LONG TIME FINITE DIMENSIONALITY IN CHARGED FLUIDS

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ABSTRACT. We consider long time dynamics of solutions of 2D periodic Nernst-Planck-Navier-Stokes systems forced by body charges and body forces. We show that, in the absence of body charges, but in the presence of fluid body forces, the charge density of the ions converges exponentially in time to zero, and the ion concentrations converge exponentially in time to equal time independent constants. This happens while the fluid continues to be dynamically active for all time. In the general case of body charges and body forces, the solutions converge in time to an invariant finite dimensional compact set in phase space.

1. INTRODUCTION

Electrodiffusion of ions in fluids, described by the Nernst-Planck-Navier-Stokes (NPNS) equations [12], is a broad subject, extensively studied in the chemical-physics, bio-physics and engineering literature. From mathematical point of view, the Nernst-Planck system without added charges and without fluid possesses global smooth solutions which converge to unique stationary states in bounded domains in two dimensions [2, 3, 10]. These results are obtained in situations in which boundaries are impermeable to the ions, where the relevant blocking boundary conditions require the vanishing of the normal fluxes of ions through the boundary. The NPNS system with blocking boundary conditions and with no applied voltage at the boundary is globally well posed in 2D [15]. Furthermore, the NPNS system was proved to have globally smooth and stable solutions in 2D with blocking boundary conditions and nonzero applied voltage [5]. In [16], weak solutions in three dimensions were shown to exist for homogeneous Neumann boundary conditions for the potential. Recently, in [11], the authors established the existence of weak solutions in the whole space, $\Omega = \mathbb{R}^3$. All these results concern situations without forcing in which there is a unique stable stationary solution.

Numerical simulations [14, 17] and experiments [13] show that instabilities occur in regimes when the system is forced. The lack of stability was suggested to lead to chaotic, and even turbulent behavior [9], analogous to fluid turbulence.

In this paper we consider the issue of long time dynamics of solutions of the NPNS system with forcing of two kinds: added charges and fluid body forces. Two ionic species, with concentrations c_1 and c_2, with valences $z_1 = 1$ and $z_2 = -1$ respectively, and with equal diffusivities $D > 0$, evolve according the Nernst-Planck equations

$$(\partial_t + u \cdot \nabla)c_i = D\mathrm{div}(\nabla c_i + z_i c_i \nabla\Phi), \tag{1}$$

$i = 1, 2$. The ionic species concentrations $c_i(x,t)$ are nonnegative functions of the two variables, position x and time t. The potential Φ obeys the Poisson equation

$$-\epsilon\Delta\Phi = \rho + N \tag{2}$$

driven by the charge density

$$\rho = c_1 - c_2 \tag{3}$$

and by the added charge density N, which we take smooth and time independent. The constant $\epsilon > 0$ is proportional to the square of the Debye length. The velocity u of the fluid obeys the Navier-Stokes equations

$$\partial_t u + u \cdot \nabla u - \nu\Delta u + \nabla p = -(\rho + N)\nabla\Phi + f \tag{4}$$

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with the divergence free condition

$$\nabla \cdot u = 0. \tag{5}$$

The variable p represents the pressure. The positive constant ν is the kinematic viscosity. The body forces f are time independent, smooth, and divergence free. We consider the NPNS system in the two dimensional periodic domain

$$\mathbb{T}^2 = [-\pi, \pi] \times [-\pi, \pi] \tag{6}$$

with periodic boundary conditions.

Our main results are as follows. In the absence of forcing of any kind ($f = N = 0$), we prove that solutions are global and regular. The velocity converges exponentially in time to zero, the concentrations converge exponentially in time to equal constant values and the charge density converges exponentially in time to zero. In the case of body forces, but in the absence of added charge densities ($f \neq 0$, $N = 0$), we prove that the solutions are global, regular and the ionic concentrations still converge exponentially in time to equal constant values, while the charge density converges exponentially in time to zero. This is interesting in view of the fact that the Navier-Stokes evolution is forced and the velocity does not cease to be dynamically active. In all cases of forced equations, including $f \neq 0$ and $N \neq 0$, we prove that all solutions converge in time to a global attractor, which is an invariant compact set in phase space with finite Hausdorff and fractal dimension.

The paper is organized as follows. Section 2 is devoted to preliminaries. We describe the asymptotic behavior of eigenvalues of the dissipative operator $\mathcal{A} = (\nu A, -D\Delta, -D\Delta)$, where A is the Stokes operator and Δ is the Laplacian. We recall a Gronwall lemma that gives, under suitable assumptions, exponentially decaying bounds. In section 3, we prove, as in [6], that

$$\int_0^T \int_{\mathbb{T}^2} \left(|c_1(x)|^2 + |c_2(x)|^2\right) dx dt < \infty, \tag{7}$$

for all $T > 0$ is a necessary and sufficient condition for the persistence of global regular solutions of the NPNS system (1)–(5). Under condition (7), the nonnegativity of the initial ionic concentrations is preserved for all positive times. In section 4, we discuss the case where no body forces f are present in the fluid and no added charge densities N take part in generating the electric field. We prove that the concentrations decay exponentially in all L^p spaces ($p \in [2; \infty)$) independently of the velocity u, implying, together with the exponential decay of the L^p norms of u, the existence of a single point attractor. We prove further that the solutions decay exponentially in H^2. In section 5, we consider added body forces, and we establish that the concentrations converge exponentially to equal constant steady states, and the charge density vanishes in the limit of large times. We address the evolution of the system in a phase space corresponding to strong solutions (H^1). We show that there exists a compact set (a ball in a the stronger norm H^2) which is an absorbing ball. This means that starting from any initial data w_0 in phase space, there exists a time t_0, depending locally uniformly on the norm of the initial data in the phase space, such that solution $\mathcal{S}(t)w_0$ belongs to the absorbing ball for times larger than t_0. We study further the properties of the nonlinear solution map $\mathcal{S}(t)$ corresponding to the NPNS system. We establish Lipschitz continuity of $\mathcal{S}(t)$ in various norms, including a smoothing property for positive times (see Theorem 5). We prove the injectivity of the solution map $\mathcal{S}(t)$ in Appendix A. Exponential decay of volume elements is proved in Appendix B. The existence of a finite dimensional global attractor is thus established for the case $N = 0$, $f \neq 0$. The global attractor is a set which is invariant under the solution map, and such that all solutions converge to it as time tends to infinity. In section 6 we treat the general case with an added charge density N. In this case the concentrations and the charge density are no longer convergent in time, but we still obtain the properties of existence of a compact absorbing ball, Lipschitz continuity and smoothing properties of the solution map. The injectivity and decay of volume elements are valid as well, and we obtain the existence of a global attractor with finite Hausdorff and fractal dimension.

2. Preliminaries

We consider the Hilbert space

$$\mathcal{H} = H \oplus L^2 \oplus L^2 \tag{8}$$

where H is the space of L^2 periodic vector fields which are divergence free and have mean zero. We define

$$\mathcal{A}w = (\nu Au, -D\Delta c_1, -D\Delta c_2) \tag{9}$$

where Δ is the Laplacian operator with periodic boundary conditions on $\mathbb{T}^2$, and $A = \mathbb{P}(-\Delta)$ is the Stokes operator. Here, $\mathbb{P}$ denotes the Leray-Hopf projector. We recall that $\mathbb{P}$ and $-\Delta$ commute on $\mathbb{T}^2$. The domain of definition of $\mathcal{A}$ is

$$\mathcal{D}(\mathcal{A}) = (H^2 \cap H) \oplus H^2 \oplus H^2. \tag{10}$$

By the spectral theorem for Hilbert spaces, and since 0 is not an eigenvalue, there is an orthonormal basis of $\mathcal{H}$ formed by a sequence of eigenvectors ω_k of $\mathcal{A}$ with corresponding eigenvalues μ_k counted with multiplicity such that $0 < \mu_1 \le \mu_2 \le \cdots \le \mu_k \to \infty$.

Proposition 1. *There exists a constant $C > 0$ such that $\mu_k \ge Ck$ for all $k \ge 1$.*

Proof. We denote by $\{\lambda_j\}$ the eigenvalues of $-\Delta$ with periodic boundary conditions on $\mathbb{T}^2$ counted with multiplicity, $0 < \lambda_1 \le \lambda_2 \le \dots$. There exists a constant $c > 0$ such that $j \le c\lambda_j$ for all $j \in \mathbb{N}$, and $\{\nu\lambda_j\}$ and $\{D\lambda_j\}$ are the eigenvalues of νA and $D(-\Delta)$ respectively counted with multiplicity. We write

$$\{\mu_i : i = 1, \dots, N\} = \{\nu\lambda_i : i = 1, \dots, j\} \cup \{D\lambda_i : i = 1, \dots, k\} \tag{11}$$

and we note that if $\mu_N = \nu\lambda_j$, then $j \le \frac{c}{\nu}\mu_N$, whereas if $\mu_N = D\lambda_k$, then $k \le \frac{c}{D}\mu_N$. Consequently, $N = j + k \le c\left(\frac{1}{\nu} + \frac{1}{D}\right)\mu_N$, which completes the proof of the lemma.

We use the following Poincaré inequality for L^p spaces [8]:

Proposition 2. *Let $p = 2m$, $m \ge 1$, $0 \le \alpha \le 2$, and let $q \in C^\infty$ have zero mean on $\mathbb{T}^2$. Then*

$$\int_{\mathbb{T}^2} q^{p-1}(x)\Lambda^\alpha q(x)dx \ge \frac{1}{p}\|\Lambda^{\alpha/2}(q^{p/2})\|_{L^2}^2 + \lambda\|q\|_{L^p}^p \tag{12}$$

holds, with an explicit constant $\lambda > 0$, which is independent of p.

We recall a uniform Gronwall lemma [1].

Lemma 1. *Let $y(t) \ge 0$ obey a differential inequality*

$$\frac{d}{dt}y + c_1 y \le F_1 + F(t)$$

with initial datum $y(0) = y_0$ with F_1 a nonnegative constant, and $F(t) \ge 0$ obeying

$$\int_t^{t+1} F(s)ds \le g_0 e^{-c_2 t} + F_2$$

where c_1, c_2, g_0 are positive constants and F_2 is a nonnegative constant. Then

$$y(t) \le y_0 e^{-c_1 t} + g_0 e^{c_1 + c}(t+1)e^{-ct} + \frac{1}{c_1}F_1 + \frac{e^{c_1}}{1 - e^{-c_1}}F_2$$

holds with $c = \min\{c_1, c_2\}$.

3. Existence and uniqueness of solutions

We consider the system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u - (\rho + N)\nabla \Phi + f \\ \nabla \cdot u = 0 \\ \rho = c_1 - c_2 \\ -\epsilon \Delta \Phi = \rho + N \\ \partial_t c_1 + u \cdot \nabla c_1 = D\Delta c_1 + D\nabla \cdot (c_1 \nabla \Phi) \\ \partial_t c_2 + u \cdot \nabla c_2 = D\Delta c_2 - D\nabla \cdot (c_2 \nabla \Phi) \end{cases} \tag{13}$$

in $\mathbb{T}^2 \times [0, \infty)$, where ν, D, ϵ are positive constants. The body forces f are smooth, divergence free, time independent, and have mean zero. The added charge density N is smooth and time independent. We assume that the initial fluid velocity u_0 has mean zero. We also assume that the initial concentrations $c_1(x,0)$ and $c_2(x,0)$ have space averages $\bar{c}_1$ and $\bar{c}_2$ satisfying

$$\bar{c}_2 - \bar{c}_1 = \bar{N} \tag{14}$$

where $\bar{N}$ is the space average of the charge density N.

Remark 1. *We note that ρ maintains a space average equal to $-\bar{N}$ whereas u maintains a space average equal to zero for all $t \geq 0$. This follows by integrating the equations satisfied by ρ and u and by using*

$$\int (\rho + N)\nabla \Phi = -\frac{1}{\epsilon}\int (\rho + N)\nabla \Lambda^{-2}(\rho + N) = -\frac{1}{\epsilon}\int \Lambda^{-1/2}(\rho + N) R \Lambda^{-1/2}(\rho + N) = 0 \tag{15}$$

where the last equality holds because the Riesz operator $R = \nabla \Lambda^{-1}$ is antisymmetric.

We use the following convention regarding constants: we denote by C a positive constant that might depend on the parameters of the problem or universal constants, C_N a positive constant depending, in addition, on the charge density N. Following the same pattern, we denote by $C_{N,f}$ a constant depending on N and f. These constants may change from line to line along the proofs.

