\rho^2} t^{-1} dt \int_\Omega \left(1 + \frac{|x-y|}{\sqrt{t}}\right) H_D(x,y,t)(\chi(y)\delta_h\theta(y) - \chi(x)\delta_h\theta(x))dy \\ &\le C\sqrt{\rho}\left(\int_0^{\rho^2} t^{-\frac{3}{2}} dt \int_\Omega H_D(x,y,t)(\chi(y)\delta_h\theta(y) - \chi(x)\delta_h\theta(x))^2 dy\right)^{\frac{1}{2}}. \end{aligned} \tag{48}$$

Thus, we have

$$|u_h(x)| \le C\sqrt{\rho D(f)(x)}, \tag{49}$$

with $f = \chi\delta_h\theta$, because

$$D(f)(x) = f(x)\Lambda_D f(x) - \frac{1}{2}\Lambda_D(f(x)^2) \ge \frac{c_1}{2}\int_0^{\rho^2} t^{-\frac{3}{2}} dt \int_\Omega H_D(x,y,t)(f(x) - f(y))^2 dy$$

in view of (19).

Next we estimate the terms in (42). For v_1 we write

$$H_D(x+h,y,t) - H_D(x,y,t) = h \cdot \int_0^1 \nabla_x H_D(x+\lambda h, y, t)d\lambda$$

and use (36) and the fact that $\int_\epsilon^{\rho^2} t^{-\frac{3}{2}-\frac{d}{2}} e^{-\frac{p^2}{Kt}} dt \le Cp^{-d-1}$ for $p > 0$:

$$|v_1(x)| \le |h|\|\theta\|_{L^\infty}\int_0^1 d\lambda \int_\epsilon^{\rho^2} t^{-\frac{1}{2}} dt \int_\Omega |\nabla_x^\perp \nabla_x H_D(x+\lambda h,y,t)|(1-\chi(y))dy \tag{50}$$

$$\le C|h|\|\theta\|_{L^\infty}\int_0^1 d\lambda \int_\Omega \frac{1-\chi(y)}{|x+\lambda h - y|^{d+1}}dy. \tag{51}$$

From (26) with $j = 1$ we get

$$|v_1(x)| \le C\|\theta\|_{L^\infty}\frac{|h|}{d(x)} \tag{52}$$

for $d(x) \ge \ell$. In order to estimate v_2 we write

$$H_D(x+h,y,t) - H_D(x,y-h,t) = h \cdot \int_0^1 (\nabla_x + \nabla_y)H_D(x+\lambda h, y + (\lambda - 1)h, t)d\lambda \tag{53}$$

and use (38)

$$|v_2(x)| \le |h| \|\theta\|_{L^\infty} \int_0^1 d\lambda \int_\epsilon^{\rho^2} t^{-\frac{1}{2}} dt \int_\Omega |\nabla_x^\perp (\nabla_x + \nabla_y) H_D(x + \lambda h, y + (\lambda - 1)h, t)|\chi(y) dy$$
$$\le C|h| \|\theta\|_{L^\infty} \int_0^1 d\lambda \int_\epsilon^{\rho^2} t^{-\frac{3}{2}} e^{-\frac{d(x)^2}{4Kt}} dt \tag{54}$$

and thus

$$|v_2(x)| \le C\|\theta\|_{L^\infty} \frac{|h|}{d(x)} \tag{55}$$

holds for $d(x) \ge \ell$. For v_3 we use (47), the upper bound $H_D(x, y, t) \le Ct^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{Kt}}$, and the fact that $\int_\epsilon^{\rho^2} t^{-1-\frac{d}{2}} \left(1 + \frac{p}{\sqrt{t}}\right) e^{-\frac{p^2}{Kt}} dt \le Cp^{-d}$ for $p > 0$ to get

$$|v_3(x)| \le |h| \|\theta\|_{L^\infty} \int_\epsilon^{\rho^2} t^{-1} dt \int_\Omega \left(1 + \frac{|x - y|}{\sqrt{t}}\right) H_D(x, y, t) |\nabla \chi(y)| dy$$
$$\le C|h| \|\theta\|_{L^\infty} \int_\Omega \frac{|\nabla \chi(y)|}{|x - y|^d} dy. \tag{56}$$

Then, from (27) we obtain

$$|v_3(x)| \le C\|\theta\|_{L^\infty} \frac{|h|}{d(x)} \tag{57}$$

for $d(x) \ge \ell$. Next, we split

$$v_4(x) = v_5(x) + v_6(x)$$

where

$$v_5(x) = \int_\epsilon^{\rho^2} t^{-\frac{1}{2}} \int_\Omega \nabla_x^\perp H_D(x, y, t) \chi(y) dy$$

and

$$v_6(x) = \int_\epsilon^{\rho^2} t^{-\frac{1}{2}} \int_\Omega \nabla_x^\perp H_D(x, y, t)(1 - \chi(y)) dy.$$

