ON THE SPACE ANALYTICITY OF THE NERNST-PLANCK-NAVIER-STOKES SYSTEM

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ABSTRACT. We consider the forced Nernst-Planck-Navier-Stokes system for n ionic species with different diffusivities and valences. We prove the local existence of analytic solutions with periodic boundary conditions in two and three dimensions. In the case of two spatial dimensions, the local solution extends uniquely and remains analytic on any time interval $[0,T]$. In the three dimensional case, we give necessary and sufficient conditions for the global in time existence of analytic solutions. These conditions involve quantitatively only low regularity norms of the fluid velocity and concentrations.

1. INTRODUCTION

We consider an electrodiffusion model describing the evolution of n ionic species in a d-dimensional fluid. The evolution of each ionic concentration c_i, $i \in \{1,\dots,n\}$ is described according to a Nernst-Planck equation

$$(\partial_t + u\cdot\nabla)c_i = D_i \mathrm{div}(\nabla c_i + z_i c_i \nabla\Phi) \tag{1}$$

where z_i are the valences of the ionic species and the constants $D_i > 0$ denote the diffusivity of the ions. The potential Φ satisfies the Poisson equation

$$-\epsilon\Delta\Phi = \rho + N \tag{2}$$

where

$$\rho = \sum_{i=1}^{n} z_i c_i, \tag{3}$$

ϵ is a positive constant proportional to the square of the Debye length, and N is an added smooth, time independent charge density. The velocity u evolves according to the Navier-Stokes equations

$$\partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla p = -(\rho + N)\nabla\Phi + f \tag{4}$$

with the divergence free condition

$$\nabla\cdot u = 0. \tag{5}$$

Here, p represents the pressure of the fluid, ν is a positive constant denoting the kinematic viscosity, and f is a time independent, smooth and divergence free body force in the fluid. In this paper, the Nernst-Planck-Navier-Stokes (NPNS) system (1)–(5) is considered in the d-dimensional torus $\mathbb{T}^d = [0,2\pi]^d$ with periodic boundary conditions.

Global existence of weak solutions for the NPNS system in two and three dimensions has been shown for homogeneous Neumann boundary conditions in [8] and for homogeneous Dirichlet boundary conditions in [7]. The most important physical applications of the system involve inhomogeneous boundary conditions. In this regard, global existence of smooth solutions for the NPNS system has been proved in [2] for blocking and uniform selective boundaries in two dimensional domains. Blocking boundary conditions require the vanishing of normal fluxes for the concentrations, and impose inhomogeneous Dirichlet boundary conditions for the electric potential. The selective boundary conditions are inhomogeneous Dirichlet boundary conditions relating the electrical potential to the concentrations. Boltzmann states are certain steady states of the concentrations with vanishing solvent velocity. Their nonlinear stability has been obtained in [4] for blocking and uniform selective boundary conditions in three dimensions. Global existence and regularity of solutions has been obtained in [3] for general selective boundary conditions in three dimensions for the case of two ionic species and for the case of many ionic species having the same diffusivities. The asymptotic interior electroneutrality of the system in two and three dimensions in the stable cases of blocking and uniform selective boundary conditions was established in [5]. This refers to the fact that the charge density vanishes away from boundaries, in the long time limit, in the limit of small Debye screening length.

The difficulties of analysis of the NPNS system are due to nontrivial boundary effects and the intrinsic nonlinear nature of the equations. The study of the system with periodic boundary conditions focuses on the nonlinear aspects only. In [1], the NPNS system has been investigated on the two dimensional torus for two ionic species with equal

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diffusivities and opposite valences. It has been shown that global smooth solutions exist for sufficiently regular initial data. It has also been shown in [1] that a finite dimensional global attractor exists and is a singleton in the absence of forcing ($f = N = 0$). In the present paper we examine further the regularity of the system in the absence of boundary effects. We show that the solutions are in fact analytic.

This paper is organized as follows. In section 2, we prove the local existence of analytic solutions in the two dimensional and three dimensional spatially periodic cases, for any initial data in $L^p(\mathbb{T}^d)$ with $p > d$. The proof uses complexification and progressive energy estimates on wedge shaped domains and is inspired by the approach of [6]. In section 3, we consider the two dimensional case and we show that L^2 initial data lead to unique local weak solutions. We then show that this local solution can be extended to a strong analytic solution on $[0, T]$ for any $T > 0$. The proof is based on a sufficient condition, expressed in terms of L^2 norms of solutions (namely that their L^3 in time norm be finite). This sufficient condition guarantees that the solution can be uniquely extended, and remains analytic. The sufficient condition is satisfied, the concentrations are proved to have L^2 norms that are actually bounded in time. The proof of this fact is presented in the Appendix. In section 4, we show that the analyticity of the unique local solution on the three dimensional torus can be extended to any time interval $[0, T]$, provided that the solution $(u, c_1, \dots, c_n)$ of the NPNS system (1)–(5) satisfies the regularity condition

$$\int_0^T \left(\|\nabla u(t)\|_{L^2}^4 + \|c_1(t)\|_{L^2}^4 + \dots + \|c_n(t)\|_{L^2}^4 \right) dt < \infty. \tag{6}$$

This condition is natural in view of the fact that the system comprises the three dimensional Navier-Stokes equations. But even if the Navier-Stokes equations are replaced by the Stokes equations, driven by the electrical forces, the condition regarding the concentrations is not known to be always satisfied in 3D.

2. Existence of a Local Analytic Solution

We consider the system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u - (\rho + N) \nabla \Phi + f \\ \nabla \cdot u = 0 \\ \rho = z_1 c_1 + \dots + z_n c_n \\ -\epsilon \Delta \Phi = \rho + N \\ \partial_t c_i + u \cdot \nabla c_i = D_i \Delta c_i + D_i \nabla \cdot (z_i c_i \nabla \Phi), \quad i = 1, \dots, n \end{cases} \tag{7}$$

in $\mathbb{T}^d \times [0, \infty)$, where $d \in \{2, 3\}$. The body forces f are smooth, divergence-free, time independent, and have mean zero. The added charge density N is smooth and time independent. We assume that the initial fluid velocity $u(x, 0)$ and the initial charge density $\rho(x, 0) + N(x)$ have zero space averages. We also assume that $u(x, 0)$ is divergence-free.

Let $L^p = L^p(\mathbb{T}^d)$ be the space of 2π-periodic functions with the norm

$$\|f\|_{L^p} = \left(\int_{\mathbb{T}^d} |f(x)|^p dx \right)^{1/p} \tag{8}$$

for $p \in [1, \infty)$ with the usual convention when $p = \infty$.

Theorem 1. *(Local existence of an analytic solution in 2D and 3D) Let $d \in \{2, 3\}$. Let $u(x, 0) = u_0(x)$ and $c_i(x, 0) = c_i^0(x)$ for $i \in \{1, \dots, n\}$. Assume that the initial data u_0, c_i^0 are in $L^p(\mathbb{T}^d)$ with $p > d$, and denote*

$$\|u_0\|_{L^p} + \|c_1^0\|_{L^p} + \dots + \|c_n^0\|_{L^p} = M_p < \infty. \tag{9}$$

Assume that f and N are real analytic with radius of analyticity larger than or equal to $\delta > 0$. Let $f + ig$ and $N + iM$ be their analytic extensions. Then, there exists a positive time $T_0 > 0$ and a number $V > 0$ depending on p, M_p, f, N and the parameters of the problem, and a unique solution $(u, c_1, \dots, c_n) \in C([0, T_0), L^p)$ of the NPNS system (7) such that for every $t \in (0, T_0)$, $(u, c_1, \dots, c_n)$ is the restriction of the analytic function $(u + iv, c_1 + id_1, \dots, c_n + id_n)$ in the region $\mathcal{D}_t$ defined by

$$\mathcal{D}_t = \{z = (x + iy) \in \mathbb{C} \mid \ |y| < Vt\}, \quad \text{for } 0 < t < T_0. \tag{10}$$

Moreover,

$$\|u(\cdot, y, t)\|_{L^p} + \|v(\cdot, y, t)\|_{L^p} + \sum_{i=1}^n \{\|c_i(\cdot, y, t)\|_{L^p} + \|d_i(\cdot, y, t)\|_{L^p}\} \le C M_p \tag{11}$$

for $t \in (0, T_0)$ and $(x, y) \in \mathcal{D}_t$.

Proof. For simplicity of exposition we take $\nu = \epsilon = D_i = 1$ for $i \in \{1, \dots, n\}$. Let

$$u^{(0)} = p^{(0)} = c_1^{(0)} = \cdots = c_n^{(0)} = 0. \tag{12}$$

We construct sequences $u^{(m)}, p^{(m)}, c_1^{(m)}, \dots, c_n^{(m)}$ in $C([0,\infty), L^p))$ such that

$$\partial_t u^{(m)} - \Delta u^{(m)} = -(u^{(m-1)} \cdot \nabla) u^{(m-1)} - \nabla p^{(m-1)} - \rho^{(m-1)} \nabla \Phi^{(m-1)} - N \nabla \Phi^{(m-1)} + f, \tag{13}$$

$$\Delta p^{(m)} = -\sum_{1 \le j,k \le d} \partial_j \partial_k (u_j^{(m)} u_k^{(m)}) - \nabla \cdot ((\rho^{(m)} + N) \nabla \Phi^{(m)}), \tag{14}$$

$$\partial_t c_i^{(m)} - \Delta c_i^{(m)} = -(u^{(m-1)} \cdot \nabla) c_i^{(m-1)} + \nabla \cdot (z_i c_i^{(m-1)} \nabla \Phi^{(m-1)}) \tag{15}$$

for $1 \le i \le n$, and

$$-\Delta \Phi^{(m)} = \rho^{(m)} + N \tag{16}$$

with the initial conditions

$$u^{(m)}(x, 0) = u_0(x), \tag{17}$$

and

$$c_i^{(m)}(x, 0) = c_i^0(x) \tag{18}$$

for $1 \le i \le n$. The constructed sequences are real analytic with radius of analyticity at least δ for all $t > 0$. This follows by induction from the fact that the sequences are solutions of the heat and Laplace equations.

Let $u^{(m)} + iv^{(m)}$, $p^{(m)} + i\pi^{(m)}$, $c_i^{(m)} + id_i^{(m)}$, $\rho^{(m)} + i\xi^{(m)}$ and $\Phi^{(m)} + i\phi^{(m)}$ be the analytic extensions. Then,

$$\begin{aligned} \partial_t u^{(m)} - \Delta u^{(m)} = {} & -(u^{(m-1)} \cdot \nabla) u^{(m-1)} + (v^{(m-1)} \cdot \nabla) v^{(m-1)} - \nabla p^{(m-1)} - \rho^{(m-1)} \nabla \Phi^{(m-1)} \\ & + \xi^{(m-1)} \nabla \phi^{(m-1)} - N \nabla \Phi^{(m-1)} + M \nabla \phi^{(m-1)} + f, \end{aligned} \tag{19}$$

$$\begin{aligned} \partial_t v^{(m)} - \Delta v^{(m)} = {} & -(u^{(m-1)} \cdot \nabla) v^{(m-1)} - (v^{(m-1)} \cdot \nabla) u^{(m-1)} - \nabla \pi^{(m-1)} - \rho^{(m-1)} \nabla \phi^{(m-1)} \\ & - \xi^{(m-1)} \nabla \Phi^{(m-1)} - N \nabla \phi^{(m-1)} - M \nabla \Phi^{(m-1)} + g, \end{aligned} \tag{20}$$

$$\Delta p^{(m)} = -\sum_{1 \le j,k \le d} \left\{ \partial_{jk} (u_j^{(m)} u_k^{(m)} - v_j^{(m)} v_k^{(m)}) \right\} - \nabla \cdot ((\rho^{(m)} + N) \nabla \Phi^{(m)} - (\xi^{(m)} + M) \nabla \phi^{(m)}) \tag{21}$$

$$\Delta \pi^{(m)} = -2 \sum_{1 \le j,k \le d} \partial_{jk} (u_j^{(m)} v_k^{(m)}) - \nabla \cdot ((\xi^{(m)} + M) \nabla \Phi^{(m)} + (\rho^{(m)} + N) \nabla \phi^{(m)}), \tag{22}$$

$$\begin{aligned} \partial_t c_i^{(m)} - \Delta c_i^{(m)} = {} & -(u^{(m-1)} \cdot \nabla) c_i^{(m-1)} + (v^{(m-1)} \cdot \nabla) d_i^{(m-1)} \\ & + \nabla \cdot (z_i c_i^{(m-1)} \nabla \Phi^{(m-1)} - z_i d_i^{(m-1)} \nabla \phi^{(m-1)}), \end{aligned} \tag{23}$$