Theorem 1. *(Local Solution) Suppose $u_0 \in H^1$ and $c_i(0) \in L^2$. Then, there exists T_0 depending on $\|u_0\|_{H^1}, \|c_i(0)\|_{L^2}$ and the parameters of the problem such that system* (13) *has a unique solution such that $u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$ and $c_i \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ on $[0,T_0]$.*

Proof. We start by taking the L^2 inner product of the equation satisfied by u with $-\Delta u$. We use the identity

$$Tr(G^T G^2) = 0 \tag{16}$$

for the two-by-two traceless matrix G with entries $G_{ij} = \frac{\partial u_i}{\partial x_j}$, and we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 = \int (\rho + N)\nabla \Phi \cdot \Delta u - \int f \Delta u. \tag{17}$$

In view of Hölder's, Ladyzhenskaya's, Poincaré's, and Young's inequalities, we have

$$\begin{aligned} \int (\rho + N)\nabla \Phi \cdot \Delta u &\leq \|\rho + N\|_{L^2}\|\nabla \Phi\|_{L^\infty}\|\Delta u\|_{L^2} \\ &\leq C\|\Delta u\|_{L^2}[\|\rho\|_{L^2} + \|N\|_{L^2}][\|\rho\|_{L^4} + \|N\|_{L^4}] \\ &\leq \frac{\nu}{4}\|\Delta u\|_{L^2}^2 + \frac{D}{8}\|\nabla \rho\|_{L^2}^2 + C\|\rho\|_{L^2}^6 + C_N. \end{aligned} \tag{18}$$

and consequently, we obtain the differential inequality

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \nu \|\Delta u\|_{L^2}^2 \leq \frac{D}{4}\|\nabla \rho\|_{L^2}^2 + C\|\rho\|_{L^2}^6 + C_{N,f} \tag{19}$$

Let $\sigma = c_1 + c_2$. Then, σ and ρ obey the system

$$\begin{cases} \partial_t \sigma + u \cdot \nabla \sigma = D\Delta \sigma + D\nabla \cdot (\rho \nabla \Phi) \\ \partial_t \rho + u \cdot \nabla \rho = D\Delta \rho + D\nabla \cdot (\sigma \nabla \Phi). \end{cases} \tag{20}$$

Taking the L^2 inner product of the first equation with σ and of the second equation with ρ, adding them, and noting that

$$\left| \int \rho \Delta \Phi \sigma \right| \le C \|\rho\|_{L^4} \|\sigma\|_{L^4} \|\rho + N\|_{L^2} \le \frac{D}{2} [\|\nabla \rho\|_{L^2}^2 + \|\nabla \sigma\|_{L^2}^2] + C\|\sigma\|_{L^2}^4 + C\|\rho\|_{L^2}^4 + C_N \tag{21}$$

by Ladyzhenskaya's and Young's inequalities, we obtain the differential inequality

$$\frac{d}{dt}(\|\sigma\|_{L^2}^2 + \|\rho\|_{L^2}^2) + D(\|\nabla \sigma\|_{L^2}^2 + \|\nabla \rho\|_{L^2}^2) \le C[\|\sigma\|_{L^2}^4 + \|\rho\|_{L^2}^4] + C_N. \tag{22}$$

Let

$$M(t) = \|\nabla u\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|\sigma\|_{L^2}^2. \tag{23}$$

Adding (22) to (19), we obtain

$$M'(t) + \frac{D}{2}(\|\nabla \sigma(t)\|_{L^2}^2 + \|\nabla \rho(t)\|_{L^2}^2) + \nu \|\Delta u(t)\|_{L^2}^2 \le CM(t)^3 + C_{N,f}. \tag{24}$$

This latter differential inequality gives short time control of the desired norms. For uniqueness, suppose (u_1, c_1^1, c_2^1) and (u_2, c_1^2, c_2^2) are two solutions of (13). Let $\rho_1 = c_1^1 - c_2^1$, $\rho_2 = c_1^2 - c_2^2$, $\sigma_1 = c_1^1 + c_2^1$, $\sigma_2 = c_1^2 + c_2^2$. We write $u = u_1 - u_2$, $\rho = \rho_1 - \rho_2$ and $\sigma = \sigma_1 - \sigma_2$. Then u, ρ and σ obey the system

$$\begin{cases} \partial_t u + u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2 + \nabla(p_1 - p_2) = \nu \Delta u - [\rho_1 \nabla \Phi_1 - \rho_2 \nabla \Phi_2] \\ \partial_t \rho + u_1 \cdot \nabla \rho_1 - u_2 \cdot \nabla \rho_2 = D\Delta \rho + D\nabla \cdot (\sigma_1 \nabla \Phi_1 - \sigma_2 \nabla \Phi_2) \\ \partial_t \sigma + u_1 \cdot \nabla \sigma_1 - u_2 \cdot \nabla \sigma_2 = D\Delta \sigma + D\nabla \cdot (\rho_1 \nabla \Phi_1 - \rho_2 \nabla \Phi_2) \end{cases} \tag{25}$$

We take the L^2 inner product of the first equation of (25) with u to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \nu \|\nabla u\|_{L^2}^2 = -\int (u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2) \cdot u \, dx - \int (\rho_1 \nabla \Phi_1 - \rho_2 \nabla \Phi_2) \cdot u \, dx. \tag{26}$$

We estimate the term

$$\begin{aligned} \left| \int (u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2) \cdot u \, dx \right| &= \left| \int [u \cdot \nabla u_1 + u_2 \cdot \nabla u] \cdot u \, dx \right| \\ &\le C\|u\|_{L^2}^{3/2} \|\nabla u\|_{L^2}^{1/2} \|\nabla u_1\|_{L^2}^{1/2} \|\Delta u_1\|_{L^2}^{1/2} \end{aligned} \tag{27}$$

using Ladyzhenskaya's inequality. In view of elliptic regularity

$$\|\nabla \Phi\|_{L^\infty} \le C\|\rho\|_{L^4}, \tag{28}$$

we have

$$\begin{aligned} \left| \int (\rho_1 \nabla \Phi_1 - \rho_2 \nabla \Phi_2) \cdot u \, dx \right| &= \left| \int [\rho \nabla \Phi_1 + \rho_2 \nabla \Phi] \cdot u \, dx \right| \\ &\le C[\|\nabla \Phi_1\|_{L^\infty} \|\rho\|_{L^2} \|u\|_{L^2} + \|\rho_2\|_{L^2} \|\rho\|_{L^2}^{1/2} \|\nabla \rho\|_{L^2}^{1/2} \|u\|_{L^2}]. \end{aligned} \tag{29}$$

Now, we take the L^2 inner product of the second equation of (25) with ρ, and we get

$$\frac{1}{2}\frac{d}{dt}\|\rho\|_{L^2}^2 + D\|\nabla \rho\|_{L^2}^2 = -\int (u_1 \cdot \nabla \rho_1 - u_2 \cdot \nabla \rho_2) \rho + D\int \nabla \cdot (\sigma_1 \nabla \Phi_1 - \sigma_2 \nabla \Phi_2)\rho. \tag{30}$$

We have

$$\begin{aligned} \left| \int (u_1 \cdot \nabla \rho_1 - u_2 \cdot \nabla \rho_2) \rho \right| &= \left| \int [u \cdot \nabla \rho_1 + u_2 \cdot \nabla \rho] \rho \right| \\ &\le C\|\nabla \rho_1\|_{L^2} \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2} \|\rho\|_{L^2}^{1/2} \|\nabla \rho\|_{L^2}^{1/2} \end{aligned} \tag{31}$$

and

$$\left| \int \nabla \cdot (\sigma_1 \nabla \Phi_1 - \sigma_2 \nabla \Phi_2) \rho \right| \le C \left[\|\nabla \Phi_1\|_{L^\infty} \|\sigma\|_{L^2} \|\nabla \rho\|_{L^2} + \|\sigma_2\|_{L^2} \|\rho\|_{L^2}^{1/2} \|\nabla \rho\|_{L^2}^{3/2} \right]. \tag{32}$$

Finally, we take the L^2 inner product of the third equation of (25) with σ to obtain

$$\frac{1}{2} \frac{d}{dt} \|\sigma\|_{L^2}^2 + D \|\nabla \sigma\|_{L^2}^2 = - \int (u_1 \cdot \nabla \sigma_1 - u_2 \cdot \nabla \sigma_2) \, \sigma + D \int \nabla \cdot (\rho_1 \nabla \Phi_1 - \rho_2 \nabla \Phi_2) \sigma. \tag{33}$$

We estimate the first term on the right hand side of (33) as in (31). For the second term, as in (32), we have

$$\left| \int \nabla \cdot (\rho_1 \nabla \Phi_1 - \rho_2 \nabla \Phi_2) \sigma \, dx \right| \le C \left[\|\nabla \Phi_1\|_{L^\infty} \|\rho\|_{L^2} \|\nabla \sigma\|_{L^2} + \|\rho_2\|_{L^2} \|\rho\|_{L^2}^{1/2} \|\nabla \rho\|_{L^2}^{1/2} \|\nabla \sigma\|_{L^2} \right]. \tag{34}$$

Putting (26)–(34) together, and applying Young's inequality, we obtain a differential inequality of the form

$$\frac{d}{dt} \left[\|u\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|\sigma\|_{L^2}^2 \right] \le C C(t) \left[\|u\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|\sigma\|_{L^2}^2 \right] \tag{35}$$

where

$$C(t) = \|\nabla u_1\|_{L^2}^{2/3} \|\Delta u_1\|_{L^2}^{2/3} + \|\nabla \rho_1\|_{L^2}^2 + \|\nabla \sigma_1\|_{L^2}^2 + \|\rho_1 + N\|_{L^3}^2 + \|\sigma_2\|_{L^2}^4 + \|\rho_2\|_{L^2}^4 + 1. \tag{36}$$

Since

$$\int_0^t C(s) ds < \infty. \tag{37}$$

for any $t \in [0, T_0]$, we obtain uniqueness.

Theorem 1 shows existence of local solutions. The calculations can be done rigorously using Galerkin approximations. Namely, we consider an orthonormal basis of L^2 consisting of the eigenfunctions $\{\Phi_k\}_{k=1}^\infty$ of the Stokes operator

$$- \Delta \Phi_k + \nabla \xi_k = \mu_k \Phi_k \tag{38}$$

with periodic boundary condition on $\mathbb{T}^2$, and such that

$$\nabla \cdot \Phi_k = 0 \qquad \forall k \in \mathbb{N}. \tag{39}$$

The functions Φ_k are C^∞, divergence free, and have mean zero. We also consider an orthonormal basis of L^2 consisting of the eigenfunctions $\{w_k\}_{k=1}^\infty$ of the Laplacian operator

$$- \Delta w_k = \lambda_k w_k \tag{40}$$

with periodic boundary condition on $\mathbb{T}^2$. The functions w_k are C^∞ and have mean zero. Let

$$u_n = \mathbb{P}_n u = \sum_{k=1}^n (u, \Phi_k)_H \Phi_k \tag{41}$$

and

$$c_n^i = \mathbb{P}_n c_i = \sum_{k=1}^n (c_i, w_k)_{L^2} w_k + \bar{c}_i = \sum_{k=0}^n (c_i, w_k)_{L^2} w_k \tag{42}$$

be the Galerkin approximations of u and c_i for $i \in \{1, 2\}$. Here, $\bar{c}_i$ is the constant average of c_i over $\mathbb{T}^2$, and $w_0 = 1/2\pi$. We fix m and n, and we replace u, c_1 and c_2 in (13) by u_n, c_n^1 and c_n^2 respectively. We test the equation for u_n with each of the functions Φ_i and the equations for c_n^1 and c_n^2 with each of the functions w_i. This gives a system of nonlinear ODE's for the coefficients of the Galerkin approximations. A solution of this latter system exists if it is bounded in some norm. To show that, we multiply the equations of this latter system by Φ_i and w_i correspondingly and we sum. We obtain the approximate system

$$\begin{cases} \partial_t u_n + \mathbb{P}_n(u_n \cdot \nabla u_n) - \nu\Delta u_n = -\mathbb{P}_n((\rho_n + \mathbb{P}_n N)\nabla\Phi_n) + \mathbb{P}_n f \\ \partial_t c_n^1 + \mathbb{P}_n(u_n \cdot \nabla c_n^1) - D\Delta c_n^1 = D\mathbb{P}_n(\nabla \cdot (c_n^1 \nabla\Phi_n)) \\ \partial_t c_n^2 + \mathbb{P}_n(u_n \cdot \nabla c_n^2) - D\Delta c_n^2 = D\mathbb{P}_n(\nabla \cdot (c_n^2 \nabla\Phi_n)) \\ -\epsilon\Delta\Phi_n = \rho_n + \mathbb{P}_n N \\ \rho_n = c_n^1 - c_n^2 \end{cases} \tag{43}$$

with $u_n(0) = \mathbb{P}_n u_0, c_n^i(0) = \mathbb{P}_n c_i(0), i = 1,2$. Since the Φ_k's and the w_k's are C^∞, the Galerkin appoximants are smooth functions, and so we can find a priori estimates by taking suitable scalar products in L^2 and integrating in time. Then, we pass to the limit via the Aubin-Lions lemma.

Theorem 2. *Let $u_0 \in H^1$ and $c_i(0) \in H^1$. Let $T > 0$. Suppose (u, c_1, c_2) solves* (13) *on the interval $[0,T]$ with*

$$\int_0^T (\|c_1(t)\|_{L^2}^2 + \|c_2(t)\|_{L^2}^2) dt < \infty. \tag{44}$$

Then, $u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$ and $c_i \in L^\infty(0,T;H^1) \cap L^2(0,T,H^2)$.

Proof. The following calculations can be done rigorously using Galerkin approximations.

The differential inequality (22) gives

$$\frac{d}{dt}(\|\sigma\|_{L^2}^2 + \|\rho\|_{L^2}^2) \le C(\|\sigma\|_{L^2}^2 + \|\rho\|_{L^2}^2)^2 + C_N. \tag{45}$$

Thus, under the assumption (44), we obtain that $c_i \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$. Moreover, the differential inequality (19) allows us to conclude that $u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$.

Now, we taking the L^2 inner product of the equation satisfied by ρ in (20) with $-\Delta\rho$, and we obtain the equation

$$\frac{1}{2}\frac{d}{dt}\|\nabla\rho\|_{L^2}^2 + D\|\Delta\rho\|_{L^2}^2 = \int (u \cdot \nabla\rho)\Delta\rho - D\int \nabla \cdot (\sigma\nabla\Phi)\Delta\rho. \tag{46}$$

We estimate

$$\left|\int \sigma\Delta\Phi\Delta\rho\right| \le \frac{1}{4}\|\Delta\rho\|_{L^2}^2 + C\|\sigma\|_{L^2}^2\|\nabla\sigma\|_{L^2}^2 + C\|\nabla\rho\|_{L^2}^4 + C_N, \tag{47}$$

$$\left|\int (\nabla\sigma \cdot \nabla\Phi)\Delta\rho\right| \le \frac{1}{4}\|\Delta\rho\|_{L^2}^2 + C\|\nabla\rho\|_{L^2}^4 + C\|\nabla\sigma\|_{L^2}^4 + C_N \tag{48}$$

and

$$\left|\int (u \cdot \nabla\rho)\Delta\rho\right| = \left|\int \nabla u \nabla\rho\nabla\rho\right| \le \frac{D}{4}\|\Delta\rho\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla\rho\|_{L^2}^2 \tag{49}$$

where we used elliptic regularity together with Ladyzhenskaya's inequality and Poincaré's inequality applied to the mean zero function $\rho + N$.