For v_5 we write

$$v_5(x) = \int_\epsilon^{\rho^2} t^{-\frac{1}{2}} dt \int_\Omega \left((\nabla_x^\perp + \nabla_y^\perp) H_D(x, y, t) \right) \chi(y) dy$$
$$+ \int_\epsilon^{\rho^2} t^{-\frac{1}{2}} dt \int_\Omega H_D(x, y, t) \nabla_y^\perp \chi(y) dy, \tag{58}$$

where we used integration by parts for the second term on the right hand side to make $\nabla_y^\perp$ fall on χ. From (37), we have

$$|v_5(x)| \le C \int_\epsilon^{\rho^2} t^{-1} e^{-\frac{d(x)^2}{Kt}} dt + C \int_\epsilon^{\rho^2} t^{-\frac{1}{2}-\frac{d}{2}} dt \int_\Omega e^{-\frac{|x-y|^2}{Kt}} |\nabla \chi(y)| dy$$
$$\le C \left(1 + \log_+ \left(\frac{\rho}{d(x)}\right)\right) + C\rho \int_\Omega \frac{|\nabla \chi(y)|}{|x - y|^d} dy. \tag{59}$$

By (27) and because $\rho \le cd(x)$ we obtain

$$|v_5(x)| \le C \tag{60}$$

for $d(x) \ge \ell$, with C depending on Ω but not on ℓ. Finally, by (47) and (26)

$$|v_6(x)| \le C \int_\Omega \frac{(1 - \chi(y))}{|x - y|^d} dy \le C \tag{61}$$

for $d(x) \geq \ell$, with a constant independent of ℓ. Thus,

$$|v_4(x)| \leq C \tag{62}$$

for $d(x) \leq \ell$, with a constant C depending only on Ω.

Summarizing, we obtain

$$|v_h(x)| \leq C\|\theta\|_{L^\infty}\frac{|h|}{d(x)} + C|\delta_h\theta(x)| \tag{63}$$

for $d(x) \geq \ell$. Combining the estimates (39), (49) and (63), the proof of the theorem is established.

We next state the uniform in ϵ bound for the gradient.

THEOREM 4. *Let χ be a good cutoff with scale ℓ and let u be given by (28). Then*

$$|\nabla u(x)| \leq C\left(\sqrt{\rho D(f)} + \|\theta\|_{L^\infty(\Omega)}\left(\frac{1}{d(x)} + \frac{1}{\rho}\right) + |\nabla\theta(x)|\right) \tag{64}$$

holds for $d(x) \geq \ell$, $\rho \leq cd(x)$ and $f = \chi\nabla\theta$ with a constant C depending only Ω and not on ϵ.

The proof is omitted because it is very similar to the one of Theorem 3.

2.2.2. *Uniform Hölder Bounds.* We prove the following uniform interior Hölder bound:

THEOREM 5. *Let $\theta_\epsilon(x,t)$ be a solution of (5)–(6) in the smooth bounded domain Ω. There exists a constant $0 < \alpha < 1$ depending only on $\|\theta_0\|_{L^\infty(\Omega)}$, and a constant $\Gamma > 0$ depending on the domain Ω such that*

$$\sup_{t\geq 0}\|\theta_\epsilon(t)\|_{C^\alpha(\Omega)} \leq \Gamma\|\theta_0\|_{C^\alpha(\Omega)} \tag{65}$$

holds uniformly in ϵ.

Proof of Theorem 5. For simplicity of notation we drop the subscript ϵ on θ_ϵ and u_ϵ. Let χ be a good cutoff with scale $\ell > 0$ and let $|h| \leq \frac{\ell}{16}$. From (5) we obtain the equation for the localized finite difference $\chi\delta_h\theta$:

$$(\partial_t + u\cdot\nabla + (\delta_h u)\cdot\nabla_h)(\delta_h\theta) + \Lambda_D(\chi\delta_h\theta) + C_h(\theta) = 0 \tag{66}$$

in Ω when $d(x) \geq \ell$ and where $C_h(\theta)$ is the commutator given by

$$C_h(\theta) = \chi\delta_h\Lambda_D\theta - \Lambda_D(\chi\delta_h\theta). \tag{67}$$