$$\begin{aligned} \partial_t d_i^{(m)} - \Delta d_i^{(m)} = {} & -(v^{(m-1)} \cdot \nabla) c_i^{(m-1)} - (u^{(m-1)} \cdot \nabla) d_i^{(m-1)} \\ & + \nabla \cdot (z_i c_i^{(m-1)} \nabla \phi^{(m-1)} + z_i d_i^{(m-1)} \nabla \Phi^{(m-1)}), \end{aligned} \tag{24}$$

$$-\Delta \Phi^{(m)} = \rho^{(m)} + N, \tag{25}$$

$$-\Delta \phi^{(m)} = \xi^{(m)} + M. \tag{26}$$

The idea of the proof is based on [6]. Let

$$\begin{aligned} & \widetilde{u}_\alpha^{(m)}(x,t) = u^{(m)}(x, \alpha t, t),\ \widetilde{v}_\alpha^{(m)}(x,t) = v^{(m)}(x, \alpha t, t),\ \widetilde{p}_\alpha^{(m)}(x,t) = p^{(m)}(x, \alpha t, t), \\ & \widetilde{\pi}_\alpha^{(m)}(x,t) = \pi^{(m)}(x, \alpha t, t),\ \widetilde{\rho}_\alpha^{(m)}(x,t) = \rho^{(m)}(x, \alpha t, t),\ \widetilde{\xi}_\alpha^{(m)}(x,t) = \xi^{(m)}(x, \alpha t, t), \\ & \widetilde{c}_{i,\alpha}^{(m)}(x,t) = c_i^{(m)}(x, \alpha t, t),\ \widetilde{d}_{i,\alpha}^{(m)}(x,t) = d_i^{(m)}(x, \alpha t, t),\ \widetilde{\Phi}_\alpha^{(m)}(x,t) = \Phi^{(m)}(x, \alpha t, t), \\ & \widetilde{\phi}_\alpha^{(m)}(x,t) = \phi^{(m)}(x, \alpha t, t),\ \widetilde{f}_\alpha(x,t) = f(x, \alpha t),\ \widetilde{g}_\alpha(x,t) = g(x, \alpha t), \\ & \widetilde{N}_\alpha(x,t) = N(x, \alpha t),\ \widetilde{M}_\alpha(x,t) = M(x, \alpha t) \end{aligned} \tag{27}$$

We drop the α's to simplify the notation, and we denote the partial derivative $\frac{\partial}{\partial x_j}$ by ∂_j. By the chain rule and the Cauchy-Riemann equations, we have

$$\partial_t \widetilde{u}^{(m)} - \Delta \widetilde{u}^{(m)} = -\sum_{j=1}^d \alpha_j \partial_j \widetilde{v}^{(m)} - (\widetilde{u}^{(m-1)} \cdot \nabla)\widetilde{u}^{(m-1)} + (\widetilde{v}^{(m-1)} \cdot \nabla)\widetilde{v}^{(m-1)} - \nabla \widetilde{p}^{(m-1)} - \widetilde{\rho}^{(m-1)} \nabla \widetilde{\Phi}^{(m-1)} + \widetilde{\xi}^{(m-1)} \nabla \widetilde{\phi}^{(m-1)} - \widetilde{N} \nabla \widetilde{\Phi}^{(m-1)} + \widetilde{M} \nabla \widetilde{\phi}^{(m-1)} + \widetilde{f}, \tag{28}$$

$$\partial_t \widetilde{v}^{(m)} - \Delta \widetilde{v}^{(m)} = \sum_{j=1}^d \alpha_j \partial_j \widetilde{u}^{(m)} - (\widetilde{u}^{(m-1)} \cdot \nabla)\widetilde{v}^{(m-1)} - (\widetilde{v}^{(m-1)} \cdot \nabla)\widetilde{u}^{(m-1)} - \nabla \widetilde{\pi}^{(m-1)} - \widetilde{\rho}^{(m-1)} \nabla \widetilde{\phi}^{(m-1)} - \widetilde{\xi}^{(m-1)} \nabla \widetilde{\Phi}^{(m-1)} - \widetilde{N} \nabla \widetilde{\phi}^{(m-1)} - \widetilde{M} \nabla \widetilde{\Phi}^{(m-1)} + \widetilde{g}, \tag{29}$$

$$\Delta \widetilde{p}^{(m)} = -\sum_{1 \le j,k \le d} \partial_{jk} (\widetilde{u}_j^{(m)} \widetilde{u}_k^{(m)} - \widetilde{v_j}^{(m)} \widetilde{v}_k^{(m)}) - \nabla \cdot ((\widetilde{\rho}^{(m)} + \widetilde{N}) \nabla \widetilde{\Phi}^{(m)} - (\widetilde{\xi}^{(m)} + \widetilde{M}) \nabla \widetilde{\phi}^{(m)}), \tag{30}$$

$$\Delta \widetilde{\pi}^{(m)} = -2 \sum_{1 \le j,k \le d} \partial_{jk} \widetilde{u}_j^{(m)} \widetilde{v}_k^{(m)} - \nabla \cdot ((\widetilde{\xi}^{(m)} + \widetilde{M}) \nabla \widetilde{\Phi}^{(m)} + (\widetilde{\rho}^{(m)} + \widetilde{N}) \nabla \widetilde{\phi}^{(m)}), \tag{31}$$

$$\partial_t \widetilde{c}_i^{(m)} - \Delta \widetilde{c}_i^{(m)} = -\sum_{j=1}^d \alpha_j \partial_j \widetilde{d}_i^{(m)} - (\widetilde{u}^{(m-1)} \cdot \nabla)\widetilde{c}_i^{(m-1)} + (\widetilde{v}^{(m-1)} \cdot \nabla)\widetilde{d}_i^{(m-1)} + \nabla \cdot (z_i \widetilde{c}_i^{(m-1)} \nabla \widetilde{\Phi}^{(m-1)} - z_i \widetilde{d}_i^{(m-1)} \nabla \widetilde{\phi}^{(m-1)}), \tag{32}$$

$$\partial_t \widetilde{d}_i^{(m)} - \Delta \widetilde{d}_i^{(m)} = \sum_{j=1}^d \alpha_j \partial_j \widetilde{c}_i^{(m)} - (\widetilde{v}^{(m-1)} \cdot \nabla)\widetilde{c}_i^{(m-1)} - (\widetilde{u}^{(m-1)} \cdot \nabla)\widetilde{d}_i^{(m-1)} + \nabla \cdot (z_i \widetilde{c}_i^{(m-1)} \nabla \widetilde{\phi}^{(m-1)} + z_i \widetilde{d}_i^{(m-1)} \nabla \widetilde{\Phi}^{(m-1)}), \tag{33}$$

$$-\Delta \widetilde{\Phi}^{(m)} = \widetilde{\rho}^{(m)} + \widetilde{N}, \tag{34}$$

$$-\Delta \widetilde{\phi}^{(m)} = \widetilde{\xi}^{(m)} + \widetilde{M}. \tag{35}$$

The initial conditions are

$$\widetilde{u}^{(m)}(x,0) = u_0(x),\ \widetilde{v}^{(m)}(x,0) = 0,\ \widetilde{c}_i^{(m)}(x,0) = c_i^0(x),\ \widetilde{d}_i^{(m)}(x,0) = d_i^0(x). \tag{36}$$

Let

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \sum_{k \in \mathbb{Z}^d} \exp\left(-\frac{|x-k|^2}{4t}\right), \quad x \in \mathbb{T}^d \tag{37}$$

be the fundamental solution of the d-dimensional heat equation with periodic boundary conditions. Then,

$$\begin{aligned} \widetilde{u}^{(m)}(x,t) &= \int \Gamma(x-w,t) u_0(w)\, dw \\ &- \int_0^t \int \sum_{j=1}^d \left\{ \partial_j \Gamma(x-w,t-s) \left(\alpha_j \widetilde{v}^{(m)} + \widetilde{u}_j^{(m-1)} \widetilde{u}^{(m-1)} - \widetilde{v}_j^{(m-1)} \widetilde{v}^{(m-1)} \right)(w,s) \right\} dwds \\ &- \int_0^t \int \left\{ \Gamma(x-w,t-s) \left(\nabla \widetilde{p}^{(m-1)} + \widetilde{\rho}^{(m-1)} \nabla \widetilde{\Phi}^{(m-1)} - \widetilde{\xi}^{(m-1)} \nabla \widetilde{\phi}^{(m-1)} \right)(w,s) \right\} dwds \\ &- \int_0^t \int \left\{ \Gamma(x-w,t-s) \left(\widetilde{N} \nabla \widetilde{\Phi}^{(m-1)} - \widetilde{M} \nabla \widetilde{\phi}^{(m-1)} - \widetilde{f} \right)(w,s) \right\} dwds, \end{aligned} \tag{38}$$

$$\widetilde{v}^{(m)}(x,t) = \int_0^t \int \sum_{j=1}^d \Big\{\partial_j\Gamma(x-w,t-s)\left(\alpha_j\widetilde{u}^{(m)} - \widetilde{v}_j^{(m-1)}\widetilde{u}^{(m-1)} - \widetilde{u}_j^{(m-1)}\widetilde{v}^{(m-1)}\right)(w,s)\Big\}\,dwds$$
$$- \int_0^t \int \Big\{\Gamma(x-w,t-s)\left(\nabla\widetilde{\pi}^{(m-1)} + \widetilde{\rho}^{(m-1)}\nabla\widetilde{\phi}^{(m-1)} + \widetilde{\xi}^{(m-1)}\nabla\widetilde{\Phi}^{(m-1)}\right)(w,s)\Big\}\,dwds$$
$$- \int_0^t \int \Big\{\Gamma(x-w,t-s)\left(\widetilde{N}\nabla\widetilde{\phi}^{(m-1)} + \widetilde{M}\nabla\widetilde{\Phi}^{(m-1)} - \widetilde{g}\right)(w,s)\Big\}\,dwds, \tag{39}$$

$$\widetilde{c}_i^{(m)}(x,t) = \int \Gamma(x-w,t)c_i^0(w)\,dw$$
$$- \int_0^t \int \Big\{\sum_{j=1}^d \partial_j\Gamma(x-w,t-s)\left(\alpha_j\widetilde{d}_i^{(m)} + \widetilde{u}_j^{(m-1)}\widetilde{c}_i^{(m-1)} - \widetilde{v}_j^{(m-1)}\widetilde{d}_i^{(m-1)}\right)(w,s)\Big\}\,dwds$$
$$+ \int_0^t \int \Big\{\nabla\Gamma(x-w,t-s)\cdot z_i\left(\widetilde{c}_i^{(m-1)}\nabla\widetilde{\Phi}^{(m-1)} - \widetilde{d}_i^{(m-1)}\nabla\widetilde{\phi}^{(m-1)}\right)(w,s)\Big\}\,dwds, \tag{40}$$

and

$$\widetilde{d}_i^{(m)}(x,t) = \int_0^t \int \Big\{\sum_{j=1}^d \partial_j\Gamma(x-w,t-s)\left(\alpha_j\widetilde{c}_i^{(m)} - \widetilde{v}_j^{(m-1)}\widetilde{c}_i^{(m-1)} - \widetilde{u}_j^{(m-1)}\widetilde{d}_i^{(m-1)}\right)(w,s)\Big\}\,dwds$$
$$+ \int_0^t \int \Big\{\nabla\Gamma(x-w,t-s)\cdot z_i\left(\widetilde{c}_i^{(m-1)}\nabla\widetilde{\phi}^{(m-1)} + \widetilde{d}_i^{(m-1)}\nabla\widetilde{\Phi}^{(m-1)}\right)(w,s)\Big\}\,dwds, \tag{41}$$

We denote

$$\|u\|_{L^{p,q}} = \left(\int_0^T \|u(\cdot,t)\|_{L^p}^q\right)^{1/q} \tag{42}$$

when $q \in [1,\infty)$ and

$$\|u\|_{L^{p,\infty}} = \sup_{0\le t\le T}\|u(\cdot,t)\|_{L^p}. \tag{43}$$

We recall well-known bounds on the Gaussian [6]:

Lemma 1. *Let*

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{\frac{d}{2}}}\sum_{k\in\mathbb{Z}^d}\exp\left(-\frac{|x-k|^2}{4t}\right), \quad x\in\mathbb{T}^d \tag{44}$$

be the fundamental solution of the d-dimensional heat equation with periodic boundary conditions. Then there is a constant $C>0$ depending on d such that

(i) $\Gamma(x,t) \le Ct^{-\frac{d}{2}}e^{-\frac{|x|^2}{4t}}$ *for* $x\in\mathbb{T}^d$ *and* $0<t\le 1$.
(ii) $\Gamma(x,t)\le C$ *for* $x\in\mathbb{T}^d$ *and* $t\ge 1$.
(iii) $\|\Gamma(\cdot,t)\|_{L^1}\le C$ *for* $t>0$,
(iv) $\|\nabla\Gamma\|_{L^{q,1}(S_T)} \le C_q(T^{\frac{q+d-dq}{2q}} + T^{\frac{1}{2}})$ *for* $1\le q<d/(d-1)$, *where* $S_T = \mathbb{T}^d\times[0,T], T>0$.