Finally, we take the L^2 inner product of the equation obeyed by σ in (20) with $-\Delta\sigma$ to get

$$\frac{1}{2}\frac{d}{dt}\|\nabla\sigma\|_{L^2}^2 + D\|\Delta\sigma\|_{L^2}^2 = \int (u \cdot \nabla\sigma)\Delta\sigma - D\int \nabla \cdot (\rho\nabla\Phi)\Delta\sigma \tag{50}$$

and proceeding in the same fashion as above, we obtain

$$\left|\int \rho\Delta\Phi\Delta\sigma\right| \le \frac{1}{4}\|\Delta\sigma\|_{L^2}^2 + C\|\rho\|_{L^2}^4 + C\|\nabla\rho\|_{L^2}^4 + C_N, \tag{51}$$

$$\left|\int (\nabla\rho \cdot \nabla\Phi)\Delta\sigma\right| \le \frac{1}{4}\|\Delta\sigma\|_{L^2}^2 + C\|\nabla\rho\|_{L^2}^4 + C_N \tag{52}$$

and

$$\left|\int (u\cdot\nabla\sigma)\Delta\sigma\right| \le \frac{D}{4}\|\Delta\sigma\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2\|\nabla\sigma\|_{L^2}^2. \tag{53}$$

Putting (46)–(53) together, we conclude that $c_i \in L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$ with bounds depending on the initial data and T.

Remark 2. *Note that if we assume that $u_0 \in H^2$ and $c_i(0)\in H^2$, then the regularity of the solutions is upgraded to $u\in L^\infty(0,T_0;H^2)\cap L^2(0,T_0;H^3)$ and $c_i\in L^\infty(0,T_0;H^2)\cap L^2(0,T_0,H^3)$.*

Remark 3. *Under the conditions of Theorem 2, if $c_i(0)\ge 0$, then $c_i(t)\ge 0$ for $0\le t\le T$ (see* [5]*).*

4. NPNS SYSTEM WITHOUT BODY FORCES NOR CHARGE DENSITIES

In this section, we treat the case where $f = N = 0$. We consider the system

$$\begin{cases}\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u - \rho\nabla\Phi\\ \nabla\cdot u = 0\\ \rho = c_1 - c_2\\ -\epsilon\Delta\Phi = \rho\\ \partial_t c_1 + u\cdot\nabla c_1 = D\Delta c_1 + D\nabla\cdot(c_1\nabla\Phi)\\ \partial_t c_2 + u\cdot\nabla c_2 = D\Delta c_2 - D\nabla\cdot(c_2\nabla\Phi)\end{cases} \tag{54}$$

in $\mathbb{T}^2\times[0,\infty)$. We prove global regularity and asymptotic behavior of solutions. We start with a priori L^2 bounds.

Proposition 3. *Let $u_0\in H, c_i(0)\in L^2$. We assume that $c_i(t)\ge 0$ holds for all $t\ge 0$. Then, there exists an absolute constant $C>0$ such that*

$$\|\sigma(t)-\bar\sigma\|_{L^2}^2 + \|\rho(t)\|_{L^2}^2 \le (2\|\sigma_0\|_{L^2}^2 + 2\|\bar\sigma\|_{L^2}^2 + \|\rho_0\|_{L^2}^2)e^{-2CDt} \tag{55}$$

holds for all $t\ge 0$. Moreover,

$$\int_t^{t+T}\left(\|\nabla\rho(s)\|_{L^2}^2 + \|\nabla\sigma(s)\|_{L^2}^2 + \frac{1}{\epsilon}\|\rho(s)\|_{L^3}^3\right)ds \le \frac{1}{2D}(2\|\sigma_0\|_{L^2}^2 + 2\|\bar\sigma\|_{L^2}^2 + \|\rho_0\|_{L^2}^2)Te^{-2CDt} \tag{56}$$

holds for any $t\ge 0, T>0$.

Proof. We recall that σ and ρ obey

$$\begin{cases}\partial_t\sigma + u\cdot\nabla\sigma = D\Delta\sigma + D\nabla\cdot(\rho\nabla\Phi)\\ \partial_t\rho + u\cdot\nabla\rho = D\Delta\rho + D\nabla\cdot(\sigma\nabla\Phi).\end{cases} \tag{57}$$

We take the L^2 inner product of the first equation of system (57) with σ and of the second equation with ρ, we add them and we use the fact that

$$\int \rho\Delta\Phi\sigma = -\frac{1}{\epsilon}\int\sigma(\rho)^2 \tag{58}$$

and that $c_i\ge 0$ for $i=1,2$, to obtain the differential inequality

$$\frac{d}{dt}(\|\sigma-\bar\sigma\|_{L^2}^2 + \|\rho\|_{L^2}^2) + 2D(\|\nabla\sigma\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2) + \frac{2D}{\epsilon}\|\rho\|_{L^3}^3 \le 0. \tag{59}$$

In view of Poincaré's inequality, we get (55). Going back to (59) and integrating, we obtain (56).

Theorem 3. *Let $u_0\in H^1$ be divergence free, and let $c_i(0)\in H^1$ be nonnegative $c_i(0)\ge 0$. Let $T>0$. Then there exists a unique solution (u,c_1,c_2) satisfying $u\in L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$ and $c_i\in L^\infty(0,T;H^1)\cap L^2(0,T,H^2)$. Moreover $c_i(t)\ge 0$ holds on $[0,T]$.*

Proof. By the local existence theorem (Theorem 1) there exists $T_0 > 0$ depending only on the norms of initial data in H^1 such that the solution exists and belongs to H^1. The condition (44) holds, and therefore, by Remark 3 $c_i(t) \geq 0$. The inequality (59) is valid on $[0, T_0]$. By Theorem 2 the solution is bounded in H^1. We apply the local existence theorem again, starting from T_0, and deduce that the solution can be extended for $T_1 > T_0$. The inequality (59) holds on $[0, T_1]$. Because the inequality (59) holds as long as $c_i \geq 0$, reasoning by contradiction we see that the solution extends to the whole interval $[0, T]$.

Corollary 1. *Under the assumptions of Proposition 3 there exists a positive constant $a = a(D, \nu)$ depending on D and ν, and a positive constant $A = A(\|\rho_0\|_{L^2}, \|\sigma_0\|_{L^2}, \|u_0\|_{L^2})$ depending on $\|\rho_0\|_{L^2}, \|\sigma_0\|_{L^2}, \|u_0\|_{L^2}$, the parameters of the problem and universal constants, such that*

$$\|u(t)\|_{L^2} \leq Ae^{-at} \tag{60}$$

holds for all $t \geq 0$.

Proof. We take the L^2 inner product of the first equation in (54) with u, and we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \nu\|\nabla u\|_{L^2}^2 = -\int \rho\nabla\Phi \cdot u. \tag{61}$$

We estimate

$$\left|\int \rho\nabla\Phi \cdot u\, dx\right| \leq \|\rho\|_{L^2}\|\nabla\Phi\|_{L^\infty}\|u\|_{L^2} \leq C\|\rho\|_{L^2}\|\rho\|_{L^3}\|u\|_{L^2} \tag{62}$$

and thus, we obtain the differential inequality

$$\frac{d}{dt}\|u\|_{L^2} + \nu\|u\|_{L^2} \leq C\|\rho\|_{L^2}\|\rho\|_{L^3}. \tag{63}$$

By Proposition 3 and Lemma 1, using (56), we obtain (60).

Remark 4. *In the case $f = N = 0$, the global attractor exists and is the singleton $(0, \bar\sigma/2, \bar\sigma/2)$. That is, for all initial data, the solution (u, c_1, c_2) converges to $(0, \bar\sigma/2, \bar\sigma/2)$.*

Proposition 4. *Let $u_0 \in H^1$ and $c_i(0) \in H^1$. Let $p > 2$. Then, there exist positive constants a_1, a_2 depending on D, ϵ, $\bar\sigma$, and λ (the constant in Proposition 2), and positive constants $C_1^p(\|\rho_0\|_{L^p}, \|\sigma_0\|_{L^2})$ and $C_2^p(\|\sigma_0\|_{L^p}, \|\rho_0\|_{L^2})$ depending on the corresponding initial data, $\bar\sigma$, p and universal constants, such that*

$$\|\rho(t)\|_{L^p} \leq C_1^p e^{-a_1 t} \tag{64}$$

and

$$\|\sigma(t) - \bar\sigma\|_{L^p} \leq C_2^p e^{-a_2 t} \tag{65}$$

hold for all $t \geq 0$.

Proof. The equation (57) for ρ is equivalent to

$$\partial_t\rho + u \cdot \nabla\rho + \frac{D\bar\sigma}{\epsilon}\rho - D\Delta\rho = D\nabla \cdot ((\sigma - \bar\sigma)\nabla\Phi). \tag{66}$$

Taking the L^2 inner product of equation (66) with $\rho|\rho|^{p-2}$ gives

$$\frac{1}{p}\frac{d}{dt}\|\rho\|_{L^p}^p + \frac{D\bar\sigma}{\epsilon}\|\rho\|_{L^p}^p + D(p-1)\int |\nabla\rho|^2|\rho|^{p-2}dx = -D(p-1)\int (\sigma - \bar\sigma)\nabla\Phi \cdot |\rho|^{p-2}\nabla\rho. \tag{67}$$

By Hölder's inequality with exponents 2, p, 2p/(p-2), followed by Young's inequality, we get

$$\begin{aligned}\left|\int (\sigma - \bar\sigma)\nabla\Phi \cdot |\rho|^{p-2}\nabla\rho\right| &\leq \|\nabla\Phi\|_{L^\infty}\||\rho|^{\frac{p-2}{2}}\nabla\rho\|_{L^2}\|\sigma - \bar\sigma\|_{L^p}\||\rho|^{\frac{p-2}{2}}\|_{L^{\frac{2p}{p-2}}} \\ &\leq \frac{1}{2}\||\rho|^{\frac{p-2}{2}}\nabla\rho\|_{L^2}^2 + \frac{1}{2}\|\nabla\Phi\|_{L^\infty}^2\|\sigma - \bar\sigma\|_{L^p}^2\|\rho\|_{L^p}^{p-2}.\end{aligned} \tag{68}$$

In view of the Gagliardo-Nirenberg inequality, we have

$$\|\sigma - \bar\sigma\|_{L^p} \leq C_p\|\sigma - \bar\sigma\|_{H^1}^{\frac{p-2}{p}}\|\sigma - \bar\sigma\|_{L^2}^{\frac{2}{p}} \tag{69}$$

where C_p is a constant that depends on p. Therefore, we get the differential inequality

$$\frac{d}{dt}\|\rho\|_{L^p}^2 + \frac{2D\bar{\sigma}}{\epsilon}\|\rho\|_{L^p}^2 \le C_p^2 D(p-1)\|\nabla\Phi\|_{L^\infty}^2 \|\sigma-\bar{\sigma}\|_{H^1}^{\frac{2(p-2)}{p}} \|\sigma-\bar{\sigma}\|_{L^2}^{\frac{4}{p}}. \tag{70}$$

If $p = 3$, then elliptic regularity, an application of Young's inequality with exponents 3, 3/2 and Poincaré inequality imply that

$$\|\nabla\Phi\|_{L^\infty}^2 \|\sigma-\bar{\sigma}\|_{H^1}^{\frac{2}{3}} \|\sigma-\bar{\sigma}\|_{L^2}^{\frac{4}{3}} \le C\left(\|\rho\|_{L^3}^3\|\sigma-\bar{\sigma}\|_{L^2} + \|\nabla\sigma\|_{L^2}^2\|\sigma-\bar{\sigma}\|_{L^2}^2\right). \tag{71}$$

In view of (56), (70) and Lemma 1, we obtain (64) for $p = 3$.

Now, we go back to the differential inequality (70). We estimate

$$\|\nabla\Phi\|_{L^\infty}^2 \|\sigma-\bar{\sigma}\|_{H^1}^{\frac{2(p-2)}{p}} \|\sigma-\bar{\sigma}\|_{L^2}^{\frac{4}{p}} \le C\|\rho\|_{L^3}^2\|\nabla\sigma\|_{L^2}^2 \tag{72}$$

where we have used elliptic regularity and Poincaré's inequality. Therefore, (70) and an application of Lemma 1 give (64) for any $p > 2$.