In (66) we used the fact that $\chi(x) = 1$ for $d(x) \geq \ell$ and the identity for the drift $((\delta_h u(x))\cdot\nabla)\theta(x+h) = ((\delta_h u(x))\cdot\nabla_h)\delta_h\theta(x)$. By [**6**, Lemma 3] there exists a constant Γ_0 such that the commutator obeys

$$|C_h(\theta)(x)| \leq \Gamma_0\frac{|h|}{d(x)^2}\|\theta\|_{L^\infty(\Omega)} \tag{68}$$

for $d(x) \geq \ell$, $|h| \leq \frac{\ell}{16}$, and $\theta \in H_0^1(\Omega)\cap L^\infty(\Omega)$. Multiplying the equation (66) by $\delta_h\theta$ we have

$$\frac{1}{2}L_\chi(\delta_h\theta)^2 + D(f) + (\delta_h\theta)C_h(\theta) = 0 \tag{69}$$

where L_χ denotes the linear operator of transport and nonlocal diffusion

$$L_\chi g = \partial_t g + u\cdot\nabla_x g + \delta_h u\cdot\nabla_h g + \Lambda_D(\chi^2 g) \tag{70}$$

and where $D(f)(x) = (f\Lambda_D f)(x) - \frac{1}{2}(\Lambda_D f^2)(x)$ with $f = \chi\delta_h\theta$. The latter obeys the pointwise nonlinear lower bound ([**6**, Theorem 5])

$$D(f)(x) \geq \gamma_1|h|^{-1}\frac{|f_d(x)|^3}{\|\theta\|_{L^\infty}} + \gamma_1\frac{f^2(x)}{d(x)} \tag{71}$$

in Ω when $d(x) \geq \ell$ and $|h| \leq \frac{\ell}{16}$ with

$$|f_d(x)| = \begin{cases} |f(x)|, & \text{if } \ |f(x)| \geq M\|\theta\|_{L^\infty(\Omega)}\frac{|h|}{d(x)}, \\ 0, & \text{if } \ |f(x)| \leq M\|\theta\|_{L^\infty(\Omega)}\frac{|h|}{d(x)}. \end{cases} \tag{72}$$

Multiplying (69) by $|h|^{-2\alpha}$ where $\alpha > 0$ will be chosen small enough we obtain

$$\frac{1}{2}L_\chi\left(\frac{|\delta_h\theta|^2}{|h|^{2\alpha}}\right) + |h|^{-2\alpha}D(f) \le 2\alpha\frac{|\delta_h u|}{|h|}\left(\frac{|\delta_h\theta|^2}{|h|^{2\alpha}}\right) + |C_h(\theta)||\delta_h\theta||h|^{-2\alpha}. \tag{73}$$

Observe that the factor 2α in the first term on the right side comes from the differentiation $\delta_h u\cdot\nabla_h(|h|^{-2\alpha})$ and its smallness will be crucial below. We recall the bound on the finite differences $\delta_h u$ from Theorem 3

$$|\delta_h u(x)| \le C\left(\sqrt{\rho D(f)(x)} + \|\theta\|_{L^\infty}\left(\frac{|h|}{d(x)} + \frac{|h|}{\rho}\right) + |\delta_h\theta(x)|\right) \tag{74}$$

which holds in Ω for $d(x)\ge\ell$, for any ρ satisfying $\rho\le cd(x)$ (to be determined below), and with a constant C depending only on Ω and not on ϵ. Incorporating the estimates (68) and (74) in (73), using the Cauchy-Schwarz inequality $\sqrt{\rho D(f)} \le \frac{1}{2}\rho + \frac{1}{2}D(f)$, and absorbing the term containing $D(f)$ with the left hand side, we get

$$\begin{aligned}\frac{1}{2}L_\chi\left(\frac{|\delta_h\theta|^2}{|h|^{2\alpha}}\right) + \frac{1}{2}|h|^{-2\alpha}D(f) &\le C_1\alpha^2\rho|h|^{-2-2\alpha}|\delta_h\theta|^4 + C_1\alpha\|\theta\|_{L^\infty}\left(\frac{1}{d(x)}+\frac{1}{\rho}\right)|h|^{-2\alpha}|\delta_h\theta|^2\\ &+ C_1\alpha|h|^{-1-2\alpha}|\delta_h\theta|^3 + \Gamma_0\frac{1}{d(x)^2}\|\theta\|_{L^\infty}|h|^{1-2\alpha}|\delta_h\theta|\end{aligned} \tag{75}$$

in Ω for $d(x)\ge\ell$ and $|h|\le\frac{\ell}{16}$. Now we choose ρ such that

$$\rho = \begin{cases} |\delta_h\theta(x)|^{-1}|h|\|\theta\|_{L^\infty}, & \text{if } |\delta_h\theta(x)| \ge M_1\|\theta\|_{L^\infty}\frac{|h|}{d(x)},\\ d(x), & \text{if } \ |\delta_h\theta(x)| \le M_1\|\theta\|_{L^\infty}\frac{|h|}{d(x)},\end{cases} \tag{76}$$

with M_1 defined by

$$M_1 = M + \sqrt{\frac{8\Gamma_0}{\gamma_1}} + 1, \tag{77}$$

where M is the constant from (72), Γ_0 is the constant from (68) and γ_1 is the constant from (71). This choice was made in order to use the lower bound on $D(f)$ to estimate the contribution due to the inner piece u_h (see (41) above) of $\delta_h u$ and the contribution from the commutator $C_h(\theta)$.