We use the following two lemmas:

Lemma 2. *There is a positive constant C depending on d such that*

$$\|\Gamma(\cdot,t)\|_{L^2} \le \frac{C}{t^{d/4}} + C \tag{45}$$

holds for all $t>0$.

Proof. By Lemma 1, for $t \ge 1$,

$$\|\Gamma(\cdot,t)\|_{L^2} \le C, \tag{46}$$

and for $0 < t < 1$,

$$\|\Gamma(\cdot,t)\|_{L^2} \le C\left(\int \frac{1}{t^d} e^{-\frac{|x|^2}{4t}} dx\right)^{1/2} = \frac{C}{t^{d/4}}\left(\int \frac{1}{t^{d/2}} e^{-\frac{|x|^2}{4t}} dx\right)^{1/2} \le \frac{C}{t^{d/4}}. \tag{47}$$

□

Lemma 3. *Let $t > 0$. Let $d \in \{2,3\}$. Let $p > d$ (and so $1 < p/(p-1) < d/(d-1)$). Then*

$$\begin{aligned}\int_0^t \left\{\int \left(\int \Gamma(x-y,t-s)\nabla\widetilde{p}^{(m)}(y)dy\right)^p dx\right\}^{1/p} ds &\le c_p\left(t^{(q+d-dq)/2q} + t^{1/2}\right)\left(\|\widetilde{u}^{(m)}\|^2_{L^{p,\infty}} + \|\widetilde{v}^{(m)}\|^2_{L^{p,\infty}}\right)\\ &+ c(t^{1-d/4}+t)(\|\widetilde{\rho}^{(m)}\|^2_{L^{p,\infty}} + \|\widetilde{N}\|^2_{L^{p,\infty}}) + c(t^{1-d/4}+t)(\|\widetilde{\xi}^{(m)}\|^2_{L^{p,\infty}} + \|\widetilde{M}\|^2_{L^{p,\infty}})\end{aligned} \tag{48}$$

where $q = p/(p-1)$, c is a constant depending on the dimension d, and c_p is a constant depending on p and the dimension d.

Proof. Let $\mathcal{N}$ be the Newtonian potential solving the Laplace equation with periodic boundary conditions. For each $i \in \{1,\dots,d\}$, we have

$$\begin{aligned}&\int \Gamma(x-y,t-s)\partial_{y_i}\widetilde{p}^{(m)}(y,s)dy = \int \partial_{y_i}\Gamma(x-y,t-s)\widetilde{p}^{(m)}(y)dy\\ &= -\int \partial_{y_i}\Gamma(x-y,t-s)\int \mathcal{N}(y-z)\partial_{jk}(\widetilde{u}_j^{(m)}\widetilde{u}_k^{(m)} - \widetilde{v}_j^{(m)}\widetilde{v}_k^{(m)})(z,s)dzdy\\ &- \int \partial_{y_i}\Gamma(x-y,t-s)\int \mathcal{N}(y-z)\nabla\cdot((\widetilde{\rho}^{(m)}+\widetilde{N})\nabla\widetilde{\Phi}^{(m)})(z,s)dzdy\\ &- \int \partial_{y_i}\Gamma(x-y,t-s)\int \mathcal{N}(y-z)\nabla\cdot((\widetilde{\xi}^{(m)}+\widetilde{M})\nabla\widetilde{\phi}^{(m)})(z,s)dzdy = A+B+C.\end{aligned} \tag{49}$$

Since $\partial_{jk}\mathcal{N}$ is a Calderon-Zygmund kernel, we estimate

$$\begin{aligned}\|A\|_{L^p} &\le \|\nabla\Gamma(\cdot,t-s)\|_{L^q}(\|\widetilde{u}^{(m)}(\cdot,s)\|^2_{L^p} + \|\widetilde{v}^{(m)}(\cdot,s)\|^2_{L^p})\\ &\le \|\nabla\Gamma(\cdot,t-s)\|_{L^q}(\|\widetilde{u}^{(m)}\|^2_{L^{p,\infty}} + \|\widetilde{v}^{(m)}\|^2_{L^{p,\infty}})\end{aligned} \tag{50}$$

in view of Young's convolution inequality with exponents $q = p/(p-1)$, p and p. We write

$$\begin{aligned}|B| &= \left|\int \partial_{y_i}\Gamma(x-y,t-s)\int \mathcal{N}(y-z)\nabla\cdot((\widetilde{\rho}^{(m)}+\widetilde{N})\nabla\widetilde{\Phi}^{(m)})(z,s)dzdy\right|\\ &= \left|\sum_{k=1}^n \int (\widetilde{\rho}^{(m)}+\widetilde{N})\partial_{z_k}\widetilde{\Phi}^{(m)}(z,s)\left(\int \partial_{z_k}\mathcal{N}(y-z)\partial_{y_i}\Gamma(x-y,t-s)dy\right)dz\right|,\end{aligned} \tag{51}$$

where

$$\begin{aligned}&\int \partial_{z_k}\mathcal{N}(y-z)\partial_{y_i}\Gamma(x-y,t-s)dy = -\int \partial_{y_k}\mathcal{N}(y-z)\partial_{y_i}\Gamma(x-y,t-s)dy\\ &= \int \partial_{y_iy_k}\mathcal{N}(y-z)\Gamma(x-y,t-s)dy = \int \partial_{y_iy_k}\mathcal{N}(x-z-Y)\Gamma(Y,t-s)dY\\ &= (\partial_{y_iy_k}\mathcal{N} * \Gamma(\cdot,t-s))(x-z)\end{aligned} \tag{52}$$

and $\partial_{y_iy_k}\mathcal{N}$ is a Calderon-Zygmund kernel. Thus, by Young's convolution inequality with exponents p, 2 and 2, and elliptic regularity, we obtain

$$\begin{aligned}\|B\|_{L^p} &\le c\|\widetilde{\rho}^{(m)}(\cdot,s)+\widetilde{N}(\cdot,s)\|_{L^p}\|\nabla\widetilde{\Phi}^{(m)}(\cdot,s)\|_{L^2}\|\Gamma(\cdot,t-s)\|_{L^2}\\ &\le c(\|\widetilde{\rho}^{(m)}\|^2_{L^{p,\infty}} + \|\widetilde{N}\|^2_{L^{p,\infty}})\|\Gamma(\cdot,t-s)\|_{L^2}.\end{aligned} \tag{53}$$

We estimate C similarly as B. Now adding the estimates for the L^p norms of $|A|, |B|$ and $|C|$, integrating in the variable s from 0 to t, and using Lemmas 1 and 2, we obtain the desired inequalities. □

Now we go back to the proof of Theorem 1. In view of Lemmas 1 and 3 with $d \in \{2,3\}$, Young's convolution inequality, Minkowski's integral inequality and elliptic regularity, we obtain

$$\begin{aligned}\|\widetilde{u}^{(m)}\|_{L^{p,\infty}} &\le C\|u_0\|_{L^{p,\infty}} + C_1|\alpha|T^{1/2}\|\widetilde{v}^{(m)}\|_{L^{p,\infty}} + C(T^{1/2-d/2+d(p-1)/2p} + T^{1/2})(\|\widetilde{u}^{(m-1)}\|^2_{L^{p,\infty}} + \|\widetilde{v}^{(m-1)}\|^2_{L^{p,\infty}})\\ &+ C(T^{1-d/4} + T)(\|\widetilde{\rho}^{(m-1)}\|^2_{L^{p,\infty}} + \|\widetilde{N}\|^2_{L^{p,\infty}} + \|\widetilde{\xi}^{(m-1)}\|^2_{L^{p,\infty}} + \|\widetilde{M}\|^2_{L^{p,\infty}})\\ &+ CT\|\widetilde{\rho}^{(m-1)}\|_{L^{p,\infty}}(\|\widetilde{\rho}^{(m-1)}\|_{L^{p,\infty}} + \|\widetilde{N}\|_{L^{p,\infty}}) + CT\|\widetilde{\xi}^{(m-1)}\|_{L^{p,\infty}}(\|\widetilde{\xi}^{(m-1)}\|_{L^{p,\infty}} + \|\widetilde{M}\|_{L^{p,\infty}})\\ &+ CT\|\widetilde{N}\|_{L^{p,\infty}}(\|\widetilde{\rho}^{(m-1)}\|_{L^{p,\infty}} + \|\widetilde{N}\|_{L^{p,\infty}}) + CT\|\widetilde{M}\|_{L^{p,\infty}}(\|\widetilde{\xi}^{(m-1)}\|_{L^{p,\infty}} + \|\widetilde{M}\|_{L^{p,\infty}}) + CT\|\widetilde{f}\|_{L^{p,\infty}},\end{aligned}$$

$$\begin{aligned}\|\widetilde{v}^{(m)}\|_{L^{p,\infty}} &\le C_1|\alpha|T^{1/2}\|\widetilde{u}^{(m)}\|_{L^{p,\infty}} + C(T^{1/2-d/2+d(p-1)/2p} + T^{1/2})(\|\widetilde{u}^{(m-1)}\|^2_{L^{p,\infty}} + \|\widetilde{v}^{(m-1)}\|^2_{L^{p,\infty}})\\ &+ C(T^{1-d/4} + T)(\|\widetilde{\rho}^{(m-1)}\|^2_{L^{p,\infty}} + \|\widetilde{N}\|^2_{L^{p,\infty}} + \|\widetilde{\xi}^{(m-1)}\|^2_{L^{p,\infty}} + \|\widetilde{M}\|^2_{L^{p,\infty}})\\ &+ CT\|\widetilde{\rho}^{(m-1)}\|_{L^{p,\infty}}(\|\widetilde{\xi}^{(m-1)}\|_{L^{p,\infty}} + \|\widetilde{M}\|_{L^{p,\infty}}) + CT\|\widetilde{\xi}^{(m-1)}\|_{L^{p,\infty}}(\|\widetilde{\rho}^{(m-1)}\|_{L^{p,\infty}} + \|\widetilde{N}\|_{L^{p,\infty}})\\ &+ CT\|\widetilde{N}\|_{L^{p,\infty}}(\|\widetilde{\xi}^{(m-1)}\|_{L^{p,\infty}} + \|\widetilde{M}\|_{L^{p,\infty}}) + CT\|\widetilde{M}\|_{L^{p,\infty}}(\|\widetilde{\rho}^{(m-1)}\|_{L^{p,\infty}} + \|\widetilde{N}\|_{L^{p,\infty}}) + CT\|\widetilde{g}\|_{L^{p,\infty}},\end{aligned}$$

$$\begin{aligned}\|\widetilde{c_i}^{(m)}\|_{L^{p,\infty}} &\le C\|c_i^0\|_{L^{p,\infty}} + C_1|\alpha|T^{1/2}\|\widetilde{d_i}^{(m)}\|_{L^{p,\infty}}\\ &+ C(T^{1/2-d/2+d(p-1)/2p} + T^{1/2})(\|\widetilde{u}^{(m-1)}\|_{L^{p,\infty}}\|\widetilde{c_i}^{(m-1)}\|_{L^{p,\infty}} + \|\widetilde{v}^{(m-1)}\|_{L^{p,\infty}}\|\widetilde{d_i}^{(m-1)}\|_{L^{p,\infty}})\\ &+ CT^{1/2}\|\widetilde{c_i}^{(m-1)}\|_{L^{p,\infty}}(\|\widetilde{\rho}^{(m-1)}\|_{L^{p,\infty}} + \|\widetilde{N}\|_{L^{p,\infty}}) + CT^{1/2}\|\widetilde{d_i}^{(m-1)}\|_{L^{p,\infty}}(\|\widetilde{\xi}^{(m-1)}\|_{L^{p,\infty}} + \|\widetilde{M}\|_{L^{p,\infty}})\end{aligned}$$

and

$$\begin{aligned}\|\widetilde{d_i}^{(m)}\|_{L^{p,\infty}} &\le C_1|\alpha|T^{1/2}\|\widetilde{c_i}^{(m)}\|_{L^{p,\infty}}\\ &+ C(T^{1/2-d/2+d(p-1)/2p} + T^{1/2})(\|\widetilde{v}^{(m-1)}\|_{L^{p,\infty}}\|\widetilde{c_i}^{(m-1)}\|_{L^{p,\infty}} + \|\widetilde{u}^{(m-1)}\|_{L^{p,\infty}}\|\widetilde{d_i}^{(m-1)}\|_{L^{p,\infty}})\\ &+ CT^{1/2}\|\widetilde{c_i}^{(m-1)}\|_{L^{p,\infty}}(\|\widetilde{\xi}^{(m-1)}\|_{L^{p,\infty}} + \|\widetilde{M}\|_{L^{p,\infty}}) + CT^{1/2}\|\widetilde{d_i}^{(m-1)}\|_{L^{p,\infty}}(\|\widetilde{\rho}^{(m-1)}\|_{L^{p,\infty}} + \|\widetilde{N}\|_{L^{p,\infty}}).\end{aligned}$$

We note that we have bounded the absolute value of the valences $|z_i|$ for $i \in \{1,\dots,n\}$ by their maximum value which is absorbed by the constant C.