Next, we note that the equation satisfied by $\sigma - \bar{\sigma}$ is given by

$$\partial_t(\sigma-\bar{\sigma}) + u\cdot\nabla(\sigma-\bar{\sigma}) = D\Delta(\sigma-\bar{\sigma}) + D\nabla\cdot(\rho\nabla\Phi). \tag{73}$$

We take the L^2 inner product of equation (73) with $(\sigma-\bar{\sigma})|\sigma-\bar{\sigma}|^{p-2}$ and we get the equation

$$\frac{1}{p}\frac{d}{dt}\|\sigma-\bar{\sigma}\|_{L^p}^p - D\int |\sigma-\bar{\sigma}|^{p-2}(\sigma-\bar{\sigma})\Delta(\sigma-\bar{\sigma})dx = -D\int \rho\nabla\Phi\cdot\nabla((\sigma-\bar{\sigma})|\sigma-\bar{\sigma}|^{p-2})dx \tag{74}$$

By Hölder's inequality with exponents 2, p, 2p/(p-2), followed by Young's inequality, we obtain

$$\begin{aligned}\left|\int \rho\nabla\Phi\cdot\nabla((\sigma-\bar{\sigma})|\sigma-\bar{\sigma}|^{p-2})dx\right| &\le (p-1)\|\nabla\Phi\|_{L^\infty}\|\rho\|_{L^p}\||\sigma-\bar{\sigma}|^{\frac{p-2}{2}}\|_{L^{\frac{2p}{p-2}}}\||\sigma-\bar{\sigma}|^{\frac{p-2}{2}}\nabla(\sigma-\bar{\sigma})\|_{L^2} \\ &\le (p-1)\left[\frac{1}{2}\||\sigma-\bar{\sigma}|^{\frac{p-2}{2}}\nabla(\sigma-\bar{\sigma})\|_{L^2}^2 + \frac{1}{2}\|\nabla\Phi\|_{L^\infty}^2\|\rho\|_{L^p}^2\|\sigma-\bar{\sigma}\|_{L^p}^{p-2}\right].\end{aligned}$$

Thus, we have the differential inequality

$$\frac{1}{p}\frac{d}{dt}\|\sigma-\bar{\sigma}\|_{L^p}^p + \frac{D(p-1)}{2}\int |\nabla(\sigma-\bar{\sigma})|^2|\sigma-\bar{\sigma}|^{p-2}dx \le \frac{D(p-1)}{2}\|\nabla\Phi\|_{L^\infty}^2\|\rho\|_{L^p}^2\|\sigma-\bar{\sigma}\|_{L^p}^{p-2}. \tag{75}$$

We note that

$$D(p-1)\int |\nabla(\sigma-\bar{\sigma})|^2|\sigma-\bar{\sigma}|^{p-2}dx = -D\int |\sigma-\bar{\sigma}|^{p-2}(\sigma-\bar{\sigma})\Delta(\sigma-\bar{\sigma}) \ge D\lambda\|\sigma-\bar{\sigma}\|_{L^p}^p \tag{76}$$

if p is an even number greater than 2. This follows from Proposition 2. Thus, for any even number $p > 2$,

$$\frac{d}{dt}\|\sigma-\bar{\sigma}\|_{L^p}^2 + D\lambda\|\sigma-\bar{\sigma}\|_{L^p}^2 \le D(p-1)\|\rho\|_{L^3}^2\|\rho\|_{L^p}^2. \tag{77}$$

In view of Lemma 1, we obtain (65) for any even number $p > 2$. An L^p estimate when p is not even can be obtained by an application of Hölder's inequality.

Proposition 5. *Let $u_0 \in H^2, c_i(0) \in H^2$. Then, there exist positive constants c_3, c_4, c_5, c_6 depending on D, ϵ and ν, and positive constants C_3, C_4, C_5, C_6 depending on the initial data $\|u_0\|_{H^2}$, $\|c_1(0)\|_{H^2}$, $\|c_2(0)\|_{H^2}$, $\bar{\sigma}$ and universal constants, such that*

$$\|\nabla u(t)\|_{L^2}^2 \le C_3 e^{-c_3 t}, \tag{78}$$

$$\|\nabla\rho(t)\|_{L^2}^2 + \|\nabla\sigma(t)\|_{L^2}^2 \le C_4 e^{-c_4 t}, \tag{79}$$

$$\|\Delta u(t)\|_{L^2}^2 \le C_5 e^{-c_5 t}, \tag{80}$$

and

$$\|\Delta\rho(t)\|_{L^2}^2 + \|\Delta\sigma(t)\|_{L^2}^2 \le C_6 e^{-c_6 t} \tag{81}$$

hold for all $t \ge 0$.

Proof. We take the L^2 inner product of the equation satisfied by u in (54) with $-\Delta u$, and we apply Hölder's and Young's inequalities to get

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \nu\|\Delta u\|_{L^2}^2 \le C\|\rho\|_{L^3}^2\|\rho\|_{L^2}^2 \tag{82}$$

and so we obtain (78) by an application of Lemma 1.

Now, we take the L^2 inner product of equation (66) obeyed by ρ with $-\Delta\rho$ and we estimate

$$\left|\int (\sigma-\bar{\sigma})\Delta\Phi\Delta\rho\right| \le C\|\Delta\rho\|_{L^2}\|\rho\|_{L^2}^{1/2}\|\nabla\rho\|_{L^2}^{1/2}\|\sigma-\bar{\sigma}\|_{L^2}^{1/2}\|\nabla\sigma\|_{L^2}^{1/2}, \tag{83}$$

$$\left|\int (\nabla\sigma\cdot\nabla\Phi)\Delta\rho\right| \le C\|\Delta\rho\|_{L^2}\|\nabla\sigma\|_{L^2}\|\rho\|_{L^3} \tag{84}$$

and

$$\left|\int (u\cdot\nabla\rho)\Delta\rho\right| \le C\|\nabla\rho\|_{L^2}^{1/2}\|\Delta\rho\|_{L^2}^{3/2}\|\nabla u\|_{L^2} \tag{85}$$

in view of Ladyzhenskaya's inequality. This gives

$$\frac{d}{dt}\|\nabla\rho\|_{L^2}^2 + D\|\Delta\rho\|_{L^2}^2 \le C\left[(\|\rho\|_{L^2}^2 + \|\nabla u\|_{L^2}^4)\|\nabla\rho\|_{L^2}^2 + (\|\rho\|_{L^3}^2 + \|\sigma-\bar{\sigma}\|_{L^2}^2)\|\nabla\sigma\|_{L^2}^2\right]. \tag{86}$$

Next, we take the L^2 inner product of the equation satisfied by σ, and proceeding as above, we obtain

$$\frac{d}{dt}\|\nabla\sigma\|_{L^2}^2 + D\|\Delta\sigma\|_{L^2}^2 \le C\left[(\|\rho\|_{L^2}^2 + \|\rho\|_{L^3}^2)\|\nabla\rho\|_{L^2}^2 + \|\nabla u\|_{L^2}^4\|\nabla\sigma\|_{L^2}^2\right]. \tag{87}$$

Adding (87) to (86) and using (56), we obtain (79).

Then, we apply $-\Delta$ to the equation obeyed by u in (54) and we take the L^2 inner product of the resulting equation with $-\Delta u$. We obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \nu\|\nabla\Delta u\|_{L^2}^2 = -\int \Delta(u\cdot\nabla u)\cdot\Delta u - \int \Delta(\rho\nabla\Phi)\cdot\Delta u. \tag{88}$$

In view of Ladyzehsnkaya's inequality, we have

$$\left|\int \Delta(u\cdot\nabla u)\cdot\Delta u\right| \le C\|\nabla\Delta u\|_{L^2}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}. \tag{89}$$

Moreover,

$$\left|\int \Delta(\rho\nabla\Phi)\cdot\Delta u\right| \le C\|\nabla\Delta u\|_{L^2}(\|\rho\|_{L^2}\|\nabla\rho\|_{L^2} + \|\rho\|_{L^3}\|\nabla\rho\|_{L^2}). \tag{90}$$

Here we have used the fact that the Riesz transforms are bounded in L^4, so

$$\|\nabla\nabla\Phi\|_{L^4} = \frac{1}{\epsilon}\|\nabla\nabla\Lambda^{-2}\rho\|_{L^4} \le C\|\rho\|_{L^4}. \tag{91}$$

Consequently, we obtain

$$\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \nu\|\nabla\Delta u\|_{L^2}^2 \le C\left[\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}^2 + \|\rho\|_{L^2}^2\|\nabla\rho\|_{L^2}^2 + \|\rho\|_{L^3}^2\|\nabla\rho\|_{L^2}^2\right]. \tag{92}$$

In view of (82) and Lemma 1, we deduce (80).

Finally, we apply $-\Delta$ to the equations satisfied by ρ and σ in (57) and we take the L^2 inner product of the resulting equations with $-\Delta\rho$ and $-\Delta\sigma$ respectively. We obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta\rho\|_{L^2}^2 + \frac{D\bar{\sigma}}{\epsilon}\|\Delta\rho\|_{L^2}^2 + D\|\nabla\Delta\rho\|_{L^2}^2 = D\int \Delta\nabla\cdot((\sigma-\bar{\sigma})\nabla\Phi)\Delta\rho - \int \Delta(u\cdot\nabla\rho)\Delta\rho \tag{93}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\Delta\sigma\|_{L^2}^2 + D\|\nabla\Delta\sigma\|_{L^2}^2 = D\int \Delta\nabla\cdot(\rho\nabla\Phi)\Delta\sigma - \int \Delta(u\cdot\nabla\sigma)\Delta\sigma. \tag{94}$$

We estimate

$$\begin{aligned}\left|\int \Delta(u\cdot\nabla\rho)\Delta\rho\right| &\le \|\nabla\Delta\rho\|_{L^2}\|\nabla u\|_{L^4}\|\nabla\rho\|_{L^4}\\ &\le C\|\nabla\Delta\rho\|_{L^2}\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2}\|\nabla\rho\|_{L^2}^{1/2}\|\Delta\rho\|_{L^2}^{1/2}\end{aligned} \tag{95}$$

and similarly

$$\left|\int \Delta(u\cdot\nabla\sigma)\Delta\sigma\right| \le C\|\nabla\Delta\sigma\|_{L^2}\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{1/2}\|\nabla\sigma\|_{L^2}^{1/2}\|\Delta\sigma\|_{L^2}^{1/2}. \tag{96}$$

Now, we have

$$\begin{aligned}\left|\int \Delta((\sigma-\bar{\sigma})\Delta\Phi)\Delta\rho\right| &\le \|\nabla\Delta\rho\|_{L^2}\left[\|\nabla\sigma\|_{L^4}\|\Delta\Phi\|_{L^4}+\|\sigma-\bar{\sigma}\|_{L^4}\|\nabla\Delta\Phi\|_{L^4}\right]\\ &\le C\|\nabla\Delta\rho\|_{L^2}\left[\|\nabla\sigma\|_{L^2}^{1/2}\|\Delta\sigma\|_{L^2}^{1/2}\|\nabla\rho\|_{L^2}+\|\nabla\sigma\|_{L^2}\|\nabla\rho\|_{L^2}^{1/2}\|\Delta\rho\|_{L^2}^{1/2}\right]\end{aligned} \tag{97}$$

whereas

$$\begin{aligned}\left|\int \Delta(\nabla(\sigma-\bar{\sigma})\cdot\nabla\Phi)\Delta\rho\right| &\le \|\nabla\Delta\rho\|_{L^2}\left[\|\nabla\nabla\sigma\|_{L^2}\|\nabla\Phi\|_{L^\infty}+\|\nabla\sigma\|_{L^4}\|\nabla\nabla\Phi\|_{L^4}\right]\\ &\le C\|\nabla\Delta\rho\|_{L^2}\left[\|\Delta\sigma\|_{L^2}\|\rho\|_{L^3}+\|\nabla\sigma\|_{L^2}^{1/2}\|\Delta\sigma\|_{L^2}^{1/2}\|\nabla\rho\|_{L^2}\right].\end{aligned} \tag{98}$$

Here, we have used the fact that the Riesz transforms are bounded in L^2, and so

$$\|\nabla\nabla\sigma\|_{L^2} = \|\nabla\Lambda^{-1}\nabla\Lambda^{-1}\Delta\sigma\|_{L^2} \le C\|\Delta\sigma\|_{L^2} \tag{99}$$

Similarly, we have the bounds

$$\left|\int \Delta(\rho\Delta\Phi)\Delta\sigma\right| \le C\|\nabla\Delta\sigma\|_{L^2}\left[\|\nabla\rho\|_{L^2}^{1/2}\|\Delta\rho\|_{L^2}^{1/2}\|\nabla\rho\|_{L^2}+\|\nabla\rho\|_{L^2}\|\nabla\rho\|_{L^2}^{1/2}\|\Delta\rho\|_{L^2}^{1/2}\right] \tag{100}$$

and

$$\left|\int \Delta(\nabla\rho\cdot\nabla\Phi)\Delta\sigma\right| \le C\|\nabla\Delta\sigma\|_{L^2}\left[\|\Delta\rho\|_{L^2}\|\rho\|_{L^3}+\|\nabla\rho\|_{L^2}^{1/2}\|\Delta\rho\|_{L^2}^{1/2}\|\nabla\rho\|_{L^2}\right] \tag{101}$$

Putting (93)–(101) together, and applying Young's and Poincaré's inequalities, we have the differential inequality

$$\begin{aligned}&\frac{d}{dt}(\|\Delta\rho\|_{L^2}^2+\|\Delta\sigma\|_{L^2}^2)+D(\|\nabla\Delta\rho\|_{L^2}^2+\|\nabla\Delta\sigma\|_{L^2}^2)\\ &\le C\left[(\|\Delta u\|_{L^2}^2+\|\nabla\rho\|_{L^2}^2+\|\nabla\sigma\|_{L^2}^2)\|\Delta\rho\|_{L^2}^2+(\|\Delta u\|_{L^2}^2+\|\nabla\rho\|_{L^2}^2)\|\Delta\sigma\|_{L^2}^2\right]\end{aligned} \tag{102}$$

Consequently, (81) follows from (86), (87), and Lemma 1.