To the end of the proof we distinguish between two cases based on the choice of ρ in (76). The first case is when $|\delta_h\theta(x)|\ge M_1\|\theta\|_{L^\infty}\frac{|h|}{d(x)}$. Then we have

$$\begin{aligned}\frac{1}{2}L_\chi\left(\frac{|\delta_h\theta|^2}{|h|^{2\alpha}}\right) + \frac{1}{2}|h|^{-2\alpha}D(f) &\le C_1\left[(\alpha\|\theta\|_{L^\infty})^2 + (2+\frac{1}{M_1})\alpha\|\theta\|_{L^\infty}\right]\|\theta\|_{L^\infty}^{-1}|\delta_h\theta|^3|h|^{-1-2\alpha}\\ &+ \Gamma_0\frac{1}{d(x)^2}\|\theta\|_{L^\infty}|\delta_h\theta||h|^{1-2\alpha}.\end{aligned} \tag{78}$$

In view of the definition of M_1 in (77),

$$\Gamma_0\frac{1}{d(x)^2}\|\theta\|_{L^\infty}|\delta_h\theta(x)||h|^{1-2\alpha} \le \frac{\gamma_1}{8}\|\theta\|_{L^\infty}^{-1}|\delta_h\theta(x)|^3|h|^{-1-2\alpha}$$

in this case. Now we choose α small enough such that

$$C_1\left[(\alpha\|\theta\|_{L^\infty})^2 + (2+M_1^{-1})\alpha\|\theta\|_{L^\infty}\right] \le \frac{\gamma_1}{8} \tag{79}$$

holds. Thus from (78), in the case when $|\delta_h\theta(x)|\ge M_1\|\theta\|_{L^\infty}\frac{|h|}{d(x)}$, we obtain

$$\frac{1}{2}L_\chi\left(\frac{|\delta_h\theta|^2}{|h|^{2\alpha}}\right) + \frac{1}{4}|h|^{-2\alpha}D(f) \le 0 \tag{80}$$

in Ω for $d(x)\ge\ell$ and $|h|\le\frac{\ell}{16}$.

The second case is when the opposite inequality holds, namely, when $|\delta_h\theta(x)| \le M_1\|\theta\|_{L^\infty}\frac{|h|}{d(x)}$. Then, using $\rho = d(x)$, from (75) we obtain

$$\frac{1}{2}L_\chi\left(\frac{|\delta_h\theta|^2}{|h|^{2\alpha}}\right) + \frac{1}{2}|h|^{-2\alpha}D(f) \le C_1\left[M_1^2(\alpha\|\theta\|_{L^\infty})^2 + (M_1+2)\alpha\|\theta\|_{L^\infty}\right]\frac{1}{d(x)}|\delta_h\theta|^2|h|^{-2\alpha} + \Gamma_0\frac{1}{d(x)^2}\|\theta\|_{L^\infty}|\delta_h\theta||h|^{1-2\alpha}. \tag{81}$$

Now, by choosing α small (with a factor of $1/M_1^2$ smaller than in (79)) such that

$$C_1M_1^2\left[(\alpha\|\theta\|_{L^\infty})^2 + (2+M_1^{-1})\alpha\|\theta\|_{L^\infty}\right] \le \frac{\gamma_1}{8}, \tag{82}$$

we arrive at

$$\frac{1}{2}L_\chi\left(\frac{|\delta_h\theta|^2}{|h|^{2\alpha}}\right) + \frac{1}{2}|h|^{-2\alpha}D(f) \le \frac{\gamma_1}{8}\frac{1}{d(x)}\left(\frac{|\delta_h\theta|^2}{|h|^{2\alpha}}\right) + \Gamma_0M_1\|\theta\|_{L^\infty}^2d(x)^{-3}|h|^{2-2\alpha}. \tag{83}$$