Now, assume that

$$C_1|\alpha|T^{1/2} \le \frac{1}{2}. \tag{54}$$

Define the sequence $\{a_m\}_{m=1}^{\infty}$ by

$$a_m = \|\widetilde{u}^{(m)}\|_{L^{p,\infty}} + \|\widetilde{v}^{(m)}\|_{L^{p,\infty}} + \sum_{i=1}^{n}\left(\|\widetilde{c_i}^{(m)}\|_{L^{p,\infty}} + \|\widetilde{d_i}^{(m)}\|_{L^{p,\infty}}\right) \tag{55}$$

and let

$$C_{f,g,M,N} = \|\widetilde{f}\|_{L^{p,\infty}} + \|\widetilde{g}\|_{L^{p,\infty}} + \|\widetilde{M}\|^2_{L^{p,\infty}} + \|\widetilde{N}\|^2_{L^{p,\infty}}. \tag{56}$$

Using

$$\|\widetilde{\rho}^{(m-1)}\|_{L^{p,\infty}} \le \left(\max_{i=1,\dots,n}|z_i|\right)\left(\sum_{i=1}^{n}\|\widetilde{c_i}^{(m-1)}\|_{L^{p,\infty}}\right) \tag{57}$$

and

$$\|\widetilde{\xi}^{(m-1)}\|_{L^{p,\infty}} \le \left(\max_{i=1,\dots,n}|z_i|\right)\left(\sum_{i=1}^{n}\|\widetilde{d_i}^{(m-1)}\|_{L^{p,\infty}}\right), \tag{58}$$

we have

$$a_m \le Ca_0 + C(T^{1/2-d/2p} + T^{1-d/4} + T^{1/2} + T)a_{m-1}^2 + CC_{f,g,N,M}(T + T^{1/2} + T^{1-d/4}) \tag{59}$$

where $C > 1$ is a positive constant depending on p. Let

$$\begin{aligned}t_1 &= \frac{M_p}{C_{f,g,N,M}}, t_2 = \frac{M_p^2}{C_{f,g,N,M}^2}, t_3 = \frac{M_p^{4/(4-d)}}{C_{f,g,N,M}^{4/(4-d)}}, t_4 = \frac{1}{(8^2C^2M_p)^{2p/(p-d)}},\\ t_5 &= \frac{1}{(8^2C^2M_p)^{4/(4-d)}}, t_6 = \frac{1}{(8^2C^2M_p)^2}, t_7 = \frac{1}{8^2C^2M_p}.\end{aligned} \tag{60}$$

and let

$$T_1 = \min\{t_1, t_2, t_3, t_4, t_5, t_6, t_7\}. \tag{61}$$

An inductive argument gives

$$a_m \le 8CM_p \tag{62}$$

for all $m \ge 1$ and for all $0 < T < T_1$.

Therefore, if $y = \alpha t$ satisfies (54), that is

$$\left|\frac{y}{t}\right| \le \frac{1}{2T^{1/2}C_1}, \tag{63}$$

then

$$\|u^{(m)}(\cdot, y, t)\|_{L^{p,\infty}} + \|v^{(m)}(\cdot, y, t)\|_{L^{p,\infty}} + \sum_{i=1}^{n}\left(\|c_i^{(m)}(\cdot, y, t)\|_{L^{p,\infty}} + \|d_i^{(m)}(\cdot, y, t)\|_{L^{p,\infty}}\right) \le 8CM_p \tag{64}$$

provided that $T \in (0, T_1)$. We note that (63) determines the domain of analyticity $\mathcal{D}_t$.

Finally, we show that the sequence $(u^{(m)}, c_1^{(m)}, \dots, c_n^{(m)})$ is a contraction. From equations (13) and (15), we have

$$\begin{aligned}\|u^{(m)} - u^{(m-1)}\|_{L^{p,\infty}} &\le C(T^{1/2-d/2p} + T^{1/2})(\|u^{(m-1)}\|_{L^{p,\infty}} + \|u^{(m-2)}\|_{L^{p,\infty}})\|u^{(m-1)} - u^{(m-2)}\|_{L^{p,\infty}} \\ &+ CT^{1-d/4}(\|\rho^{(m-1)}\|_{L^{p,\infty}} + \|\rho^{(m-2)}\|_{L^{p,\infty}} + \|N\|_{L^{p,\infty}})\|\rho^{(m-1)} - \rho^{(m-2)}\|_{L^{p,\infty}} \\ &+ CT(\|\rho^{(m-1)}\|_{L^{p,\infty}} + \|\rho^{(m-2)}\|_{L^{p,\infty}})\|\rho^{(m-1)} - \rho^{(m-2)}\|_{L^{p,\infty}} \\ &+ CT\|N\|_{L^{p,\infty}}\|\rho^{(m-1)} - \rho^{(m-2)}\|_{L^{p,\infty}}\end{aligned} \tag{65}$$

and

$$\begin{aligned}\|c_i^{(m)} - c_i^{(m-1)}\|_{L^{p,\infty}} &\le C(T^{1/2-d/2p} + T^{1/2})\|c_i^{(m-1)}\|_{L^{p,\infty}}\|u^{(m-1)} - u^{(m-2)}\|_{L^{p,\infty}} \\ &+ C(T^{1/2-d/2p} + T^{1/2})\|u^{(m-2)}\|_{L^{p,\infty}}\|c_i^{(m-1)} - c_i^{(m-2)}\|_{L^{p,\infty}} \\ &+ CT^{1/2}\|\rho^{(m-1)} + N\|_{L^{p,\infty}}\|c_i^{(m-1)} - c_i^{(m-2)}\|_{L^{p,\infty}} \\ &+ CT^{1/2}\|c_i^{(m-2)}\|_{L^{p,\infty}}\|\rho^{(m-1)} - \rho^{(m-2)}\|_{L^{p,\infty}}.\end{aligned} \tag{66}$$

Define the sequence $\{b_m\}_{m=1}^\infty$ by

$$b_m = (u^{(m)}, c_1^{(m)}, \dots, c_n^{(m)}). \tag{67}$$

In view of (64), there exists $T_0 \in (0, T_1]$ depending on p, M_p, f, N and the parameters of the problem such that

$$\|b_m\|_{L^{p,\infty}} \le \frac{1}{2}\|b_{m-1}\|_{L^{p,\infty}}, \tag{68}$$

holds for all $t \in (0, T_0)$ and $(x, y) \in \mathcal{D}_t$. This shows that $\{b_m\}_{m=1}^\infty$ is a contraction and converges to $S = (u, c_1, \dots, c_n)$. The fact that S is a local analytic solution of the Nernst-Planck-Navier-Stokes system (7) follows from (64).

Remark 1. *We note that Theorem 1 holds in any dimension $d \ge 2$. The restriction $d \in \{2, 3\}$ is only needed in our stated version of Lemmas 2 and 3 but can be removed. Indeed, letting*

$$r \in \left(1, \frac{d}{d-2}\right), \tag{69}$$

the L^r norm of the periodic heat kernel can be bounded by

$$\|\Gamma(\cdot, t)\|_{L^r} \le C\left(\int t^{-\frac{rd}{2}} e^{-\frac{r|x|^2}{4t}} dx\right)^{1/r} \le Ct^{-\frac{d(r-1)}{2r}}\left(\int t^{-d/2} e^{-\frac{r|x|^2}{4t}} dx\right)^{1/r} \tag{70}$$

when $0 < t < 1$, which yields the bound

$$\|\Gamma(\cdot, t)\|_{L^r} \le Ct^{-\frac{d(r-1)}{2r}} + C \tag{71}$$

for any $t > 0$. Using this latter estimate, we can generalize Lemma 3 to higher dimensions. The only required modification would be an equivalent bound of the estimate (53) in term of the L^r norm of the heat kernel for a suitable

r satisfying (69). *However, $\|B\|_{L^p}$ can be estimated as*

$$\|B\|_{L^p} \le c\|\widetilde{\rho}^{(m)}(\cdot,s)+\widetilde{N}(\cdot,s)\|_{L^p}\|\nabla\widetilde{\Phi}^{(m)}(\cdot,s)\|_{L^{r'}}\|\Gamma(\cdot,t-s)\|_{L^r}$$
$$\le c(\|\widetilde{\rho}^{(m)}\|^2_{L^{p,\infty}}+\|\widetilde{N}\|^2_{L^{p,\infty}})\|\Gamma(\cdot,t-s)\|_{L^r} \tag{72}$$
$$\le c\left(Ct^{-\frac{d(r-1)}{2r}}+C\right)(\|\widetilde{\rho}^{(m)}\|^2_{L^{p,\infty}}+\|\widetilde{N}\|^2_{L^{p,\infty}}) \tag{73}$$

where r' is the Hölder conjugate exponent of r. Here we have used the elliptic regularity estimate

$$\|\nabla\widetilde{\Phi}^{(m)}\|_{L^{r'}} \le C\|\widetilde{\rho}^{(m)}+\widetilde{N}\|_{L^p} \tag{74}$$

where p is as defined in the statement of Lemma 3. The fact that the power $\frac{d(r-1)}{2r}$ is less than one allows us to integrate $\|B\|_{L^p}$ in time from 0 to t yielding similar estimate to (48).

Remark 2. *We note that the solution $(u(x,t),c_1(x,t),\dots,c_n(x,t))$ is infinitely differentiable in the space variable for any time $t\in(0,T_0)$. Moreover, the solution obeys*

$$u\in L^\infty(0,T_0;L^2)\cap L^2(0,T_0;H^1) \tag{75}$$

and

$$c_i\in L^\infty(0,T_0;L^2)\cap L^2(0,T_0;H^1) \tag{76}$$

for all $i\in\{1,\dots,n\}$. The proof is based on energy methods, and follows from considerations we are presenting in the next sections.

3. Extension of the Local Analytic Solution in 2D

Let H be the subspace of L^2 consisting of periodic, divergence free, mean zero vector fields.