We denote by $C^{0,\gamma}$ the space of γ-Hölder continuous functions on $\mathbb{T}^2$ with the norm

$$\|v\|_{C^{0,\gamma}} = \|v\|_{L^\infty} + \sup_{x,y\in\mathbb{T}^2, x\neq y}\frac{|v(x)-v(y)|}{|x-y|^\gamma}. \tag{103}$$

Corollary 2. *Let $u_0\in H^2, c_i(0)\in H^2$. Then, there exists a positive constant c_8 depending on D,ϵ,ν, and a positive constant C_8 depending on $\|u_0\|_{H^2}, \|c_i(0)\|_{H^2}$, $\|c_i(0)\|_{H^2}$, $\bar{\sigma}$ and universal constants, such that*

$$\|u(t)\|_{C^{0,1/2}} + \|\rho(t)\|_{C^{0,1/2}} + \|\sigma(t)-\bar{\sigma}\|_{C^{0,1/2}} \le C_8 e^{-c_8 t} \tag{104}$$

holds for all $t\ge 0$.

Proof. The estimate (104) follows from the bound

$$\|v\|_{C^{0,1/2}} \le C\|v\|_{W^{1,4}} \le C[\|v\|_{L^4}+\|\nabla v\|_{L^4}] \le C[\|v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{1/2}+\|\nabla v\|_{L^2}^{1/2}\|\Delta v\|_{L^2}^{1/2}], \tag{105}$$

which holds for all $v\in W^{1,4}(\mathbb{T}^2)$ with mean zero, and from Proposition 5.

Remark 5. *In Proposition 4, we assumed that $u_0 \in H^1, c_i(0) \in H^1$ which guarantee by Theorem 3 the global existence of solutions and the nonnegativity of the concentrations c_i, and obtained the exponential decay of the L^p norm of ρ and $\sigma - \bar{\sigma}$. In Corollary 2, we have assumed higher regularity of the initial data to get the exponential decay of the L^∞ norm of u, ρ and $\sigma - \bar{\sigma}$. However, if we assume in this latter corollary that the initial data are only in H^1, then from* (86), (87), *and* (82) *we deduce the existence of t_0 such that*

$$\|\Delta u(t_0)\|_{L^2}^2 + \|\Delta \rho(t_0)\|_{L^2}^2 + \|\Delta \sigma(t_0)\|_{L^2}^2 < \infty \tag{106}$$

and so we obtain (81) *and* (80) *for all $t \geq t_0$. We also note that the constants C_1^p and C_2^p in Proposition 4 are independent of u, depending only on the L^p norm of the $c_1(0)$ and $c_2(0)$, whereas the constants C_4 and C_6 in Proposition 5 depend on the H^2 norm of all initial data.*

5. ADDED BODY FORCES

In this section, we consider the Navier-Stokes equations driven the electrical force and a smooth, mean zero, divergence free body force,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u - \rho \nabla \Phi + f \\ \nabla \cdot u = 0 \\ \rho = c_1 - c_2 \\ -\epsilon \Delta \Phi = \rho \\ \partial_t c_1 + u \cdot \nabla c_1 = D \Delta c_1 + D \nabla \cdot (c_1 \nabla \Phi) \\ \partial_t c_2 + u \cdot \nabla c_2 = D \Delta c_2 - D \nabla \cdot (c_2 \nabla \Phi) \end{cases} \tag{107}$$

in $\mathbb{T}^2 \times [0, \infty)$, with $u_0, c_1(0), c_2(0) \in H^1$. We assume that u_0 has mean zero, and $c_1(0)$ and $c_2(0)$ have equal mean. We take $c_i(0) \geq 0$, and by Theorem 3 which is valid in this case as well, the concentrations c_i are nonnegative for all time $t > 0$.

Proposition 6. *Let $p \geq 2$. $u_0, c_1(0), c_2(0) \in H^1$ There exist positive constants a_1, a_2 depending on D, ϵ, $\bar{\sigma}$, and λ (the constant in Proposition 2), and positive constants $C_1^p(\|\rho_0\|_{L^p}, \|\sigma_0\|_{L^2})$ and $C_2^p(\|\sigma_0\|_{L^p}, \|\rho_0\|_{L^2})$ depending on the corresponding initial data, $\bar{\sigma}$, p and universal constants, such that*

$$\|\rho(t)\|_{L^p} \leq C_1^p e^{-a_1 t} \tag{108}$$

and

$$\|\sigma(t) - \bar{\sigma}\|_{L^p} \leq C_2^p e^{-a_2 t} \tag{109}$$

hold for all $t \geq 0$. Furthermore,

$$\int_t^{t+T} \left(\|\nabla \rho(s)\|_{L^2}^2 + \|\nabla \sigma(s)\|_{L^2}^2 + \frac{1}{\epsilon} \|\rho(s)\|_{L^3}^3 \right) ds \leq \frac{1}{2D} (2\|\sigma_0\|_{L^2}^2 + 2\|\bar{\sigma}\|_{L^2}^2 + \|\rho_0\|_{L^2}^2) T e^{-2CDt} \tag{110}$$

holds for any $t \geq 0, T > 0$.

The proof follows along the lines of the proofs of Propositions 3 and 4. Indeed, multiplying the ρ and $\sigma - \bar{\sigma}$ equations by $\rho|\rho|^{p-2}$ and $(\sigma - \bar{\sigma})|\sigma - \bar{\sigma}|^{p-2}$ respectively, the terms involving u cancel out and we conclude that the estimates for the L^p norms of ρ and σ (108) and (109) hold for any $p \geq 2$. In particular, (44) is satisfied.

The following proposition shows that adding a body force to the Navier-Stokes equation does not change the exponential decay of the H^2 norms of ρ and $\sigma - \bar{\sigma}$ but results in the velocity u being bounded in H^2.

Proposition 7. *Let $u_0 \in H^2, c_i(0) \in H^2$. Then, there exist positive constants c_3', c_4', c_5', c_6' depending on D, ϵ and ν, and positive constants C_3' and C_5' depending on the initial data $\|u_0\|_{H^2}$, $\|c_1(0)\|_{H^2}$, $\|c_2(0)\|_{H^2}$ and $\bar{\sigma}$, and positive constants C_4' and C_6' depending in addition on the forces f, and positive constants R_3 and R_5 depending on f such that*

$$\|\nabla u(t)\|_{L^2}^2 \leq C_3' e^{-c_3' t} + R_3, \tag{111}$$

$$\|\nabla\rho(t)\|_{L^2}^2 + \|\nabla\sigma(t)\|_{L^2}^2 \leq C_4' e^{-c_4' t}, \tag{112}$$

$$\|\Delta u(t)\|_{L^2}^2 \leq C_5' e^{-c_5' t} + R_5, \tag{113}$$

and

$$\|\Delta\rho(t)\|_{L^2}^2 + \|\Delta\sigma(t)\|_{L^2}^2 \leq C_6' e^{-c_6' t} \tag{114}$$

hold for all $t \geq 0$.

Moreover, there exists a positive constant $L > 0$ *depending on* $\|u_0\|_{H^1}, \|c_1(0)\|_{H^1}, \|c_2(0)\|_{H^1}, f$ *and universal constants such that*

$$\int_0^t (\|\Delta u(s)\|_{L^2}^2 + \|\Delta\rho(s)\|_{L^2}^2 + \|\Delta\sigma(s)\|_{L^2}^2) ds \leq L \tag{115}$$

for all $t \geq 0$.

We note that the estimate (115) requires only that $u_0, c_1(0), c_2(0) \in H^1$. No additional regularity of the initial data is required.

The proof is similar to the proof of Proposition 5. We omit the details.

Corollary 3. *Let* $u_0 \in H^2, c_i(0) \in H^2$. *Then, there exist positive constants* c_8' *and* c_9' *depending on* D, ϵ, ν, *and a positive constant* C_8' *depending on* $\|u_0\|_{H^2}, \|c_i(0)\|_{H^2}, \|c_i(0)\|_{H^2}$, *and* $\bar{\sigma}$, *a positive constant* C_9' *depending in addition on the body forces* f, *and a positive constant* R_9 *depending on* f *such that*

$$\|u\|_{C^{0,1/2}} \leq C_8' e^{-c_8' t} + R_9 \tag{116}$$

and

$$\|\rho(t)\|_{C^{0,1/2}} + \|\sigma(t) - \bar{\sigma}\|_{C^{0,1/2}} \leq C_9' e^{-c_9' t} \tag{117}$$

holds for all $t \geq 0$.

This follows from Proposition 7, see the proof of Corollary 2.

Theorem 4. *(Absorbing Ball) Let* $u_0, c_1(0), c_2(0) \in H^1$ *such that* u_0 *and* $(c_1 - c_2)(0)$ *have mean zero. Suppose that* (u, c_1, c_2) *solves* (107). *Then, there exists an* $R > 0$ *depending on* f, *and* $t_0 > 0$ *depending on* $\|u_0\|_{H^1}, \|c_1(0)\|_{H^1}, \|c_2(0)\|_{H^1}$ *and the parameters of the problem, such that*

$$\|\Delta u(t)\|_{L^2}^2 + \|\Delta c_1(t)\|_{L^2}^2 + \|\Delta c_2(t)\|_{L^2}^2 \leq R \tag{118}$$

holds for all $t \geq t_0$.

Proof. In view of equation (115), there exists $\tau \in [0, 1]$ such that

$$\|\Delta u(t_0)\|_{L^2}^2 + \|\Delta\rho(t_0)\|_{L^2}^2 + \|\Delta\sigma(t_0)\|_{L^2} \leq L. \tag{119}$$

Thus, the result follows from equations (113), (114), and from the parallelogram law

$$\|\Delta\rho\|_{L^2}^2 + \|\Delta\sigma\|_{L^2}^2 = 2\|\Delta c_1\|_{L^2}^2 + 2\|\Delta c_2\|_{L^2}^2. \tag{120}$$

Let $\mathcal{V} = H^1 \cap H \oplus H^1 \oplus H^1 \subset \mathcal{H}$. Let $\mathcal{V}'$ be the convex subset of $\mathcal{V}$ consisting of vectors (u, c_1, c_2) such that u is divergence free with mean zero and $c_1 \geq 0$, $c_2 \geq 0$ a.e. with $\int c_1 = \int c_2$. Let

$$\mathcal{S}(t) : \mathcal{V}' \mapsto \mathcal{V}' \tag{121}$$

be the solution map

$$\mathcal{S}(t)(u_0, c_1(0), c_2(0)) = (u(t), c_1(t), c_2(t)) \tag{122}$$

corresponding to system (107). As a consequence of Theorem 3, because the solution is absolutely continuous as a function of time with values in $\mathcal{V}'$, it follows that $\mathcal{S}(t)$ is well-defined on $\mathcal{V}'$ for every $t \geq 0$. Moreover, the uniqueness of solutions implies that

$$\mathcal{S}(t+s)w_0 = \mathcal{S}(t)(\mathcal{S}(s)w_0) \tag{123}$$

for all $t, s \geq 0$, i.e., $\mathcal{S}(t)$ is a semigroup. We proceed to investigate other properties of the map $\mathcal{S}(t)$.

We consider the natural topology on $\mathcal{H}$

$$\|w\|_{\mathcal{H}}^2 = \|u\|_{L^2}^2 + \|c_1\|_{L^2}^2 + \|c_2\|_{L^2}^2 \tag{124}$$

and the natural topology on $\mathcal{V}'$

$$\|w\|_{\mathcal{V}'}^2 = \|u\|_{H^1}^2 + \|c_1\|_{H^1}^2 + \|c_2\|_{H^1}^2. \tag{125}$$

We address the continuity of the map $\mathcal{S}(t)$.

Theorem 5. *(Continuity) Let $w_1^0 = (u_1(0), c_1^1(0), c_2^1(0)), w_2^0 = (u_2(0), c_1^2(0), c_2^2(0)) \in \mathcal{V}'$. Let $t > 0$. There exist constants $K_1(t)$, $K_2(t)$ and $K_3(t)$, locally uniformly bounded as functions of $t \geq 0$, and locally bounded as initial data w_1^0, w_2^0 are varied in $\mathcal{V}'$, such that $\mathcal{S}(t)$ is Lipschitz continuous in $\mathcal{H}$ obeying*

$$\|\mathcal{S}(t)w_1^0 - \mathcal{S}(t)w_2^0\|_{\mathcal{H}}^2 \leq K_1(t)\|w_1^0 - w_2^0\|_{\mathcal{H}}^2, \tag{126}$$

$\mathcal{S}(t)$ is Lipschitz continuous in $\mathcal{V}'$ obeying

$$\|\mathcal{S}(t)w_1^0 - \mathcal{S}(t)w_2^0\|_{\mathcal{V}'}^2 \leq K_2(t)\|w_1^0 - w_2^0\|_{\mathcal{V}'}^2, \tag{127}$$

and $\mathcal{S}(t)$ is Lipschitz continuous for $t > 0$ from $\mathcal{H}$ to $\mathcal{V}'$ obeying

$$t\|\mathcal{S}(t)w_1^0 - \mathcal{S}(t)w_2^0\|_{\mathcal{V}'}^2 \leq K_3(t)\|w_1^0 - w_2^0\|_{\mathcal{H}}^2. \tag{128}$$

Proof. We write $\mathcal{S}(t)w_1^0 = (u_1(t), c_1^1(t), c_2^1(t))$ and $\mathcal{S}(t)w_2^0 = (u_2(t), c_1^2(t), c_2^2(t))$. Let $\rho_1 = c_1^1 - c_2^1$, $\rho_2 = c_1^2 - c_2^2$, $\sigma_1 = c_1^1 + c_2^1$, $\sigma_2 = c_1^2 + c_2^2$. We write $u = u_1 - u_2$, $\rho = \rho_1 - \rho_2$ and $\sigma = \sigma_1 - \sigma_2$.