Summarizing, in view of the inequalities (80) and (83), the damping term $\frac{\gamma_1}{d(x)}|\delta_h\theta(x)|^2$ in (71) and the choice of small α in (82), we have that

$$L_\chi\left(\frac{|\delta_h\theta(x)|^2}{|h|^{2\alpha}}\right) + \frac{1}{2}\frac{\gamma_1}{d(x)}\left(\frac{|\delta_h\theta(x)|^2}{|h|^{2\alpha}}\right) \le B \tag{84}$$

holds in Ω for $d(x) \ge \ell$ and $|h| \le \frac{\ell}{16}$ where

$$B = 2(16)^{-2+2\alpha}\Gamma_0M_1\|\theta\|_{L^\infty}^2d(x)^{-1-2\alpha} = \Gamma_1\frac{\gamma_1}{2}\|\theta\|_{L^\infty}^2d(x)^{-1-2\alpha} \tag{85}$$

with Γ_1 depending on Ω but not on ϵ. We note that

$$L_\chi\left(\frac{|\delta_h\theta(x)|^2}{|h|^{2\alpha}}\right) + \frac{1}{2}\frac{\gamma_1}{d(x)}\left(\frac{|\delta_h\theta(x)|^2}{|h|^{2\alpha}} - \Gamma_1\ell^{-2\alpha}\|\theta\|_{L^\infty}^2\right) \le 0 \tag{86}$$

holds for any $t > 0$, $x \in \Omega$ with $d(x) \ge \ell$ and $|h| \le \frac{\ell}{16}$.

We claim that for any $\delta > 0$ and any $T > 0$

$$\sup_{d(x)\ge\ell,|h|\le\frac{\ell}{16},0\le t\le T}\frac{|\delta_h\theta(x,t)|^2}{|h|^{2\alpha}} \le (1+\delta)\left[\sup_{d(x)\ge\ell,|h|\le\frac{\ell}{16}}\frac{|\delta_h\theta_0(x)|^2}{|h|^{2\alpha}} + \Gamma_1\ell^{-2\alpha}\|\theta_0\|_{L^\infty}^2\right] \tag{87}$$

holds uniformly in ϵ. Indeed, let $R = (1+\delta)\left[\sup_{d(x)\ge\ell,|h|\le\frac{\ell}{16}}\frac{|\delta_h\theta_0(x)|^2}{|h|^{2\alpha}} + \Gamma_1\ell^{-2\alpha}\|\theta_0\|_{L^\infty}^2\right]$ and assume by contradiction that there exist $\tilde{t} \le T$, $\tilde{x} \in \Omega$ and $\tilde{h}$ with $d(\tilde{x}) \ge \ell$ and $|\tilde{h}| \le \frac{\ell}{16}$ such that the opposite inequality of (87) holds, namely

$$\frac{|\theta(\tilde{x}+\tilde{h},\tilde{t}) - \theta(\tilde{x},\tilde{t})|^2}{|\tilde{h}|^{2\alpha}} > R. \tag{88}$$

Because θ is smooth, and by continuity in t,

$$\sup_{d(x)\ge\ell,|h|\le\frac{\ell}{16}}\frac{|\delta_h\theta(x,t)|^2}{|h|^{2\alpha}} \le (1+\delta)\sup_{d(x)\ge\ell,|h|\le\frac{\ell}{16}}\frac{|\delta_h\theta_0(x)|^2}{|h|^{2\alpha}}$$

for a short time $0 \le t \le t_1$. Also, in particular, θ is bounded in C^1, and there exists a constant C such that

$$\sup_{d(x)\ge\ell,|h|\le\frac{\ell}{16}}\frac{|\delta_h\theta(x)|^2}{|h|^2} \le C$$

on the time interval $[0,T]$. It follows that there exists $\delta_1 > 0$ such that

$$\sup_{d(x)\ge\ell,|h|\le\delta_1}\frac{|\delta_h\theta(x)|^2}{|h|^{2\alpha}} \le C\delta_1^{2-2\alpha} \le \frac{R}{2}.$$

In view of these considerations, we must have $\tilde{t} > t_1$, $|\tilde{h}| \ge \delta_1$. Moreover, for any $\tilde{t} > t_1$, the maximum

$$s(\tilde{t}) = \sup_{d(x)\ge \ell, |h|\le \frac{\ell}{16}} \frac{|\delta_h \theta(\tilde{t})|^2}{|h|^{2\alpha}}$$

is attained on a compact set: there exist $\bar{x} \in \Omega$ with $d(\bar{x}) \ge \ell$ and $\bar{h} \ne 0$ with $\delta_1 \le |\bar{h}| \le \frac{\ell}{16}$ such that

$$\frac{|\theta(\bar{x}+\bar{h},\tilde{t}) - \theta(\bar{x},\tilde{t})|^2}{|\bar{h}|^{2\alpha}} = s(\tilde{t}) > R. \tag{89}$$