Definition 1. *A solution $(u,c_1,\dots,c_n)$ of* (7) *is said to be a weak solution on $[0,T]$ if*

$$u\in L^\infty(0,T;H)\cap L^2(0,T;H^1\cap H), \tag{77}$$
$$c_i\in L^\infty(0,T;L^2)\cap L^2(0,T;H^1) \tag{78}$$

for all $i\in\{1,\dots,n\}$, and $(u,c_1,\dots,c_n)$ solves (7) *in the sense of distributions.*

Theorem 2. *(Local Solution in 2D) Let $d=2$. Let $u_0\in L^2$ be divergence free and have mean zero. Let $c_i(0)\in L^2$ for all $i\in\{1,\dots,n\}$. Then there exists a positive time T_2 depending on the initial data and the parameters of the problem such that the system* (7) *has a unique weak solution on $[0,T_2]$.*

Proof: We provide a priori bounds. For each $i\in\{1,\dots,n\}$, we take the L^2 inner product of the equation obeyed by c_i with c_i, to obtain

$$\frac{1}{2}\frac{d}{dt}\|c_i\|^2_{L^2}+D_i\|\nabla c_i\|^2_{L^2} = -D_i\int_{\mathbb{T}^2} z_ic_i\nabla\Phi\cdot\nabla c_i. \tag{79}$$

In view of elliptic regularity and Gagliardo-Nirenberg's inequality, we have

$$\|\nabla\Phi\|_{L^\infty}\le C\|\rho+N\|_{L^3}\le C\|\rho+N\|^{2/3}_{L^2}\|\nabla(\rho+N)\|^{1/3}_{L^2}$$
$$\le C\left\{\sum_{j=1}^n\|c_j\|^{2/3}_{L^2}+\|N\|^{2/3}_{L^2}\right\}\left\{\sum_{k=1}^n\|\nabla c_k\|^{1/3}_{L^2}+\|\nabla N\|^{1/3}_{L^2}\right\} \tag{80}$$

and thus

$$\left|\int_{\mathbb{T}^2} z_ic_i\nabla\Phi\cdot\nabla c_i\right|\le C\left\{\sum_{j=1}^n\|c_j\|^{2/3}_{L^2}+\|N\|^{2/3}_{L^2}\right\}\left\{\sum_{k=1}^n\|\nabla c_k\|^{1/3}_{L^2}+\|\nabla N\|^{1/3}_{L^2}\right\}\|c_i\|_{L^2}\|\nabla c_i\|_{L^2} \tag{81}$$

by Hölder's inequality. Adding the differential inequalities obtained for each ionic concentration and applying Young's inequality, we have

$$\frac{d}{dt}\left\{\sum_{i=1}^n\|c_i\|^2_{L^2}\right\}+\sum_{i=1}^n D_i\|\nabla c_i\|^2_{L^2}\le C\sum_{i=1}^n\|c_i\|^5_{L^2}+C_N \tag{82}$$

where C_N is some positive constant depending on the added charge density N, the parameters of the problem, and some universal constants. Now, we take the L^2 inner product of the equation obeyed by u with u to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2}+\nu\|\nabla u\|^2_{L^2} = \int_{\mathbb{T}^2} fu - \int_{\mathbb{T}^2}(\rho+N)\nabla\Phi u. \tag{83}$$

We estimate

$$\left|\int_{\mathbb{T}^2}(\rho+N)\nabla\Phi u\right| \le \|\rho+N\|_{L^2}\|\nabla\Phi\|_{L^\infty}\|u\|_{L^2} \le C\|\rho+N\|_{L^2}^{5/3}\|\nabla(\rho+N)\|_{L^2}^{1/3}\|u\|_{L^2}$$
$$\le C\left\{\sum_{j=1}^n \|c_j\|_{L^2}^{5/3}+\|N\|_{L^2}^{5/3}\right\}\left\{\sum_{k=1}^n \|\nabla c_k\|_{L^2}^{1/3}+\|\nabla N\|_{L^2}^{1/3}\right\}\|\nabla u\|_{L^2} \tag{84}$$

by Hölder's inequality, elliptic regularity, Gagliardo-Nirenberg's inequality and Poincaré's inequality. This gives the differential inequality

$$\frac{d}{dt}\|u\|_{L^2}^2+\nu\|\nabla u\|_{L^2}^2 \le C\|f\|_{L^2}^2+\sum_{i=1}^n \frac{D_i}{2}\|\nabla c_i\|_{L^2}^2 + C\sum_{i=1}^n\|c_i\|_{L^2}^5 + C_N. \tag{85}$$

Adding (82) and (85), we obtain

$$\frac{d}{dt}\left\{\|u\|_{L^2}^2+\sum_{i=1}^n\|c_i\|_{L^2}^2\right\}+\sum_{i=1}^n\frac{D_i}{2}\|\nabla c_i\|_{L^2}^2+\nu\|\nabla u\|_{L^2}^2 \le C\sum_{i=1}^n\|c_i\|_{L^2}^5+C_{N,f}. \tag{86}$$

This estimate applied to Galerkin approximations, use of the Aubin-Lions lemma and passage to the limit yields weak solutions.

For uniqueness, suppose $(u_1, c_1^1, \dots, c_n^1)$ and $(u_2, c_1^2, \dots, c_n^2)$ are two weak solutions of the NPNS system (7) with initial data $u_1(0)=u_2(0), c_i^1(0)=c_i^2(0)$ for all $i=1,\dots,n$. Let $u=u_1-u_2$, $c_i=c_i^1-c_i^2$ for $i=1,\dots,n$, $\rho=\rho_1-\rho_2$ and $\Phi=\Phi_1-\Phi_2$. Then $(u, c_1, \dots, c_n)$ obeys the system

$$\begin{cases}\partial_t u + u\cdot\nabla u_1 + u_2\cdot\nabla u + \nabla(p_1-p_2) = \nu\Delta u - N\nabla\Phi - \rho\nabla\Phi_1 - \rho_2\nabla\Phi\\ \nabla\cdot u = 0\\ \rho = z_1c_1+\dots+z_nc_n\\ -\epsilon\Delta\Phi=\rho\\ \partial_t c_i + u\cdot\nabla c_i^1 + u_2\cdot\nabla c_i = D_i\Delta c_i + D_i\nabla\cdot(z_ic_i\nabla\Phi_1)+D_i\nabla\cdot(z_ic_i^2\nabla\Phi), \quad i=1,\dots,n.\end{cases} \tag{87}$$

We take the L^2 inner product of the u and c_i equations in (87) with u and c_i respectively, we add the resulting equations and we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{\|u\|_{L^2}^2+\sum_{i=1}^n\|c_i\|_{L^2}^2\right\}+\nu\|\nabla u\|_{L^2}^2+\sum_{i=1}^n D_i\|\nabla c_i\|_{L^2}^2$$
$$= -\int_{\mathbb{T}^2}(u\cdot\nabla u_1)\cdot u - \int_{\mathbb{T}^2} N\nabla\Phi\cdot u - \int_{\mathbb{T}^2}\rho\nabla\Phi_1\cdot u - \int_{\mathbb{T}^2}\rho_2\nabla\Phi\cdot u$$
$$-\int_{\mathbb{T}^2}\sum_{i=1}^n(u\cdot\nabla c_i^1)c_i - \int_{\mathbb{T}^2}\sum_{i=1}^n D_i(z_ic_i\nabla\Phi_1)\cdot\nabla c_i - \int_{\mathbb{T}^2}\sum_{i=1}^n D_i(z_ic_i^2\nabla\Phi)\cdot\nabla c_i. \tag{88}$$

In view of Ladyzhenskaya's interpolation inequality applied to the mean zero function u, we have

$$\left|\int_{\mathbb{T}^2}(u\cdot\nabla u_1)\cdot u\right| \le C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u_1\|_{L^2}. \tag{89}$$

Using the continuous embedding $H^1\subset L^6$, we bound

$$\left|\int_{\mathbb{T}^2}N\nabla\Phi\cdot u\right| \le \|N\|_{L^3}\|\nabla\Phi\|_{L^6}\|u\|_{L^2} \le C\|N\|_{L^3}\|\rho\|_{L^2}\|u\|_{L^2}. \tag{90}$$

We estimate

$$\left|\int_{\mathbb{T}^2}\rho\nabla\Phi_1\cdot u\right| \le \|\nabla\Phi_1\|_{L^\infty}\|\rho\|_{L^2}\|u\|_{L^2} \le C\|\nabla\rho_1+\nabla N\|_{L^2}\|\rho\|_{L^2}\|u\|_{L^2} \tag{91}$$

in view of elliptic regularity $\|\nabla\Phi_1\|_{L^\infty} \le C\|\rho_1+N\|_{L^3}$, the Gagliardo-Nirenberg interpolation inequality and the Poincaré inequality applied to the mean zero function ρ_1+N. Using the fact that $\|\nabla\Phi\|_{L^6}\le C\|\rho\|_{L^2}$, we have

$$\left|\int_{\mathbb{T}^2}\rho_2\nabla\Phi\cdot u\right| \le C\|\rho_2\|_{L^2}^{2/3}(\|\rho_2\|_{L^2}^{1/3}+\|\nabla\rho_2\|_{L^2}^{1/3})\|\rho\|_{L^2}\|u\|_{L^2}. \tag{92}$$

Now, we use Hölder's inequality with exponents $2,4,4$ and Ladyzhenskaya's inequality applied to the mean zero functions u and c_i to estimate

$$\left|\int_{\mathbb{T}^2}\sum_{i=1}^n(u\cdot\nabla c_i^1)c_i\right| \le C\sum_{i=1}^n\|\nabla c_i^1\|_{L^2}\|c_i\|_{L^2}^{1/2}\|\nabla c_i\|_{L^2}^{1/2}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}. \tag{93}$$

Since $\rho_1 + N$ has mean zero, the Gagliardo-Nirenberg and Poincaré inequalities give the bound

$$\left| \int_{\mathbb{T}^2} \sum_{i=1}^n D_i(z_i c_i \nabla \Phi_1) \cdot \nabla c_i \right| \le C \sum_{i=1}^n \|\nabla \rho_1 + \nabla N\|_{L^2} \|c_i\|_{L^2} \|\nabla c_i\|_{L^2}. \tag{94}$$

Finally, we estimate

$$\left| \int_{\mathbb{T}^2} \sum_{i=1}^n D_i(z_i c_i^2 \nabla \Phi) \cdot \nabla c_i \right| \le C \|c_i^2\|_{L^3} \|\nabla \Phi\|_{L^6} \|\nabla c_i\|_{L^2} \le C \|c_i^2\|_{L^2}^{2/3} (\|c_i^2\|_{L^2}^{1/3} + \|\nabla c_i^2\|_{L^2}^{1/3}) \|\rho\|_{L^2} \|\nabla c_i\|_{L^2}. \tag{95}$$

Let

$$M(t) = \|u\|_{L^2}^2 + \sum_{i=1}^n \|c_i\|_{L^2}^2. \tag{96}$$

Then $M(t)$ obeys the differential inequality

$$M'(t) \le C K(t) M(t) \tag{97}$$

where

$$K(t) = \|\nabla u_1\|_{L^2}^2 + \sum_{i=1}^n \left\{ \|c_i^2\|_{L^2}^2 + \|\nabla c_i^1\|_{L^2}^2 + + \|\nabla c_i^2\|_{L^2}^2 \right\} + C_N. \tag{98}$$

This gives uniqueness.

Remark 3. *The uniqueness of the weak solution together with Remark 2 implies its analyticity on* $(0, \min\{T_0, T_2\})$*, provided that the initial data is in* $L^p(\mathbb{T}^2)$ *for some* $p > 2$.

Definition 2. *A solution* $(u, c_1, \ldots, c_n)$ *of* (7) *is said to be a strong solution on* $[0, T]$ *if*

$$u \in L^\infty(0, T; H^1 \cap H) \cap L^2(0, T; H^2 \cap H) \tag{99}$$

and

$$c_i \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2) \tag{100}$$

for all $i \in \{1, \ldots, n\}$.

Proposition 1. *Let* $d = 2$. *Let* $u_0 \in H^1$ *be divergence free and have mean zero. Let* $c_i(0) \in H^1$ *for all* $i \in \{1, \ldots, n\}$. *Suppose* $(u, c_1, \ldots, c_n)$ *solves the system* (7) *on* $[0, T]$ *in the sense of distributions and obeys*

$$\int_0^T \left(\|c_1(t)\|_{L^2}^3 + \cdots + \|c_n(t)\|_{L^2}^3 \right) dt < \infty. \tag{101}$$

Then $(u, c_1, \ldots, c_n)$ *is unique on* $[0, T]$ *and is a strong solution of* (7) *on* $[0, T]$. *If, in addition,* $c_i(0) \ge 0$ *for* $i \in \{1, \ldots, n\}$*, then* $c_i(x, t) \ge 0$ *for a.e.* $x \in \mathbb{T}^2$ *and for all* $t \in [0, T]$.

Proof: The differential inequality (82) implies that

$$\frac{d}{dt} \left\{ \sum_{i=1}^n \|c_i\|_{L^2}^2 \right\} + \sum_{i=1}^n D_i \|\nabla c_i\|_{L^2}^2 \le C \left(\sum_{i=1}^n \|c_i\|_{L^2}^3 \right) \sum_{i=1}^n \|c_i\|_{L^2}^2 + C_N \tag{102}$$

and thus $c_i \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ for all $i \in \{1, \ldots, n\}$. Integrating (85) in time from 0 to t, we conclude that $u \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$. This implies that $(u, c_1, \ldots, c_n)$ is unique.