We note that u, ρ and σ obey system (25). Following the proof of uniqueness in Theorem 1, we obtain a differential inequality of the form

$$\frac{d}{dt}\left[\|u\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|\sigma\|_{L^2}^2\right] + \nu\|\nabla u\|_{L^2}^2 + D\|\nabla\rho\|_{L^2}^2 + D\|\nabla\sigma\|_{L^2}^2 \leq k_1(t)\left[\|u\|_{L^2}^2 + \|\rho\|_{L^2}^2 + \|\sigma\|_{L^2}^2\right] \tag{129}$$

where

$$k_1(t) = C[\|\nabla u_1\|_{L^2}^{2/3}\|\Delta u_1\|_{L^2}^{2/3} + \|\nabla\rho_1\|_{L^2}^2 + \|\nabla\sigma_1\|_{L^2}^2 + \|\rho_1\|_{L^3}^2 + \|\sigma_2\|_{L^2}^4 + \|\rho_2\|_{L^2}^4 + 1]. \tag{130}$$

Letting

$$K_1(t) = 4\exp\left\{\int_0^t k_1(s)ds\right\}, \tag{131}$$

we obtain (126).

Now, we take the L^2 inner product of the three equations of system (25) with $-\Delta u$, $-\Delta\rho$ and $-\Delta\sigma$ respectively, and we add them. We obtain the differential inequality

$$\begin{aligned}
&\frac{d}{dt}\left[\|\nabla u\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2\right] + \nu\|\Delta u\|_{L^2}^2 + D\|\Delta\rho\|_{L^2}^2 + D\|\Delta\sigma\|_{L^2}^2\\
&\leq C\left[\|u_1\cdot\nabla u_1 - u_2\cdot\nabla u_2\|_{L^2}^2 + \|u_1\cdot\nabla\rho_1 - u_2\cdot\nabla\rho_2\|_{L^2}^2 + \|u_1\cdot\nabla\sigma_1 - u_2\cdot\nabla\sigma_2\|_{L^2}^2\right]\\
&+ C\left[\|\rho_1\nabla\Phi_1 - \rho_2\nabla\Phi_2\|_{L^2}^2 + \|\nabla\cdot(\sigma_1\nabla\Phi_1 - \sigma_2\nabla\Phi_2)\|_{L^2}^2 + \|\nabla\cdot(\rho_1\nabla\Phi_1 - \rho_2\nabla\Phi_2)\|_{L^2}^2\right].
\end{aligned} \tag{132}$$

We estimate

$$\|u_1\cdot\nabla u_1 - u_2\cdot\nabla u_2\|_{L^2}^2 = \|u\cdot\nabla u_1 + u_2\cdot\nabla u\|_{L^2}^2 \leq C[\|\nabla u_1\|_{L^4}^2\|\nabla u\|_{L^2}^2 + \|u_2\|_{L^\infty}^2\|\nabla u\|_{L^2}^2], \tag{133}$$

$$\|u_1\cdot\nabla\rho_1 - u_2\cdot\nabla\rho_2\|_{L^2}^2 = \|u\cdot\nabla\rho_1 + u_2\cdot\nabla\rho\|_{L^2}^2 \leq C[\|\nabla\rho_1\|_{L^4}^2\|\nabla u\|_{L^2}^2 + \|u_2\|_{L^\infty}^2\|\nabla\rho\|_{L^2}^2] \tag{134}$$

and

$$\|u_1\cdot\nabla\sigma_1 - u_2\cdot\nabla\sigma_2\|_{L^2}^2 = \|u\cdot\nabla\sigma_1 + u_2\cdot\nabla\sigma\|_{L^2}^2 \leq C[\|\nabla\sigma_1\|_{L^4}^2\|\nabla u\|_{L^2}^2 + \|u_2\|_{L^\infty}^2\|\nabla\sigma\|_{L^2}^2] \tag{135}$$

using Poincaré and Ladyzhenskaya's interpolation inequalities. Using in addition elliptic regularity, we have

$$\|\rho_1\nabla\Phi_1 - \rho_2\nabla\Phi_2\|_{L^2}^2 = \|\rho\nabla\Phi_1 + \rho_2\nabla\Phi\|_{L^2}^2 \leq C[\|\nabla\Phi_1\|_{L^\infty}^2 + \|\rho_2\|_{L^2}^2]\|\nabla\rho\|_{L^2}^2. \tag{136}$$

We also estimate

$$\begin{aligned}\|\nabla\cdot(\sigma_1\nabla\Phi_1-\sigma_2\nabla\Phi_2)\|_{L^2}^2 &= \|\sigma\Delta\Phi_1+\sigma_2\Delta\Phi+\nabla\sigma\cdot\nabla\Phi_1+\nabla\sigma_2\cdot\nabla\Phi\|_{L^2}^2\\ &\le C[\|\rho_1\|_{L^\infty}^2\|\sigma\|_{L^2}^2+\|\nabla\Phi_1\|_{L^\infty}^2\|\nabla\sigma\|_{L^2}^2]\\ &\quad+C(\|\sigma_2\|_{L^\infty}^2+\|\nabla\sigma_2\|_{L^2}^2)\|\nabla\rho\|_{L^2}^2 \end{aligned}\tag{137}$$

and

$$\begin{aligned}\|\nabla\cdot(\rho_1\nabla\Phi_1-\rho_2\nabla\Phi_2)\|_{L^2}^2 &= \|\rho\Delta\Phi_1+\rho_2\Delta\Phi+\nabla\rho\cdot\nabla\Phi_1+\nabla\rho_2\cdot\nabla\Phi\|_{L^2}^2\\ &\le C[\|\rho_1\|_{L^\infty}^2+\|\rho_2\|_{L^\infty}^2+\|\nabla\Phi_1\|_{L^\infty}^2+\|\nabla\rho_2\|_{L^2}^2]\|\nabla\rho\|_{L^2}^2. \end{aligned}\tag{138}$$

In view of (129), we obtain a differential inequality of the form

$$\frac{d}{dt}[\|u\|_{H^1}^2+\|\rho\|_{H^1}^2+\|\sigma\|_{H^1}^2]\le k_2(t)[\|u\|_{H^1}^2+\|\rho\|_{H^1}^2+\|\sigma\|_{H^1}^2]\tag{139}$$

where

$$\begin{aligned}k_2(t)=k_1(t)&+C[\|\nabla u_1\|_{L^4}^2+\|\nabla\rho_1\|_{L^4}^2+\|\nabla\sigma_1\|_{L^4}^2+\|\nabla\rho_2\|_{L^2}^2+\|\nabla\sigma_2\|_{L^2}^2]\\ &+C[\|u_2\|_{L^\infty}^2+\|\sigma_2\|_{L^\infty}^2+\|\rho_2\|_{L^\infty}^2].\end{aligned}\tag{140}$$

Letting

$$K_2(t)=4\exp\left\{\int_0^t k_2(s)ds\right\},\tag{141}$$

we obtain (127).

The derivation of (128) is a little different. The sum of the equations resulting from taking L^2 inner product of the u, ρ and σ equations with $-\Delta u$, $-\Delta\rho$ and $-\Delta\sigma$ respectively gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}[\|\nabla u\|_{L^2}^2+\|\nabla\rho\|_{L^2}^2+\|\nabla\sigma\|_{L^2}^2]+\nu\|\Delta u\|_{L^2}^2+D\|\Delta\rho\|_{L^2}^2+D\|\Delta\sigma\|_{L^2}^2\\ &=\int(u\cdot\nabla u_1+u_2\cdot\nabla u)\cdot\Delta u+\int(u\cdot\nabla\rho_1+u_2\cdot\nabla\rho)\Delta\rho+\int(u\cdot\nabla\sigma_1+u_2\cdot\nabla\sigma)\Delta\sigma\\ &+\int(\rho\nabla\Phi_1+\rho_2\nabla\Phi)\cdot\Delta u-D\int(\nabla\cdot(\sigma\nabla\Phi_1+\sigma_2\nabla\Phi))\Delta\rho-D\int(\nabla\cdot(\rho\nabla\Phi_1+\rho_2\nabla\Phi))\Delta\sigma.\end{aligned}\tag{142}$$

In order to get (128), we let $w(t)=(u(t),\rho(t),\sigma(t))$, and we show that w obeys a differential inequality of the type

$$\frac{d}{dt}\|w\|_{H^1}^2\le Z_1(t)\|w\|_{H^1}^2+Z_2(t)\|w\|_{L^2}^2\tag{143}$$

such that

$$\|w(t)\|_{L^2}^2\le Z_3(t)\|w_0\|_{L^2}^2\tag{144}$$

and

$$\int_0^t\|w(s)\|_{H^1}^2ds\le C(Z_4(t)+1)\|w_0\|_{L^2}^2\tag{145}$$

where $Z_1(t),Z_3(t)$ and $Z_4(t)$ are locally bounded functions in time, $Z_2(t)$ is a locally integrable function in time, and C is a positive constant. Then, multiplying (143) by t and integrating by parts in time from 0 to t, we obtain

$$t\|w(t)\|_{H^1}^2\le C'(Z_5(t)+1)\|w_0\|_{L^2}^2\tag{146}$$

where $Z_5(t)$ is a locally bounded function in time, and $C'>0$ is a positive constant.

We start by integrating (129). Using (126), we obtain

$$\int_0^t(\|\nabla u(s)\|_{L^2}^2+\|\nabla\rho(s)\|_{L^2}^2+\|\nabla\sigma(s)\|_{L^2}^2)ds\le C\left[1+\int_0^t k_1(s)K_1(s)ds\right]\|w_1^0-w_2^0\|_{L^2}^2.\tag{147}$$

This is the analogue of (145). Then, we estimate

$$\left|\int (u\cdot\nabla u_1 + u_2\cdot\nabla u)\cdot\Delta u\right| \le C\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla u_1\|_{L^2}^{1/2}\|\Delta u_1\|_{L^2}^{1/2}\|\Delta u\|_{L^2} + C\|u_2\|_{L^2}^{1/2}\|\nabla u_2\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\Delta u\|_{L^2}^{3/2}, \tag{148}$$

$$\left|\int (u\cdot\nabla \rho_1 + u_2\cdot\nabla \rho)\Delta \rho\right| \le C\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla \rho_1\|_{L^2}^{1/2}\|\Delta \rho_1\|_{L^2}^{1/2}\|\Delta \rho\|_{L^2} + C\|u_2\|_{L^2}^{1/2}\|\nabla u_2\|_{L^2}^{1/2}\|\nabla \rho\|_{L^2}^{1/2}\|\Delta \rho\|_{L^2}^{3/2} \tag{149}$$

and

$$\left|\int (u\cdot\nabla \sigma_1 + u_2\cdot\nabla \sigma)\Delta \sigma\right| \le C\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}\|\nabla \sigma_1\|_{L^2}^{1/2}\|\Delta \sigma_1\|_{L^2}^{1/2}\|\Delta \sigma\|_{L^2} + C\|u_2\|_{L^2}^{1/2}\|\nabla u_2\|_{L^2}^{1/2}\|\nabla \sigma\|_{L^2}^{1/2}\|\Delta \sigma\|_{L^2}^{3/2}. \tag{150}$$

In view of the fact that

$$\|\nabla\Phi\|_{L^4} \le C\|\rho\|_{L^2}, \tag{151}$$

we have

$$\left|\int (\rho\nabla\Phi_1 + \rho_2\nabla\Phi)\cdot\Delta u\right| \le C[\|\rho_1\|_{L^3}\|\rho\|_{L^2}\|\Delta u\|_{L^2} + \|\rho_2\|_{L^4}\|\rho\|_{L^2}\|\Delta u\|_{L^2}] \tag{152}$$

Moreover,

$$\left|\int (\nabla\cdot(\sigma\nabla\Phi_1 + \sigma_2\nabla\Phi))\Delta\rho\right| \le C[\|\sigma\|_{L^2}^{1/2}\|\nabla\sigma\|_{L^2}^{1/2}\|\nabla\rho_1\|_{L^2} + \|\sigma_2\|_{L^\infty}\|\rho\|_{L^2}]\|\Delta\rho\|_{L^2} + C[\|\rho_1\|_{L^3}\|\nabla\sigma\|_{L^2} + \|\nabla\sigma_2\|_{L^2}\|\nabla\rho\|_{L^2}]\|\Delta\rho\|_{L^2} \tag{153}$$

and

$$\left|\int (\nabla\cdot(\rho\nabla\Phi_1 + \rho_2\nabla\Phi))\Delta\sigma\right| \le C[\|\nabla\rho\|_{L^2}\|\nabla\rho_1\|_{L^2} + \|\rho_2\|_{L^\infty}\|\rho\|_{L^2}]\|\Delta\sigma\|_{L^2} + C[\|\rho_1\|_{L^3}\|\nabla\rho\|_{L^2} + \|\nabla\rho_2\|_{L^2}\|\nabla\rho\|_{L^2}]\|\Delta\sigma\|_{L^2}. \tag{154}$$

We apply Young's inequality and we use (129) to obtain

$$\begin{aligned}
&\frac{d}{dt}[\|u\|_{H^1}^2 + \|\rho\|_{H^1}^2 + \|\sigma\|_{H^1}^2]\\
&\le C[k_1 + \|\nabla u_1\|_{L^2}^2\|\Delta u_1\|_{L^2}^2 + \|\nabla\rho_1\|_{L^2}^2\|\Delta\rho_1\|_{L^2}^2 + \|\nabla\sigma_1\|_{L^2}^2\|\Delta\sigma_1\|_{L^2}^2]\|u\|_{L^2}^2\\
&+ C[k_1 + \|\nabla\rho_1\|_{L^2}^2 + \|\nabla\rho_2\|_{L^2}^2 + \|\sigma_2\|_{L^\infty}^2 + \|\rho_2\|_{L^\infty}^2](\|\rho\|_{L^2}^2 + \|\sigma\|_{L^2}^2)\\
&+ C(1 + \|\nabla u_2\|_{L^2}^4 + \|\nabla\rho_1\|_{L^2}^2 + \|\nabla\rho_2\|_{L^2}^2 + \|\nabla\sigma_2\|_{L^2}^2)(\|\nabla u\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2).
\end{aligned} \tag{155}$$

This is a differential inequality of type (143), with $w(t) = (u(t), \rho(t), \sigma(t))$ satisfying (144) and (145). Therefore, we obtain (128).