At this maximum, from (86), we have that

$$\frac{d}{dt}\left(\frac{|\theta(\bar{x}+\bar{h},t) - \theta(\bar{x},t)|^2}{|\bar{h}|^{2\alpha}}\right)\Big|_{t=\tilde{t}} < 0$$

by using (89) and the fact that the terms in L_χ involving ∇_x and ∇_h vanish and Λ_D is nonnegative. Therefore there exists $t' < \tilde{t}$ such that $s(t') > s(\tilde{t})$. This implies that $\inf\{t > t_1 \mid s(t) > R\} = t_1$ which is absurd because we made sure that $s(t_1) < R$. Because δ and T are arbitrary, from (87) we arrive at

$$\sup_{d(x)\ge \ell, |h|\le \frac{\ell}{16}, t\ge 0} \frac{|\delta_h \theta(x)|^2}{|h|^{2\alpha}} \le \left[\sup_{d(x)\ge \ell, |h|\le \frac{\ell}{16}} \frac{|\delta_h \theta_0(x)|^2}{|h|^{2\alpha}} + \Gamma_1 \ell^{-2\alpha}\|\theta_0\|_{L^\infty}^2\right], \tag{90}$$

where Γ_1 does not depend on ℓ. For any fixed $x \in \Omega$ we may take ℓ such that $\ell \le d(x) \le 2\ell$. Then (90) implies

$$d(x)^{2\alpha}\frac{|\delta_h \theta(x,t)|^2}{|h|^{2\alpha}} \le \left[\|\theta_0\|_{C^\alpha}^2 + \Gamma_1 2^{2\alpha}\|\theta_0\|_{L^\infty}^2\right], \tag{91}$$

which completes the proof of the theorem.

2.2.3. *Uniform Gradient bounds.* Here we establish global interior bounds for the gradient $\nabla_\epsilon \theta$, which are uniform in ϵ.

THEOREM 6. *Let $\theta_\epsilon(x,t)$ be a solution of (5)–(6) in Ω. There exists a constant C which depends on Ω and $\|\theta_0\|_{W^{1,\infty}(\Omega)}$ such that*

$$\sup_{x\in\Omega,\, t\ge 0} d(x)|\nabla\theta_\epsilon(x,t)| \le C$$

holds uniformly in ϵ.

Proof of Theorem 6. We omit the subscript ϵ on the variables. Let χ be a good cutoff with scale $\ell > 0$. We apply the gradient operator to (5):

$$(\partial_t + u\cdot\nabla)\nabla\theta + (\nabla u)^*\nabla\theta + \nabla\Lambda_D\theta = 0 \tag{92}$$

where $(\nabla u)^*$ denotes the transposed matrix of ∇u. Then $g = \nabla\theta$ obeys

$$(\partial_t + u\cdot\nabla)g + \Lambda_D(\chi g) + C_\chi(\theta) + (\nabla u)^* g = 0 \tag{93}$$

in Ω, where $C_\chi(\theta) = \nabla\Lambda_D\theta - \Lambda_D\chi\nabla\theta$. We multiply (93) by g. Because $\chi(x) = 1$ for $d(x) \ge \ell$, we obtain

$$\frac{1}{2}L_\chi(g^2) + D(f) + gC_\chi(\theta) + g(\nabla u)^* g = 0 \tag{94}$$

in Ω with $d(x) \ge \ell$, where $f = \chi g$, $D(f) = f\Lambda_D f - \frac{1}{2}\Lambda_D\left(f^2\right)$ and

$$L_\chi(\phi) = \partial_t\phi + u\cdot\nabla\phi + \Lambda_D(\chi^2\phi). \tag{95}$$

We recall the commutator bound from [**6**]: there exists a constant Γ_3, independent of ℓ such that

$$|C_\chi(\theta)(x)| \le \frac{\Gamma_3}{d(x)^2}\|\theta\|_{L^\infty(\Omega)} \tag{96}$$

in Ω for $d(x) \geq \ell$. By (96) and (64) we obtain

$$\frac{1}{2}L_\chi(g^2) + D(f) \leq \frac{\Gamma_3}{d(x)^2}|g|\|\theta\|_{L^\infty(\Omega)} + C\left(\sqrt{\rho D(f)} + \|\theta\|_{L^\infty(\Omega)}\left(\frac{1}{d(x)} + \frac{1}{\rho}\right) + |\nabla\theta(x)|\right)g^2 \quad (97)$$

for $d(x) \geq \ell$ and $\rho \leq cd(x)$ to be determined below. Using the Cauchy-Schwarz inequality for the term involving $D(f)$ on the right side of (97), we get

$$L_\chi(g^2) + D(f) \leq \frac{2\Gamma_3}{d(x)^2}|g|\|\theta\|_{L^\infty(\Omega)} + C_4\rho g^4 + C_4\|\theta\|_{L^\infty(\Omega)}\left(\frac{1}{d(x)} + \frac{1}{\rho}\right)g^2 + C_4|g|^3 \quad (98)$$