Now we upgrade the regularity of the solution. We take the L^2 inner product of the u-equation in (7) with $-\Delta u$. We use the fact that $tr(M^T M^2) = 0$ where M is the 2 by 2 traceless matrix with entries $M_{ij} = \frac{\partial u_i}{\partial x_j}$. We obtain the equation

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 = -\int_{\mathbb{T}^2} f \cdot \Delta u + \int_{\mathbb{T}^2} (\rho + N) \nabla \Phi \cdot \Delta u. \tag{103}$$

In view of the elliptic regularity $\|\nabla \Phi\|_{L^\infty} \le C \|\rho + N\|_{L^4}$ and Ladyzhenskaya's interpolation inequality applied to the mean zero function $\rho + N$, we estimate

$$\left| \int_{\mathbb{T}^2} (\rho + N) \nabla \Phi \cdot \Delta u \right| \le C \|\Delta u\|_{L^2} \|\rho + N\|_{L^2}^{3/2} \|\nabla \rho + \nabla N\|_{L^2}^{1/2}. \tag{104}$$

Using Young's inequality, we obtain the differential inequality

$$\frac{d}{dt} \|\nabla u\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 \le C \|f\|_{L^2}^2 + C \|\rho + N\|_{L^2}^3 \|\nabla \rho + \nabla N\|_{L^2} \tag{105}$$

and we conclude that $u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$. Finally, we take the L^2 inner product of the c_i-equation in (7) with $-\Delta c_i$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla c_i\|_{L^2}^2 + D_i\|\nabla c_i\|_{L^2}^2 = -D_i\int_{\mathbb{T}^2} z_i(\nabla c_i \cdot \nabla\Phi)\Delta c_i - D_i\int_{\mathbb{T}^2} z_i c_i \Delta\Phi\Delta c_i. \tag{106}$$

In view of elliptic regularity and Ladyzhenskaya's inequality, we estimate

$$\left|\int_{\mathbb{T}^2} D_i z_i(\nabla c_i\cdot\nabla\Phi)\Delta c_i\right| \le C\|\Delta c_i\|_{L^2}\|\nabla c_i\|_{L^2}\|\rho+N\|_{L^2}^{1/2}\|\nabla\rho+\nabla N\|_{L^2}^{1/2}. \tag{107}$$

Using in addition the Poincaré inequality applied to the mean zero function $\rho + N$, we have

$$\left|\int_{\mathbb{T}^2} D_i z_i c_i\Delta\Phi\Delta c_i\right| \le C\|\Delta c_i\|_{L^2}\|c_i\|_{L^2}^{1/2}(\|c_i\|_{L^2}^{1/2} + \|\nabla c_i\|_{L^2}^{1/2})\|\nabla\rho+\nabla N\|_{L^2}. \tag{108}$$

We obtain the differential inequality

$$\frac{d}{dt}\sum_{i=1}^n\|\nabla c_i\|_{L^2}^2 + \sum_{i=1}^n D_i\|\Delta c_i\|_{L^2}^2 \le C\sum_{i=1}^n\|c_i\|_{L^2}^4 + C\left(\sum_{i=1}^n\|\nabla c_i\|_{L^2}^2\right)^2 + C_N \tag{109}$$

and thus $c_i \in L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$ for all $i\in\{1,\dots,n\}$. The nonnegativity of the ionic concentrations for all positive times follows from the fact that the initial concentrations are nonnegative and the regularity of solutions (see [2]). This completes the proof of Proposition 1.

Remark 4. *We note that Theorem 2 guarantees the existence of a time $T > 0$ such that a weak solution exists on $[0,T]$ and satisfies condition* (101)*.*

The following proposition will be used to extend the local weak solution on $[0,T_2]$ into a strong solution on $[0,T]$ for any $T > 0$.

Proposition 2. *Let $d = 2$ and $T > 0$. Let $u_0 \in L^2$ be divergence free and have mean zero. Let $c_i(0) \in L^2$ for all $i \in \{1,\dots,n\}$. Suppose $(u, c_1,\dots,c_n)$ solves* (7) *on $[0,T]$ in the sense of distributions such that $c_i(x,t) \ge 0$ for a.e. $x \in \mathbb{T}^2$ and for all $t\in[0,T]$. Then there exists a positive constant $\Gamma > 0$ depending on the initial data, the time T, and the parameters of the problems such that*

$$\sup_{0\le t\le T}\|u(t)\|_{L^2} + \sup_{0\le t\le T}\sum_{i=1}^n\|c_i(t)\|_{L^2} \le \Gamma \tag{110}$$

holds.

The proof of Proposition 2 is based on [2] and is presented in Appendix A.
We obtain the following extension theorem:

Theorem 3. *(Global analytic solution in 2D) Let $d = 2$. Let $T > 0$. Let $u_0 \in H^1$ be divergence free and have mean zero. Let $c_i(0) \in H^1$ for all $i\in\{1,\dots,n\}$. Then there exists a unique strong analytic solution $\mathcal{S} = (u, c_1,\dots,c_n)$ on $[0,T]$. Moreover, for any $p > 2$, the $L^p(\mathbb{T}^2)$ norm of $\mathcal{S}$ is uniformly bounded in time by a constant depending only on the initial data, p, the fixed time T and the parameters of the problem.*

Proof: The existence of a unique strong solution $\mathcal{S}$ on $[0,T]$ follows from Theorem 2 and Propositions 1 and 2.
Now, fix $p > 2$. Since $u_0, c_i(0) \in H^1$, then $u_0, c_i(0) \in L^p$ in view of the continuous Sobolev embedding $H^1(\mathbb{T}^2) \subset L^p(\mathbb{T}^2)$. Thus, by Theorem 1, there exists a time $T_0 > 0$ and a solution $\mathcal{S}' = (\tilde{u}, \tilde{c}_1, ..., \tilde{c}_n) \in C([0,T_0], L^p)$ of the NPNS system (7) such that the solution $\mathcal{S}'$ is analytic on $(0,T_0)$. By Remark 2, $\mathcal{S}'$ is a weak solution on $(0,T_0)$, and by the uniqueness of weak solutions, we conclude that $\mathcal{S} = \mathcal{S}'$ on $(0,T_0)$. In view of Proposition 1, we have that the $H^1(\mathbb{T}^2)$ norm and hence the $L^p(\mathbb{T}^2)$ norm of the solution $\mathcal{S}$ is uniformly bounded in time by some constant that depends only on the initial data, the fixed time $T > 0$ and the parameters of the problem. This allows us to extend the analyticity and the uniform L^p boundedness properties of the local solution from the time interval $(0,T_0)$ into the interval $(0,T)$ by repeated application of Theorem 1.

4. Extension of the Local Analytic Solution in 3D

Theorem 4. *(Local Solution in 3D) Let $d = 3$. Let $u_0 \in H^1$ be divergence free and have mean zero. Let $c_i(0) \in L^2$ for all $i \in \{1, \dots, n\}$. Then there exists a positive time T_3 depending on the initial data and the parameters of the problem such that the system* (7) *has a unique solution $(u, c_1, \dots, c_n)$ on $[0, T_3]$ such that*

$$u \in L^\infty(0, T_3; H^1 \cap H) \cap L^2(0, T_3; H^2 \cap H), \tag{111}$$

$$c_i \in L^\infty(0, T_3; L^2) \cap L^2(0, T_3; H^1) \tag{112}$$

for all $i \in \{1, \dots, n\}$.

Proof: The proof is based on Galerkin approximations, energy estimates, and the Aubin-Lions lemma. For simplicity of exposition we perform only energy estimates. For each $i \in \{1, \dots, n\}$, we take the L^2 inner product of the equation obeyed by c_i with c_i. We estimate

$$\left|\int_{\mathbb{T}^3} D_i z_i c_i \nabla\Phi \cdot \nabla c_i\right| \le C\left\{\sum_{j=1}^n \|c_j\|_{L^2}^{1/2} + \|N\|_{L^2}^{1/2}\right\}\left\{\sum_{k=1}^n \|\nabla c_k\|_{L^2}^{1/2} + \|\nabla N\|_{L^2}^{1/2}\right\}\|c_i\|_{L^2}\|\nabla c_i\|_{L^2} \tag{113}$$

in view of the bound

$$\|\nabla\Phi\|_{L^\infty} \le C\|\rho + N\|_{L^2}^{1/2}\|\nabla(\rho + N)\|_{L^2}^{1/2}. \tag{114}$$

We obtain the differential inequality

$$\frac{d}{dt}\left\{\sum_{i=1}^n \|c_i\|_{L^2}^2\right\} + \sum_{i=1}^n D_i\|\nabla c_i\|_{L^2}^2 \le C\sum_{i=1}^n \|c_i\|_{L^2}^6 + C_N. \tag{115}$$

Now we take the L^2 inner product of the u-equation in (7) with $-\Delta u$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \nu\|\Delta u\|_{L^2}^2 = \int_{\mathbb{T}^3}(u\cdot\nabla u)\cdot\Delta u - \int_{\mathbb{T}^3} f\cdot\Delta u + \int_{\mathbb{T}^3}(\rho + N)\nabla\Phi\cdot\Delta u. \tag{116}$$

We bound

$$\left|\int_{\mathbb{T}^3}(\rho + N)\nabla\Phi\cdot\Delta u\right| \le C\|\Delta u\|_{L^2}\|\rho + N\|_{L^2}^{3/2}\|\nabla\rho + \nabla N\|_{L^2}^{1/2}. \tag{117}$$

Using the fact that u is divergence free and integrating by parts, we have

$$\left|\int_{\mathbb{T}^3}(u\cdot\nabla u)\cdot\Delta u\right| \le \|\nabla u\|_{L^4}^2\|\nabla u\|_{L^2} \le C\|\Delta u\|_{L^2}^{3/2}\|\nabla u\|_{L^2}^{3/2}. \tag{118}$$

Hence, we obtain

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \nu\|\Delta u\|_{L^2}^2 \le \|f\|_{L^2}^2 + C\|\nabla u\|_{L^2}^6 + C\|\rho + N\|_{L^2}^3\|\nabla\rho + \nabla N\|_{L^2} \tag{119}$$

Putting (115) and (119), we deduce the differential inequality

$$\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2 + \sum_{i=1}^n \|c_i\|_{L^2}^2\right) + \nu\|\Delta u\|_{L^2}^2 + \sum_{i=1}^n \frac{D_i}{2}\|\nabla c_i\|_{L^2}^2 \le C\left(\|\nabla u\|_{L^2}^2 + \sum_{i=1}^n \|c_i\|_{L^2}^2\right)^3 + C_{N,f} \tag{120}$$

yielding a local solution $(u, c_1, \dots, c_n)$ on some short time interval $[0, T_3]$ satisfying (111) and (112).

We proceed to show uniqueness. That is, suppose that $(u_1, c_1^1, \dots, c_n^1)$ and $(u_2, c_1^2, \dots, c_n^2)$ solve (7) in the sense of distributions, have equal initial data, and satisfy (111) and (112). Let $u = u_1 - u_2$, $c_i = c_i^1 - c_i^2$ for $i = 1, \dots, n$, $\rho = \rho_1 - \rho_2$ and $\Phi = \Phi_1 - \Phi_2$. Then $(u, c_1, \dots, c_n)$ obeys the system (87). We take the L^2 inner product of the u and c_i equations in (87) with $-\Delta u$ and c_i respectively, we add the resulting equations and we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\|\nabla u\|_{L^2}^2 + \sum_{i=1}^n \|c_i\|_{L^2}^2\right\} + \nu\|\Delta u\|_{L^2}^2 + \sum_{i=1}^n D_i\|\nabla c_i\|_{L^2}^2 \\
&= \int_{\mathbb{T}^3}(u\cdot\nabla u_1)\cdot\Delta u + \int_{\mathbb{T}^2} N\nabla\Phi\cdot\Delta u + \int_{\mathbb{T}^3}\rho\nabla\Phi_1\cdot\Delta u + \int_{\mathbb{T}^2}\rho_2\nabla\Phi\cdot\Delta u \\
&- \int_{\mathbb{T}^3}\sum_{i=1}^n (u\cdot\nabla c_i^1)c_i - \int_{\mathbb{T}^3}\sum_{i=1}^n D_i(z_i c_i\nabla\Phi_1)\cdot\nabla c_i - \int_{\mathbb{T}^3}\sum_{i=1}^n D_i(z_i c_i^2\nabla\Phi)\cdot\nabla c_i.
\end{aligned} \tag{121}$$

In view of the continuous embedding $H^1 \subset L^6$, we estimate

$$\left|\int_{\mathbb{T}^3}(u\cdot\nabla u_1)\cdot\Delta u\right| \le C\|\Delta u\|_{L^2}\|u\|_{L^3}\|\nabla u_1\|_{L^6} \le C\|\Delta u\|_{L^2}\|\nabla u\|_{L^2}\|\Delta u_1\|_{L^2} \tag{122}$$

and

$$\left|\int_{\mathbb{T}^3} N\nabla\Phi \cdot \Delta u\right| \le \|N\|_{L^3}\|\nabla\Phi\|_{L^6}\|\Delta u\|_{L^2} \le C\|N\|_{L^3}\|\rho\|_{L^2}\|\Delta u\|_{L^2}. \tag{123}$$