We proceed to show that the solution map $\mathcal{S}(t)$ is injective on $\mathcal{V}'$.

Theorem 6. *(Backward Uniqueness) Let $w_1^0, w_2^0 \in \mathcal{V}'$. If there exists $T > 0$ such that $\mathcal{S}(T)w_1^0 = \mathcal{S}(T)w_2^0$, then $w_1^0 = w_2^0$.*

The proof is given in Appendix A below.

Now, we fix $M > 0$, and we let $\mathcal{V}_M$ to be the subset of $\mathcal{V}'$ consisting of vectors (u, c_1, c_2) such that u is divergence free with mean zero and c_1 and c_2 are nonnegative functions a.e. with equal space averages less than or equal to M. As a consequence of Theorem 4, there exists $R_1 > 0$ depending only on f such that for any initial data $w_0 = (u_0, c_1(0), c_2(0)) \in \mathcal{V}_M$, there exists $t_0 > 0$ depending on $\|u_0\|_{H^1}, \|c_1(0)\|_{H^1}, \|c_2(0)\|_{H^1}$ and the parameters of the problem such that

$$\mathcal{S}(t)w_0 \in \mathcal{B}_{R_1}^M = \{w = (u, c_1, c_2) \in \mathcal{V}_M : \|u\|_{H^2} + \|c_1 - \bar{c}_1\|_{H^2} + \|c_2 - \bar{c}_2\|_{H^2} \le R_1\} \tag{156}$$

holds for all $t \ge t_0$.

Remark 6. *We note that there exists $T > 0$ depending only on R_1 and M and the parameters of the problem such that*

$$\mathcal{S}(t)B_{R_1}^M \subset B_{R_1}^M \tag{157}$$

for all $t \geq T$.

Remark 7. *$\mathcal{B}_{R_1}^M$ is compact in $\mathcal{H}$ because the space averages of all the concentrations c_1 and c_2 such that $(u, c_1, c_2) \in \mathcal{V}_M$ are uniformly bounded by M.*

Remark 8. *The set $\mathcal{V}_M$ is convex. Consequently, $\mathcal{B}_{R_1}^M$ is a convex set, and so it is connected.*

The properties of the map $\mathcal{S}(t)$ listed and proved above, together with the connectedness and compactness properties of $\mathcal{B}_{R_1}^M$, imply the existence of a global attractor.

Theorem 7. *(Global Attractor) Let*

$$X_M = \bigcap_{t>0} \mathcal{S}(t)\mathcal{B}_{R_1}^M \tag{158}$$

Then:

(a) *X_M is compact in $\mathcal{H}$.*
(b) *$\mathcal{S}(t)X_M = X_M$ for all $t \geq 0$.*
(c) *If Z is bounded in $\mathcal{V}_M$ in the norm of of $\mathcal{V}$, and $\mathcal{S}(t)Z = Z$ for all $t \geq 0$, then $Z \subset X_M$.*
(d) *For every $w_0 \in \mathcal{V}_M$, $\lim_{t\to\infty} dist_{\mathcal{H}}(\mathcal{S}(t)w_0, X_M) = 0$.*
(e) *X_M is connected.*

The proof is omitted and follows the proof of the analogous result in [4].

We end this section by showing that X_M has finite fractal dimension. The abstract formulation of the system is

$$\begin{cases} \partial_t u + \nu A u + B(u,u) + \mathbb{P}(\rho\nabla\Phi) = f, \\ \partial_t c_1 + u\cdot\nabla c_1 - D\Delta c_1 - D\nabla\cdot(c_1\nabla\Phi) = 0, \\ \partial_t c_2 + u\cdot\nabla c_2 - D\Delta c_2 + D\nabla\cdot(c_2\nabla\Phi) = 0, \\ -\epsilon\Delta\Phi = \rho, \\ \rho = c_1 - c_2 \end{cases} \tag{159}$$

where $\mathbb{P}$ is the Leray-Hopf projector, $A = \mathbb{P}(-\Delta)$ is the Stokes operator, and $B(u,v) = \mathbb{P}(u.\nabla v)$.

We consider a solution $\widetilde{w} = S(t)\widetilde{w}_0 = (\widetilde{u}(t), \widetilde{c}_1(t), \widetilde{c}_2(t))$ of (159) with initial data $\widetilde{w}_0$ in $\mathcal{B}_{R_1}^M$. We consider the linearization of $S(t)$ along $\widetilde{w}(t)$

$$w_0 \mapsto w(t) = S'(t, \widetilde{w})w_0 \tag{160}$$

viewed as an operator on $\mathcal{H}$. The function $w(t) = (u(t), c_1(t), c_2(t))$ solves

$$\partial_t w + \mathcal{A}w + L(\widetilde{w})w = 0 \tag{161}$$

where

$$\mathcal{A}w = (\nu A u, -D\Delta c_1, -D\Delta c_2) \tag{162}$$

and

$$L(\widetilde{w})w = (L_1(\widetilde{w})w, L_2(\widetilde{w})w, L_3(\widetilde{w})w) \tag{163}$$

with

$$L_1(\widetilde{w})w = B(\widetilde{u}, u) + B(u, \widetilde{u}) + \mathbb{P}(\rho\nabla\widetilde{\Phi} + \widetilde{\rho}\nabla\Phi)), \tag{164}$$

$$L_2(\widetilde{w})w = u\cdot\nabla\widetilde{c}_1 + \widetilde{u}\cdot\nabla c_1 - D\nabla\cdot(c_1\nabla\widetilde{\Phi} + \widetilde{c}_1\nabla\Phi), \tag{165}$$

$$L_3(\widetilde{w})w = u\cdot\nabla\widetilde{c}_2 + \widetilde{u}\cdot\nabla c_2 + D\nabla\cdot(c_2\nabla\widetilde{\Phi} + \widetilde{c}_2\nabla\Phi). \tag{166}$$

We consider the scalar product in $\wedge^n\mathcal{H}$ given by

$$(w_1 \wedge \ldots \wedge w_n, y_1 \wedge \ldots \wedge y_n)_{\wedge^n\mathcal{H}} = det(w_i, y_j)_{\mathcal{H}} \tag{167}$$

and the volume elements given by

$$V_n(t) = \|w_1(t) \wedge ... \wedge w_n(t)\|_{\wedge^n \mathcal{H}}. \tag{168}$$

We note that the monomial $w_1(t) \wedge ... \wedge w_n(t)$ evolves according to the equation

$$\partial_t (w_1(t) \wedge ... \wedge w_n(t)) + (\mathcal{A} + L(\widetilde{w}))_n (w_1(t) \wedge ... \wedge w_n(t)) = 0 \tag{169}$$

where

$$(A + L(\widetilde{w}))_n (w_1(t) \wedge ... \wedge w_n(t)) = (A + L(\widetilde{w}))w_1 \wedge ... \wedge w_n + ... + w_1 \wedge ... \wedge (\mathcal{A} + L(\widetilde{w}))w_n. \tag{170}$$

Thus, the volume element evolves according to the ODE

$$\frac{d}{dt} V_n + \mathrm{Trace}((\mathcal{A} + L(\widetilde{w}))Q_n) V_n = 0 \tag{171}$$

where Q_n is the orthogonal projection in $\mathcal{H}$ onto the linear space spanned by the vectors $w_1, ..., w_n$.

Theorem 8. *(Decay of Volume Elements) There exists a positive integer N_0 depending on R_1 and M such that for any $\widetilde{w}_0 \in \mathcal{B}_{R_1}$, and for any $n \geq N_0$, and for any $w_1(0), ..., w_n(0) \in \mathcal{H}$*

$$\|S'(t, \widetilde{w}) w_1(0) \wedge ... \wedge S'(t, \widetilde{w}) w_n(0)\|_{\Lambda^n \mathcal{H}} \leq V_n(0) e^{-cnt} \tag{172}$$

holds for any $t \geq t_0$ with t_0 depending on R_1.

The proof is given in Appendix B.

As a consequence, and following the proof of the similar result in [4], we conclude that

Theorem 9. *The global attractor X_M has a finite fractal dimension in $\mathcal{H}$.*

We end this section with the following result:

Theorem 10. *The global attractor X_M has a finite fractal dimension in $\mathcal{V}$.*

Proof. Since $\mathcal{B}_{R_1}^M$ is bounded in H^2, we conclude by Rellich compactness theorem that $\mathcal{S}(t) B_{R_1}^M$ is compact in $\mathcal{V}$ for all $t \geq T$, see Remark 6. Hence, the property (128), together with the fact that X_M has a finite fractal dimension in $\mathcal{H}$, allows us to conclude that X_M has a finite fractal dimension in $\mathcal{V}$.

6. ADDED BODY FORCES AND ADDED CHARGE DENSITY

In this section, we consider the general case

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u - (\rho + N) \nabla \Phi + f \\ \nabla \cdot u = 0 \\ \rho = c_1 - c_2 \\ -\epsilon \Delta \Phi = \rho + N \\ \partial_t c_1 + u \cdot \nabla c_1 = D \Delta c_1 + D \nabla \cdot (c_1 \nabla \Phi) \\ \partial_t c_2 + u \cdot \nabla c_2 = D \Delta c_2 - D \nabla \cdot (c_2 \nabla \Phi) \end{cases} \tag{173}$$

where the body forces f are smooth, divergence-free, time independent, and have mean zero, and the added charge density N is smooth and time independent. We assume that u_0 has mean zero, and that the initial concentrations $c_1(x,0)$ and $c_2(x,0)$ have space averages $\bar{c}_1$ and $\bar{c}_2$ satisfying $\bar{c}_2 - \bar{c}_1 = \bar{N}$. We consider initial data $(u_0, c_1(0), c_2(0)) \in H^1$. We also assume that the initial concentrations are nonnegative functions and we recall that this property is preserved for all positive times t by Theorem 3, which holds in this case as well.

Proposition 8. *Let $u_0 \in H$ and $c_i(0) \in L^2$. Then, there exists $C > 0$ such that*

$$\|\sigma(t) - \bar{\sigma}\|_{L^2}^2 + \|\rho(t) - \bar{\rho}\|_{L^2}^2 \le (\|\sigma_0 - \bar{\sigma}\|_{L^2}^2 + \|\rho_0 - \bar{\rho}\|_{L^2}^2)e^{-Dt} + \|\bar{\sigma}\|_{L^2}^2 + C\|N\|_{L^6}^6 \tag{174}$$

holds for all $t \ge 0$. Moreover,

$$\int_t^{t+T} \left(\|\nabla\rho(s)\|_{L^2}^2 + \|\nabla\sigma(s)\|_{L^2}^2 + \frac{1}{\epsilon}\|\rho(s)\|_{L^3}^3\right) ds \le \frac{1}{D}((\|\sigma_0 - \bar{\sigma}\|_{L^2}^2 + \|\rho_0 - \bar{\rho}\|_{L^2}^2)e^{-Dt} + C(T+1)(\|\bar{\sigma}\|_{L^2}^2 + \|N\|_{L^6}^6) \tag{175}$$

holds for any $t \ge 0, T > 0$.

Proof. We recall that σ and ρ obey

$$\begin{cases} \partial_t \sigma + u \cdot \nabla\sigma = D\Delta\sigma + D\nabla \cdot (\rho\nabla\Phi) \\ \partial_t \rho + u \cdot \nabla\rho = D\Delta\rho + D\nabla \cdot (\sigma\nabla\Phi). \end{cases} \tag{176}$$

We take the L^2 inner product of the equations obeyed by σ and ρ with σ and ρ respectively, we add, and use the fact that

$$\int \rho\Delta\Phi\sigma = -\frac{1}{\epsilon}\int \sigma(\rho)^2 - \frac{1}{\epsilon}\int N\rho\sigma \tag{177}$$

to get the equation

$$\frac{1}{2}\frac{d}{dt}(\|\sigma\|_{L^2}^2 + \|\rho\|_{L^2}^2) + D(\|\nabla\sigma\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2) + \frac{D}{\epsilon}\int \sigma(\rho)^2 = -\frac{D}{\epsilon}\int N\rho\sigma. \tag{178}$$

We estimate

$$\begin{aligned} \left|\frac{D}{\epsilon}\int N\rho\sigma\right| &\le \frac{D}{\epsilon}\|N\|_{L^6}\|\rho\|_{L^3}\|\sigma\|_{L^2} \le \frac{D}{2\epsilon}\|\rho\|_{L^3}^3 + \frac{D}{4}\|\sigma\|_{L^2}^2 + C\|N\|_{L^6}^6 \\ &\le \frac{D}{2\epsilon}\|\rho\|_{L^3}^3 + \frac{D}{2}\|\sigma - \bar{\sigma}\|_{L^2}^2 + \frac{D}{2}\|\bar{\sigma}\|_{L^2}^2 + C\|N\|_{L^6}^6 \end{aligned} \tag{179}$$

in view of Hölder's and Young's inequalities. We obtain the differential inequality

$$\frac{1}{2}\frac{d}{dt}(\|\sigma - \bar{\sigma}\|_{L^2}^2 + \|\rho - \bar{\rho}\|_{L^2}^2) + \frac{D}{2}(\|\nabla\sigma\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2) + \frac{D}{2\epsilon}\|\rho\|_{L^3}^3 \le \frac{D}{2}\|\bar{\sigma}\|_{L^2}^2 + C\|N\|_{L^6}^6. \tag{180}$$

In view of Poincaré inequality, we get

$$\frac{d}{dt}(\|\sigma - \bar{\sigma}\|_{L^2}^2 + \|\rho - \bar{\rho}\|_{L^2}^2) + D(\|\sigma - \bar{\sigma}\|_{L^2}^2 + \|\rho - \bar{\rho}\|_{L^2}^2) \le D\|\bar{\sigma}\|_{L^2}^2 + C\|N\|_{L^6}^6. \tag{181}$$

This gives (174). Integrating (180), we obtain (175).