for $d(x) \geq \ell$. Next, we use the super-cubic pointwise lower bound [**6**, Theorem 4] where $0 < \alpha < 1$ is fixed and given in Theorem 5:

$$D(f) \geq \gamma_2\|\theta\|_{C^\alpha(\Omega)}^{-\frac{1}{1-\alpha}}|g|^{3+\frac{\alpha}{1-\alpha}}d(x)^{\frac{\alpha}{1-\alpha}} + \frac{\gamma_1}{d(x)}g^2 \quad (99)$$

in Ω when $d(x) \geq \ell$ for $|f(x)| \geq M\|\theta\|_{L^\infty(\Omega)}d(x)^{-1}$. Observe that $|g| = |f|$ when $d(x) \geq \ell$. We distinguish between two cases. First, in the case $|g(x)| \geq M\|\theta\|_{L^\infty(\Omega)}d(x)^{-1}$, we choose

$$\rho^{-1} = C_5|g(x)|. \quad (100)$$

Then the right hand side of (98) becomes at most cubic in g:

$$L_\chi(g^2) + D(f) \leq K|g|^3, \quad (101)$$

where

$$K = \frac{2\Gamma_3}{M^2\|\theta_0\|_{L^\infty(\Omega)}} + C_4\left(\frac{1}{C_5} + \frac{1}{M} + C_5\|\theta\|_{L^\infty(\Omega)} + 1\right).$$

In view of the super-cubic bound (99), in this case we have that

$$L_\chi(g^2) + |g|^3\left(\gamma_2\left(\|\theta\|_{C^\alpha(\Omega)}^{-\frac{1}{\alpha}}|g|d(x)\right)^{\frac{\alpha}{1-\alpha}} - K\right) \leq 0 \quad (102)$$

holds for $d(x) \geq \ell$. Now, in the opposite case, $|g(x)| \leq M\|\theta\|_{L^\infty}d(x)^{-1}$, we choose

$$\rho(x) = d(x) \quad (103)$$

and obtain from (98)

$$L_\chi(g^2) + D(f) \leq \frac{K_1}{d(x)^3}, \quad (104)$$

where

$$K_1 = C_4M^4\|\theta\|_{L^\infty(\Omega)}^4 + C_4M^3\|\theta\|_{L^\infty(\Omega)}^3 + 2C_4M^2\|\theta\|_{L^\infty(\Omega)}^3 + 2M\Gamma_3\|\theta\|_{L^\infty(\Omega)}^2. \quad (105)$$

By the convex damping inequality [**6**, Theorem 4]

$$D(f) \geq \frac{\gamma_1}{d(x)}g^2,$$

we obtain in this case

$$L_\chi(g^2) + \frac{1}{d(x)}\left(\gamma_1 g^2(x) - \frac{K_1}{d(x)^2}\right) \leq 0. \quad (106)$$

Putting together (65), (102) and (106), we obtain uniformly in ϵ

$$\sup_{d(x)\geq\ell}|\nabla\theta(x,t)| \leq C\left[\|\nabla\theta_0\|_{L^\infty(\Omega)} + \frac{P(\|\theta\|_{L^\infty(\Omega)})}{\ell}\right] \quad (107)$$

where $P(\|\theta\|_{L^\infty(\Omega)})$ is a polynomial of degree four. The proof is completed by choosing ℓ depending on x, because the constants in (107) do not depend on ℓ.

3. Proof of Theorem 2

Let $T > 0$ and let $\epsilon_n \to 0$. We denote $\theta_n = \theta_{\epsilon_n}$ the solutions to the corresponding ϵ_n-approximate equations

$$\partial_t \theta_n + u_n \cdot \nabla \theta_n + \Lambda_D \theta_n = 0 \tag{108}$$

where $u_n = \nabla^\perp \psi_n = \nabla^\perp \int_{\epsilon_n}^\infty t^{-\frac{1}{2}} e^{t\Delta} \theta_n dt$ with initial data $\theta_n(0) = \theta_0$ in $L^2(\Omega)$. From the energy inequality

$$\frac{1}{2}\|\theta_n(T)\|_{L^2}^2 + \int_0^T \|\Lambda_D^{\frac{1}{2}} \theta_n\|_{L^2}^2 dt \le \frac{1}{2}\|\theta_0\|_{L^2}^2 \tag{109}$$

it follows that θ_n are uniformly bounded in $L^\infty(0,T; L^2(\Omega))$ and $L^2(0,T;\mathcal{D}(\Lambda_D^{\frac{1}{2}}))$. Clearly

$$\|u_n\|_{L^2} = \|\nabla^\perp \psi_n\|_{L^2} = \|\Lambda_D \psi_n\|_{L^2} = \|\theta_n\|_{L^2} \le \|\theta_0\|_{L^2}$$