Using elliptic regularity, the 3D Gagliardo-Nirenberg inequality, and Poincaré's inequality, we have

$$\left|\int_{\mathbb{T}^3} \rho\nabla\Phi_1 \cdot \Delta u\right| \le \|\nabla\Phi_1\|_{L^\infty}\|\rho\|_{L^2}\|\Delta u\|_{L^2} \le C\|\nabla\rho_1 + \nabla N\|_{L^2}\|\rho\|_{L^2}\|\Delta u\|_{L^2}, \tag{124}$$

and

$$\left|\int_{\mathbb{T}^3} \rho_2\nabla\Phi \cdot \Delta u\right| \le C\|\rho_2\|_{L^2}^{1/2}(\|\rho_2\|_{L^2}^{1/2} + \|\nabla\rho_2\|_{L^2}^{1/2})\|\rho\|_{L^2}\|\Delta u\|_{L^2}. \tag{125}$$

We also estimate

$$\left|\int_{\mathbb{T}^3} \sum_{i=1}^n (u\cdot\nabla c_i^1)c_i\right| \le \sum_{i=1}^n \|\nabla c_i^1\|_{L^2}\|c_i\|_{L^3}\|u\|_{L^6} \le C\sum_{i=1}^n \|\nabla c_i^1\|_{L^2}\|\nabla c_i\|_{L^2}\|\nabla u\|_{L^2}, \tag{126}$$

$$\left|\int_{\mathbb{T}^3} \sum_{i=1}^n D_i(z_ic_i\nabla\Phi_1)\cdot\nabla c_i\right| \le C\sum_{i=1}^n \|\nabla\rho_1 + \nabla N\|_{L^2}\|c_i\|_{L^2}\|\nabla c_i\|_{L^2}, \tag{127}$$

and

$$\begin{aligned}\left|\int_{\mathbb{T}^3} \sum_{i=1}^n D_i(z_ic_i^2\nabla\Phi)\cdot\nabla c_i\right| &\le C\sum_{i=1}^n \|c_i^2\|_{L^3}\|\nabla\Phi\|_{L^6}\|\nabla c_i\|_{L^2}\\ &\le C\sum_{i=1}^n \|c_i^2\|_{L^2}^{1/2}(\|c_i^2\|_{L^2}^{1/2} + \|\nabla c_i^2\|_{L^2}^{1/2})\|\rho\|_{L^2}\|\nabla c_i\|_{L^2}.\end{aligned} \tag{128}$$

Let

$$M_1(t) = \|\nabla u\|_{L^2}^2 + \sum_{i=1}^n \|c_i\|_{L^2}^2. \tag{129}$$

Then $M_1(t)$ obeys the differential inequality

$$M_1'(t) \le CK_1(t)M_1(t) \tag{130}$$

where

$$K_1(t) = \|\Delta u_1\|_{L^2}^2 + \sum_{i=1}^n \left\{\|\nabla c_i^2\|_{L^2}^2 + \|c_i^2\|_{L^2}^2 + \|\nabla c_i^1\|_{L^2}^2\right\} + C_N. \tag{131}$$

This gives uniqueness.

Remark 5. *If we upgrade the regularity of the initial velocity from $u_0 \in L^p$ into $u_0 \in H^1$, then it can be shown, using energy estimates, that the unique analytic local solution derived in Theorem 1 satisfies*

$$u \in L^\infty(0,\tilde{T}_0,H^1) \cap L^2(0,\tilde{T}_0,H^2) \tag{132}$$

for some positive time $\tilde{T}_0 < T_0$. Without loss of generality, we can assume that the solution in Theorem 1 obeys the regularity condition (132) *when $u_0 \in H^1$.*

Proposition 3. *Let $d = 3$. Let $u_0 \in H^1$ be divergence free and have mean zero. Let $c_i(0) \in H^1$ for all $i \in \{1,\dots,n\}$. Suppose that $(u,c_1,\dots,c_n)$ solves the NPNS system* (7) *on $[0,T]$ in the sense of distributions and obeys*

$$\int_0^T \left(\|\nabla u(t)\|_{L^2}^4 + \|c_1(t)\|_{L^2}^4 + \dots + \|c_n(t)\|_{L^2}^4\right)dt < \infty. \tag{133}$$

Then $(u,c_1,\dots,c_n)$ is a strong solution of (7) *on $[0,T]$, and hence it is unique.*

Proof: By the differential inequality (115), we have that $c_i \in L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$ for all $i \in \{1,\dots,n\}$, whereas the differential inequality (119) implies that $u \in L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$. Now, we upgrade the regularity of the ionic concentrations. We take the L^2 inner product of the c_i-equation in (7) with $-\Delta c_i$. We estimate

$$\left|\int_{\mathbb{T}^3} D_iz_i(\nabla c_i\cdot\nabla\Phi)\Delta c_i\right| \le C\|\Delta c_i\|_{L^2}\|\nabla c_i\|_{L^2}\|\rho+N\|_{L^2}^{1/4}\|\nabla\rho+\nabla N\|_{L^2}^{3/4} \tag{134}$$

and

$$\left|\int_{\mathbb{T}^3} D_iz_ic_i\Delta\Phi\Delta c_i\right| \le C\|\Delta c_i\|_{L^2}\|c_i\|_{L^2}^{1/4}(\|c_i\|_{L^2}^{3/4} + \|\nabla c_i\|_{L^2}^{3/4})\|\nabla\rho+\nabla N\|_{L^2} \tag{135}$$

using the Gagliardo-Nirenberg and Poincaré inequalities. We obtain

$$\frac{d}{dt}\sum_{i=1}^{n}\|\nabla c_i\|_{L^2}^2 + \sum_{i=1}^{n} D_i\|\Delta c_i\|_{L^2}^2 \le C\sum_{i=1}^{n}\|c_i\|_{L^2}^4 + C\left(\sum_{i=1}^{n}\|\nabla c_i\|_{L^2}^2\right)^2 + C_N \tag{136}$$

and thus $c_i \in L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$ for all $i \in \{1,\dots,n\}$. This ends the proof of Proposition 3.

Theorem 5. *(Extension of the local analytic solution in 3D) Let $d = 3$. Let $T > 0$. Let $u_0 \in H^1$ be divergence free and have mean zero. Let $c_i(0) \in H^1$ for all $i \in \{1,\dots,n\}$. Suppose $\mathcal{S} = (u, c_1,\dots,c_n)$ solves* (7) *on $[0,T]$ in the sense of distributions and satisfies*

$$\int_0^T \left(\|\nabla u(t)\|_{L^2}^4 + \|c_1(t)\|_{L^2}^4 + \dots + \|c_n(t)\|_{L^2}^4\right) dt < \infty. \tag{137}$$

Then the solution $\mathcal{S}$ is analytic on $[0,T]$, and for any $p > 3$, its $L^p(\mathbb{T}^3)$ norm is uniformly bounded in time by some constant depending only on the initial data, p, the fixed time T, the parameters of the problem, and some universal constants.

The proof is similar to the proof of Theorem 3 and is based on the uniqueness of the solutions. We omit further details.

5. Appendix A

In this appendix, we present the proof of Proposition 2. We use the following two elementary lemmas:

Lemma 4. *Let $M > 0$. There exist universal constants $C_1, C_2 > 0$ depending only on M such that*

$$|Ma\log(Ma)| \le C_1 + C_2|a\log(a)| \tag{138}$$

and

$$|a\log(a)| \le C_1 + C_2|Ma\log(Ma)| \tag{139}$$

hold for all $a \ge 0$. The following estimate

$$(x_1+\dots+x_n)\log(x_1+\dots+x_n) \le nx_1\log(nx_1)+\dots+nx_n\log(nx_n) \tag{140}$$

also holds for all $x_1,\dots,x_n \ge 0$.

Proof: Using

$$\lim_{a\to\infty}\left|\frac{Ma\log(Ma)}{a\log a}\right| = M, \tag{141}$$

$$\lim_{a\to 0^+}\left|\frac{Ma\log(Ma)}{a\log a}\right| = M, \tag{142}$$

and the continuity of the function $f(a) = Ma\log(Ma)$ on compact subsets of $(0,\infty)$, we obtain (138). The bound (139) follows from (138). The nondecreasing property of the logarithm yields

$$(x_1+\dots+x_n)\log(x_1+\dots+x_n) \le \left(\max_{1\le i\le n} nx_i\right)\log\left(\max_{1\le i\le n} nx_i\right) \le \sum_{i=1}^{n} nx_i\log(nx_i) \tag{143}$$

for all $x_1,\dots,x_n \ge 0$. This gives (140).

Lemma 5. *Let $T > 0$. Suppose $F(x,t)$ has mean zero over $\mathbb{T}^2$ and satisfies*

$$\int_{\mathbb{T}^2} |F(x,t)|\log|F(x,t)|dx \le C \tag{144}$$

for all $t \in [0,T]$ where C depends only on T and universal constants. Let $v(x,t)$ be the solution of

$$-\Delta v = F \tag{145}$$

with periodic boundary conditions. Then there exists a constant $C_3 > 0$ such that

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{T}^2}|v(x)| \le C_3 \tag{146}$$

holds.

Proof: The solution v is given by

$$v(x,t) = \int_{\mathbb{T}^2} \mathcal{N}(x-y)F(y,t)dy \tag{147}$$

where $\mathcal{N}$ is the Newtonian potential solving the $2D$ Laplace equation with periodic boundary conditions. We note that

$$|\mathcal{N}(x-y)| \le C_4 + C_5|\log|x-y|| \tag{148}$$

for all $x,y \in \mathbb{T}^2$. Indeed, if $\chi(x)$ is a smooth compactly supported function in $|x| \le 1$ that is identically 1 in $|x| \le 1/2$, and if $\Psi(x)$ is the function defined by

$$\Psi(x) = \frac{1}{2\pi}\chi(x)\log(|x|), \tag{149}$$

then it can be shown that

$$\Delta(\Psi - \chi_1) = \delta(x) - \frac{1}{4\pi^2} \tag{150}$$

for $x \in [-\pi,\pi]^2$, with δ the Dirac distribution at the origin and χ_1 a smooth 2π-periodic function. The function χ_1 is obtained using the Poisson summation formula [9] for the $C_0^\infty(\mathbb{R}^2)$ function $\psi(x) = \frac{1}{2\pi}(2\nabla\chi(x)\cdot\nabla\log(|x|) + \Delta\chi(x)\log(|x|))$,

$$\sum_{n\in\mathbb{Z}^2} \psi(x+2\pi n) = \frac{1}{4\pi^2}\sum_{n\in\mathbb{Z}^2} \widehat{\psi}(n)e^{in\cdot x} \tag{151}$$

where $\widehat{\psi}$ is the Fourier transform of ψ in $\mathbb{R}^2$. Namely, we observe that the integral $\int_{\mathbb{R}^2}\psi(x)dx = -1$ and set

$$\chi_1(x) = -\frac{1}{4\pi^2}\sum_{n\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{|n|^2}\widehat{\psi}(n)e^{in\cdot x}. \tag{152}$$

Integrating (150) against the mean zero function F over the torus $\mathbb{T}^2$ shows that the Newtonian potential obeys

$$\mathcal{N} = \Psi - \chi_1 \tag{153}$$

yielding the estimate (148).

In view of the estimate

$$|\log|x-y|||F(y,t)| \le |F(y,t)|\log|F(y,t)| - |F(y,t)| + e^{|\log|x-y||} \le |F(y,t)|\log|F(y,t)| + e^{|\log|x-y||} \tag{154}$$

that holds for all $x,y \in \mathbb{T}^2$ and $t \in [0,T]$, and using the assumption (144), we obtain (146).

Now we prove Proposition 2.

Proof of Proposition 2: The proof is divided into four steps. Throughout the proof, Γ_i denotes a constant depending only on $T, \|u_0\|_{L^2}, \mathcal{E}(0), \|c_i(0)\|_{L^2}, N, f$, the parameters of the problem and some universal constants. We recall that the ionic concentrations $c_i(x,t)$ are nonnegative for all $t \in [0,T]$..

Step 1: Energy Bounds. We define the energy

$$\mathcal{E}(t) = \int_{\mathbb{T}^2} E(x,t)dx \tag{155}$$

where

$$E(x,t) = \sum_{i=1}^n (c_i\log(c_i) - c_i + 1) + \frac{1}{2}(\rho+N)\Phi. \tag{156}$$

We note that $\mathcal{E}(t) \ge 0$ for all $t \ge 0$. This follows from the inequality $x\log(x) - x + 1 \ge 0$ that holds for all $x \ge 0$, and from the fact that

$$\int_{\mathbb{T}^2}(\rho+N)\Phi dx = -\epsilon\int_{\mathbb{T}^2}\Phi\Delta\Phi dx = \epsilon\int_{\mathbb{T}^2}|\nabla\Phi|^2 dx \ge 0. \tag{157}$$

The densities of the first variation of $\mathcal{E}$ are given by

$$\frac{\delta\mathcal{E}}{\delta c_i} = \log c_i + z_i\Phi \tag{158}$$

and hence the ionic concentrations evolve according to

$$\partial_t c_i + u\cdot\nabla c_i = D_i\nabla\cdot(c_i\nabla(\log c_i + z_i\Phi)) = D_i\nabla\cdot\left(c_i\nabla\frac{\delta\mathcal{E}}{\delta c_i}\right). \tag{159}$$

for all $i \in \{1,\dots,n\}$. Let $D_t = \partial_t + u\cdot\nabla$ be the material derivative with respect to u. Then

$$D_t\left(\sum_{i=1}^n(c_i\log c_i - c_i)\right) = \sum_{i=1}^n \log c_i D_t c_i = \sum_{i=1}^n \frac{\delta\mathcal{E}}{\delta c_i}D_t c_i - \Phi\sum_{i=1}^n z_i D_t c_i = \sum_{i=1}^n \frac{\delta\mathcal{E}}{\delta c_i}D_t c_i - \Phi D_t\rho \tag{160}$$

and so

$$D_t E = \sum_{i=1}^{n} \frac{\delta \mathcal{E}}{\delta c_i} D_t c_i - \Phi D_t \rho + \frac{1}{2} D_t((\rho + N)\Phi). \tag{161}$$

Integrating in the space variable over the torus $\mathbb{T}^2$ and using the divergence free condition for the velocity u, we obtain

$$\frac{d}{dt}\mathcal{E} = \sum_{i=1}^{n} \int_{\mathbb{T}^2} \frac{\delta \mathcal{E}}{\delta c_i} D_i \nabla \cdot \left(c_i \nabla \frac{\delta \mathcal{E}}{\delta c_i} \right) dx - \int_{\mathbb{T}^2} \Phi D_t \rho dx + \frac{1}{2} \int_{\mathbb{T}^2} \partial_t \left((\rho + N)\Phi \right) dx. \tag{162}$$

In view of the self-adjointness of $-\Delta$ and the fact that N is time indepedent, we have

$$\frac{1}{2} \int_{\mathbb{T}^2} \partial_t \left((\rho + N)\nabla\Phi \right) dx = \frac{1}{2} \int_{\mathbb{T}^2} \Phi \partial_t (\rho + N) dx + \frac{1}{2} \int_{\mathbb{T}^2} (\rho + N) \partial_t \Phi dx = \int_{\mathbb{T}^2} \Phi \partial_t \rho dx, \tag{163}$$

hence

$$\begin{aligned} \frac{d}{dt}\mathcal{E} &= \sum_{i=1}^{n} \int_{\mathbb{T}^2} \frac{\delta \mathcal{E}}{\delta c_i} D_i \nabla \cdot \left(c_i \nabla \frac{\delta \mathcal{E}}{\delta c_i} \right) dx - \int_{\mathbb{T}^2} \Phi u \cdot \nabla \rho dx \\ &= \sum_{i=1}^{n} \int_{\mathbb{T}^2} \frac{\delta \mathcal{E}}{\delta c_i} D_i \nabla \cdot \left(c_i \nabla \frac{\delta \mathcal{E}}{\delta c_i} \right) dx + \int_{\mathbb{T}^2} \rho \nabla \Phi \cdot u dx. \end{aligned} \tag{164}$$

Now we take the L^2 inner product of the equation obeyed the velocity u in (7) with u to obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \nu \|\nabla u\|_{L^2}^2 = - \int_{\mathbb{T}^2} \rho \nabla \Phi \cdot u dx - \int_{\mathbb{T}^2} N \nabla \Phi \cdot u dx + \int_{\mathbb{T}^2} f u dx \tag{165}$$

which is equivalent to

$$\frac{d}{dt} \left\{ \frac{1}{2} \|u\|_{L^2}^2 + \mathcal{E} \right\} + \nu \|\nabla u\|_{L^2}^2 + \mathcal{D} = - \int_{\mathbb{T}^2} N \nabla \Phi \cdot u dx + \int_{\mathbb{T}^2} f u dx \tag{166}$$

in view of (164), where

$$\mathcal{D} = \sum_{i=1}^{n} D_i \int_{\mathbb{T}^2} c_i \left| \nabla \frac{\delta \mathcal{E}}{\delta c_i} \right|^2 dx. \tag{167}$$

Using Hölder's inequality, we have

$$\left| \int_{\mathbb{T}^2} N \nabla \Phi \cdot u dx \right| \le \|N\|_{L^\infty} \|\nabla \Phi\|_{L^2} \|u\|_{L^2} \tag{168}$$

and

$$\left| \int_{\mathbb{T}^2} f u dx \right| \le \|f\|_{L^2} \|u\|_{L^2} \tag{169}$$

which, after applying Young's inequality, yields the differential inequality

$$\frac{d}{dt} \left\{ \frac{1}{2} \|u\|_{L^2}^2 + \mathcal{E} \right\} + \nu \|\nabla u\|_{L^2}^2 + \mathcal{D} \le C \|N\|_{L^\infty} \left\{ \frac{1}{2} \|u\|_{L^2}^2 + \mathcal{E} \right\} + \frac{1}{2} \|f\|_{L^2}^2. \tag{170}$$

Therefore,

$$\|u\|_{L^2}^2 + \mathcal{E} + \int_0^T \left(\|\nabla u\|_{L^2}^2 + \mathcal{D} \right) dt \le \Gamma_0. \tag{171}$$

This ends the proof of Step 1.

Step 2: Bounds for Φ in $L^\infty(0,T;L^\infty)$. Fix $i \in \{1, \dots, n\}$. As a consequence of the energy bound (171), we have

$$\int_{\mathbb{T}^2} |c_i \log c_i - c_i + 1| dx = \int_{\mathbb{T}^2} (c_i \log c_i - c_i + 1) dx \le \Gamma_0. \tag{172}$$

By the triangle inequality we obtain

$$\int_{\mathbb{T}^2} |c_i \log c_i - c_i| dx \le \Gamma_0 + 2(2\pi)^2 \tag{173}$$

and so

$$\int_{\mathbb{T}^2} \left| \frac{c_i}{e} \log \frac{c_i}{e} \right| dx \le \frac{1}{e} \left(\Gamma_0 + 2(2\pi)^2 \right). \tag{174}$$

Using Lemma 4, we conclude that

$$\int_{\mathbb{T}^2} |c_i \log c_i| dx \le \Gamma_1. \tag{175}$$

Now we estimate

$$\begin{aligned}\int_{\mathbb{T}^2} |\rho + N| \log(|\rho + N|) &\le \int_{\mathbb{T}^2} \left(|N| + \sum_{i=1}^n |z_i c_i|\right) \log\left(|N| + \sum_{i=1}^n |z_i c_i|\right) dx \\ &\le \int_{\mathbb{T}^2} \sum_{i=1}^n (n+1)|z_i c_i| \log(|z_i c_i|) dx + \int_{\mathbb{T}^2} (n+1)|N| \log|N| dx \\ &\le \Gamma_2 + \Gamma_3 \int_{\mathbb{T}^2} \sum_{i=1}^n |c_i| \log|c_i| dx \\ &\le \Gamma_4 \end{aligned} \tag{176}$$

by several applications of Lemma 4. In view of Lemma 5, we conclude that

$$\sup_{0 \le t \le T} \sup_{x \in \mathbb{T}^2} |\Phi(x)| \le \Gamma_5. \tag{177}$$

This finishes the proof of Step 2.

Step 3: Bounds for c_i in $L^2(0,T;L^2)$. We consider the auxiliary functions

$$\widetilde{c_i}(x,t) = c_i(x,t) e^{z_i \Phi(x,t)} \tag{178}$$

for $i \in \{1, \dots, n\}$, and we note that

$$\mathcal{D}(t) = \sum_{i=1}^n D_i \int_{\mathbb{T}^2} \frac{\widetilde{c_i}}{e^{z_i \Phi}} |\nabla \log \widetilde{c_i}|^2 dx. \tag{179}$$

Using the uniform in time boundedness of Φ in $L^\infty(0,T;L^\infty)$ given by (177) and the fact that

$$\int_0^T \mathcal{D}(t) \le \Gamma_0, \tag{180}$$

we obtain the bound

$$\int_0^T \int_{\mathbb{T}^2} \widetilde{c_i}^{-1} |\nabla \widetilde{c_i}|^2 dx dt \le \Gamma_6 \tag{181}$$

which implies that $\nabla \sqrt{\widetilde{c_i}} \in L^2(0,T;L^2)$. We note that

$$\|c_i(t)\|_{L^1} = \int_{\mathbb{T}^2} c_i(x,t) dx = \|c_i(0)\|_{L^1} \tag{182}$$

for all $t \in [0,T]$, and so

$$\|\sqrt{\widetilde{c_i}}\|_{L^2} = \left(\int_{\mathbb{T}^2} c_i e^{z_i \Phi}\right)^{1/2} \le \Gamma_7 \left(\int_{\mathbb{T}^2} c_i\right)^{1/2} \le \Gamma_8 \tag{183}$$

in view of (177). Therefore, we obtain

$$\int_0^T \|\sqrt{\widetilde{c_i}(t)}\|_{H^1}^2 \le \Gamma_9. \tag{184}$$

In view of Ladyzhenskaya's interpolation inequality, we have

$$\|\sqrt{\widetilde{c_i}}\|_{L^4}^4 \le C \|\sqrt{\widetilde{c_i}}\|_{L^2}^2 \|\sqrt{\widetilde{c_i}}\|_{H^1}^2 \tag{185}$$

and hence

$$\int_0^T \|\widetilde{c_i}\|_{L^2}^2 dx dt \le \Gamma_{10}. \tag{186}$$

This gives bounds for the ionic concentrations in $L^2(0,T;L^2)$, that is

$$\int_0^T \|c_i\|_{L^2}^2 dx dt \le \Gamma_{11}. \tag{187}$$

Therefore, Step 3 is completed.

Step 4: Bounds for c_i in $L^\infty(0,T;L^2)$. For each $i \in \{1, \dots, n\}$, we take the L^2 inner product of the equation obeyed by c_i in (7) with c_i and we obtain

$$\frac{1}{2} \frac{d}{dt} \|c_i\|_{L^2}^2 + D_i \|\nabla c_i\|_{L^2}^2 = -D_i z_i \int_{\mathbb{T}^2} z_i c_i \nabla \Phi \cdot \nabla c_i. \tag{188}$$

We estimate

$$\left|D_i z_i \int_{\mathbb{T}^2} z_i c_i \nabla\Phi \cdot \nabla c_i\right| \le C\|\nabla\Phi\|_{L^4}\|\nabla c_i\|_{L^2}\|c_i\|_{L^4}$$
$$\le C\|\nabla\Phi\|_{L^2}^{1/2}\|\rho + N\|_{L^2}^{1/2}\|\nabla c_i\|_{L^2}\|c_i\|_{L^2}^{1/2}\|c_i\|_{H^1}^{1/2}$$
$$\le \frac{D_i}{2}\|\nabla c_i\|_{L^2}^2 + \Gamma_{12}\sum_{j=1}^n \|c_j\|_{L^2}^4 + C_N \tag{189}$$

where C_N is a constant depending only on N and the parameters of the problem. Here, we used Hölder's inequality with exponents $4, 2, 4$, followed by an application of Ladyzhenskaya's interpolation inequality. We have also used the fact that $\nabla\Phi$ is bounded in $L^\infty(0,T;L^2)$ which follows from the boundedness of the energy (171). This yields the differential inequality

$$\frac{d}{dt}\sum_{i=1}^n \|c_i\|_{L^2}^2 + \sum_{i=1}^n D_i\|\nabla c_i\|_{L^2}^2 \le \Gamma_{13}\sum_{i=1}^n \|c_i\|_{L^2}^2 + \Gamma_{14} \tag{190}$$

which allows us to conclude that

$$\sup_{0\le t\le T}\sum_{i=1}^n \|c_i(t)\|_{L^2}^2 \le \Gamma_{15}. \tag{191}$$

This ends the proof of Proposition 2.

6. Conflict of interest

The authors state that there is no conflict of interest.

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