for all $t \in [0,T]$, hence $u_n\theta_n$ are uniformly bounded in $L^\infty(0,T; L^2(\Omega))$. Then both $u_n \cdot \nabla \theta_n = \nabla(u_n \theta_n)$ and $\Lambda_D \theta_n$ are uniformly bounded in $L^\infty(0,T; H^{-1}(\Omega))$. Using the equation (108), we obtain that $\partial_t \theta_n$ are uniformly bounded in $L^\infty([0,T]; H^{-1}(\Omega))$. We then apply an Aubin-Lions lemma based on L^2 in time, and with spaces $\mathcal{D}(\Lambda_D^{\frac{1}{2}}) \subset\subset L^2(\Omega) \subset H^{-1}(\Omega)$. Therefore, there exists a subsequence, denoted also by θ_n, converging to some function θ strongly in $L^2([0,T] \times \Omega)$ and weekly in $L^2(0,T;\mathcal{D}(\Lambda_D^{\frac{1}{2}}))$. As a consequence, using that $\nabla^\perp \Lambda_D^{-1}$ is a bounded linear operator, u_n converges strongly to $u = \nabla^\perp \int_0^\infty t^{-\frac{1}{2}} e^{t\Delta} \theta dt$ in $L^2([0,T] \times \Omega)$. From here the product $u_n\theta_n$ converges strongly to $u\theta$ in $L^1([0,T] \times \Omega)$, which implies that the nonlinear terms $u_n \cdot \nabla \theta_n$ converge to $u \cdot \nabla \theta$ in the sense of distributions, in $D'([0,T] \times \Omega)$. This establishes that θ is a weak solution to the critical SQG. More precisely, θ satisfies the weak formulation

$$\int_0^\infty \int_\Omega \theta \partial_t \phi \, dxdt + \int_0^\infty \int_\Omega u\theta \cdot \nabla \phi \, dxdt - \int_0^\infty \int_\Omega \Lambda_D^{\frac{1}{2}} \theta \Lambda_D^{\frac{1}{2}} \phi \, dxdt = 0$$

for all $\phi \in C_0^\infty([0,T] \times \Omega)$. Moreover, θ obeys the energy inequality

$$\frac{1}{2}\|\theta(T)\|_{L^2}^2 + \int_0^T \|\Lambda_D^{\frac{1}{2}} \theta\|_{L^2}^2 dt \le \frac{1}{2}\|\theta_0\|_{L^2}^2. \tag{110}$$

Now, assume that $\theta_0 \in L^\infty(\Omega)$ and $\sup_{x\in\Omega} d(x)|\nabla_x \theta_0(x)| \le C$. By Theorem 1 we know that θ_n are uniformly bounded in $L^\infty(0,T; L^\infty(\Omega))$ and $\sup_{x\in\Omega} d(x)|\nabla \theta_n(x)| \le C$ for all $t \in [0,T]$. Let $\varphi \in C_0^\infty(\Omega)$ with $\int_\Omega |\varphi| dx \le 1$. There exists $\delta > 0$ and a subdomain $\Omega_\delta \subset \Omega$ with $d(x) > \delta$ for any $x \in \Omega_\delta$ such that $\operatorname{supp} \varphi \subset \Omega_\delta$. Integrating by parts and using the strong convergence of θ_n to θ in $L^2([0,T] \times \Omega)$, we have

$$\begin{aligned}&\int_\Omega d(x)\partial_i \theta(x,t)\varphi(x) dx = -\int_{\Omega_\delta} \theta(x,t)\partial_i(d(x)\varphi(x)) dx = \lim_{n\to\infty} -\int_{\Omega_\delta} \theta_n(x,t)\partial_i(d(x)\varphi(x)) dx \\ &= \lim_{n\to\infty} \int_\Omega d(x)\partial_i \theta_n(x,t)\varphi(x) dx\end{aligned}$$

for $i = 1,2$, because the function $d(x)$ is Lipschitz continuous and $\nabla(d(x)\varphi(x))$ is bounded on Ω. Then,

$$\sup_{\varphi \in C_0^\infty(\Omega), \|\varphi\|_{L^1} \le 1} \left| \int_\Omega d(x)\partial_i \theta(x,t)\varphi(x) dx \right| \le C,$$

and by the density of $C_0^\infty(\Omega)$ in $L^1(\Omega)$ and the duality of $L^\infty(\Omega)$ and $L^1(\Omega)$, we get $\sup_{x\in\Omega} d(x)|\nabla \theta(x,t)| \le C$ for all $t \in [0,T]$. Using a similar argument, $\sup_{x\in\Omega} |\theta(x,t)| \le C$ for all $t \in [0,T]$. This concludes the proof of the theorem.

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DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PA 19122
E-mail address: ignatova@temple.edu