Proposition 9. *Let $u_0 \in H^1, c_i(0) \in H^1$. Then, there exist positive constants M_1, M_2, M_3, M_4 and M_5 depending on the initial data and the parameters of the problem, and positive constants $\xi_1, \xi_2,$ and ξ_3 depending on f, N and $\bar{\sigma}$ such that*

$$\|\nabla u\|_{L^2}^2 \le M_1(\|\nabla u_0\|_{L^2}, \|\sigma_0\|_{L^2}, \|\rho_0\|_{L^2})e^{-Dt} + \xi_1(f, N, \bar{\sigma}), \tag{182}$$

$$\|\rho\|_{L^3}^2 \le M_2(\|\rho_0\|_{L^3}, \|\sigma_0\|_{L^2})e^{-Dt} + \xi_2(f, N, \bar{\sigma}), \tag{183}$$

and

$$\|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2 \le M_3(\|\nabla\rho_0\|_{L^2}, \|\nabla\sigma_0\|_{L^2}, \|\rho_0\|_{L^3}, \|\nabla u_0\|_{L^2})e^{-Dt} + \xi_3(f, N, \bar{\sigma}) \tag{184}$$

hold for any $t \ge 0$. Moreover,

$$\int_t^{t+T} (\|\Delta\rho\|_{L^2}^2 + \|\Delta\sigma\|_{L^2}^2) ds \le M_4(\|\nabla\rho_0\|_{L^2}, \|\nabla\sigma_0\|_{L^2}, \|\rho_0\|_{L^3}, \|\nabla u_0\|_{L^2})e^{-Dt} + \xi_3(f, N, \bar{\sigma})(T+1) \tag{185}$$

and

$$\int_t^{t+T} \|\Delta u\|_{L^2}^2 ds \le M_5(\|\nabla u_0\|_{L^2}, \|\sigma_0\|_{L^2}, \|\rho_0\|_{L^2}) e^{-Dt} + \xi_1(f, N, \bar{\sigma})(T+1) \tag{186}$$

hold for any $t \ge 0$, $T > 0$.

Proof. The proof is similar to that of Proposition 5. We briefly sketch the main ideas. Taking the L^2 inner product of the u-equation with $-\Delta u$ leads to the differential inequality

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \nu\|\Delta u\|_{L^2}^2 \le C\|\rho\|_{L^2}^6 + C\|\rho\|_{L^3}^3 + C_{f,N}. \tag{187}$$

An application of Lemma 1 gives (186). Integrating (187) gives (186).

Taking the L^2 inner product of the ρ-equation (66) with $\rho|\rho|$ and estimating the resulting terms gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\rho\|_{L^3}^2 + \frac{D\bar{\sigma}}{\epsilon}\|\rho\|_{L^3}^2 &\le C\|\sigma - \bar{\sigma}\|_{H^1}^{2/3}\|\sigma - \bar{\sigma}\|_{L^2}^{4/3}\|\rho\|_{L^3}^2 + C_N \\ &\le C\|\rho\|_{L^3}^3 + C\|\nabla\sigma\|_{L^2}^2\|\sigma - \bar{\sigma}\|_{L^2}^4 + C_N. \end{aligned} \tag{188}$$

Thus, Lemma 1 gives (183).

Finally, taking the L^2 inner product of the ρ-equation (66) and of the σ-equation with $-\Delta\rho$ and $-\Delta\sigma$ respectively, adding the resulting equations, and estimating the obtained terms give the differential inequality

$$\begin{aligned} \frac{d}{dt}(\|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2) + D(\|\Delta\rho\|_{L^2} + \|\Delta\sigma\|_{L^2}^2) &\le C(\|\nabla u\|_{L^2}^4 + \|\rho\|_{L^3}^2)(\|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2) \\ &+ C\|\sigma - \bar{\sigma}\|_{L^2}^2\|\nabla\sigma\|_{L^2}^2 \\ &+ C\|\rho\|_{L^2}^2\|\nabla\rho\|_{L^2}^2 + C_N \end{aligned} \tag{189}$$

Lemma 1 gives (184). Integrating (189) gives (185).

Proposition 10. *Let $u_0 \in H^2, c_i(0) \in H^2$. Then, there exist positive constants M_6 and M_7 depending on the initial data and the parameters of the problem, and positive constants ξ_4 and ξ_5 depending on f, N and $\bar{\sigma}$ such that*

$$\|\Delta u\|_{L^2}^2 \le M_6(\|\Delta u_0\|_{L^2}, \|\nabla\sigma_0\|_{L^2}, \|\nabla\rho_0\|_{L^2})e^{-Dt} + \xi_4(f, N, \bar{\sigma}) \tag{190}$$

and

$$\|\Delta\rho\|_{L^2}^2 + \|\Delta\rho\|_{L^2}^2 \le M_7(\|\Delta\rho_0\|_{L^2}, \|\Delta\sigma_0\|_{L^2}, \|\nabla u_0\|_{L^2})e^{-Dt} + \xi_5(f, N, \bar{\sigma}) \tag{191}$$

hold for all $t \ge 0$.

Proof. The proof follows the derivation of (80) and (81) in Proposition 5. We omit the details.

Let $\mathcal{V}''$ be the convex subset of $\mathcal{V} = H^1 \oplus H \oplus H^1 \oplus H^1$ consisting of vectors (u, c_1, c_2) such that u is divergence free with mean zero and c_1 and c_2 are non-negative functions a.e. whose difference has a space average equal to $-\bar{N}$. We define the solution map

$$\mathcal{O}(t) : \mathcal{V}'' \mapsto \mathcal{V}'' \tag{192}$$

corresponding to system (173) by

$$\mathcal{O}(t)(u_0, c_1(0), c_2(0)) = (u(t), c_1(t), c_2(t)). \tag{193}$$

For each $M > 0$, we consider the convex subset $\mathcal{V}'_M$ of $\mathcal{V}''$ consisting of vectors (u, c_1, c_2) such that u is divergence free with mean zero and c_1 and c_2 are non-negative functions a.e. whose space averages are less than or equal to M and whose difference has a space average equal to $-\bar{N}$. By Proposition 10, there exists $R_2 > 0$ depending on the body forces f, the added charge density N, and the positive constant M, such that for any $w_0 = (u_0, c_1(0), c_2(0)) \in \mathcal{V}'_M$, there exists $t'_0 > 0$ depending on $\|u_0\|_{H^1}, \|c_1(0)\|_{H^1}, \|c_2(0)\|_{H^1}$ such that

$$\mathcal{O}(t)w_0 \in \mathcal{B}_{R_2}^M = \left\{w = (u, c_1, c_2) \in \mathcal{V}'_M : \|u\|_{H^2} + \|c_1 - \bar{c}_1\|_{H^2} + \|c_2 - \bar{c}_2\|_{H^2} \le R_2\right\} \tag{194}$$

for all $t \geq t_0'$. We note that the map $\mathcal{O}(t)$ has the same properties as the map $\mathcal{S}(t)$, namely the existence of a compact absorbing ball, continuity properties (cf. Theorem 5) and injectivity (cf. Theorem 6). The existence of a global attractor is proved as in Theorem 7 and its finite dimensionality follows from decay of volume elements (Theorem 8) like in Theorems 9 and 10. The proofs of these theorems are similar to the proofs of the respective results for $N = 0$, and are omitted.

Theorem 11. *There exists a global attractor X which is compact in $\mathcal{V}''$ and has finite fractal dimension, such that*

$$\lim_{t\to\infty} dist_{\mathcal{V}}(\mathcal{O}(t)w_0, X) = 0 \tag{195}$$

holds uniformly for w_0 in bounded sets in $\mathcal{V}''$.

7. Appendix A

We give the proof of the backward uniqueness property of the solution map $\mathcal{S}(t)$.

Let $w(t) = \mathcal{S}(t)w_1^0 - \mathcal{S}(t)w_2^0 = (u(t), c_1(t), c_2(t))$ and $\widetilde{w}(t) = \frac{1}{2}(\mathcal{S}(t)w_1^0 + \mathcal{S}(t)w_2^0) = (\widetilde{u}(t), \widetilde{c}_1(t), \widetilde{c}_2(t))$. Let $\rho = c_1 - c_2$, $\widetilde{\rho} = \widetilde{c}_1 - \widetilde{c}_2$, $\Phi = \frac{1}{\epsilon}\Lambda^{-2}\rho$ and $\widetilde{\Phi} = \frac{1}{\epsilon}\Lambda^{-2}\widetilde{\rho}$.

We note that $w(t)$ obeys the equation

$$\partial_t w + \mathcal{A}w + L(\widetilde{w})w = 0 \tag{196}$$

where

$$\mathcal{A}w = (\nu Au, -D\Delta c_1, -D\Delta c_2) \tag{197}$$

and

$$L(\widetilde{w})w = (L_1(\widetilde{w})w, L_2(\widetilde{w})w, L_3(\widetilde{w})w) \tag{198}$$

with

$$L_1(\widetilde{w})w = B(\widetilde{u}, u) + B(u, \widetilde{u}) + \mathbb{P}(\rho\nabla\widetilde{\Phi} + \widetilde{\rho}\nabla\Phi)), \tag{199}$$

$$L_2(\widetilde{w})w = u\cdot\nabla\widetilde{c}_1 + \widetilde{u}\cdot\nabla c_1 - D\nabla\cdot(c_1\nabla\widetilde{\Phi} + \widetilde{c}_1\nabla\Phi), \tag{200}$$

$$L_3(\widetilde{w})w = u\cdot\nabla\widetilde{c}_2 + \widetilde{u}\cdot\nabla c_2 + D\nabla\cdot(c_2\nabla\widetilde{\Phi} + \widetilde{c}_2\nabla\Phi). \tag{201}$$

We consider the evolution of the norm

$$E_0 = \|u\|_{L^2}^2 + \|c_1\|_{L^2}^2 + \|c_2\|_{L^2}^2 = \|w\|_H^2 \tag{202}$$

obtained by taking the inner product in $\mathcal{H}$ of equation (196) with $(u, c_1, c_2) = w$, and we note that E_0 obeys the equation

$$\frac{1}{2}\frac{d}{dt}E_0 + E_1 + (L(\widetilde{w})w, w)_{\mathcal{H}} = 0 \tag{203}$$

where

$$E_1 = \nu\|A^{\frac{1}{2}}u\|_H^2 + D\|\nabla c_1\|_{L^2}^2 + D\|\nabla c_2\|_{L^2}^2 = (w, \mathcal{A}w)_{\mathcal{H}}. \tag{204}$$

We observe that

$$\frac{1}{2}\frac{d}{dt}\log\left(\frac{1}{E_0}\right) = \frac{E_1}{E_0} + \frac{(L(\widetilde{w})w, w)_{\mathcal{H}}}{E_0} \tag{205}$$

Let

$$Y(t) = \log\left(\frac{1}{E_0}\right) \tag{206}$$

and so

$$\frac{1}{2}\frac{d}{dt}Y(t) = \frac{E_1}{E_0} + \frac{(L(\widetilde{w})w, w)_{\mathcal{H}}}{E_0}. \tag{207}$$

We proceed to show that $Y(t)$ cannot reach the value $+\infty$ in finite time. We start by noting that the derivative of E_1/E_0 obeys

$$\frac{d}{dt}\frac{E_1}{E_0} = E_0^{-1}\frac{d}{dt}E_1 - \frac{E_1}{E_0}\frac{d}{dt}\log E_0 = E_0^{-1}\frac{d}{dt}E_1 + \frac{E_1}{E_0}\frac{d}{dt}Y. \tag{208}$$

Taking the inner product of equation (196) in $\mathcal{H}$ with $\mathcal{A}w$ leads to

$$\frac{1}{2}\frac{d}{dt}E_1 + \|\mathcal{A}w\|_{\mathcal{H}}^2 + (L(\widetilde{w})w, \mathcal{A}w)_{\mathcal{H}} = 0 \tag{209}$$

which implies that

$$\frac{1}{2}\frac{d}{dt}\frac{E_1}{E_0} = -\frac{\|\mathcal{A}w\|_{\mathcal{H}}^2 + (L(\widetilde{w})w, \mathcal{A}w)_{\mathcal{H}}}{E_0} + \frac{E_1}{E_0}\left(\frac{E_1 + (L(\widetilde{w})w, w)_{\mathcal{H}}}{E_0}\right) \tag{210}$$

Since

$$\frac{E_1^2}{E_0^2} - \frac{\|\mathcal{A}w\|_{\mathcal{H}}^2}{E_0} = -\left\|\left(\mathcal{A} - \frac{E_1}{E_0}\right)\frac{w}{E_0^{1/2}}\right\|_{\mathcal{H}}^2, \tag{211}$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\frac{E_1}{E_0} = -E_0^{-1}\|(\mathcal{A} - E_1E_0^{-1})w\|_{\mathcal{H}}^2 - E_0^{-1}\left(L(\widetilde{w})w, \left(A - E_1E_0^{-1}\right)w\right)_{\mathcal{H}}. \tag{212}$$

Now, we claim that

$$|(L(\widetilde{w})w, w)_{\mathcal{H}}| \le A_1(t)E_1 + A_0(t)E_0 \tag{213}$$

with

$$\int_0^T (A_0(t) + A_1(t))dt < \infty. \tag{214}$$

To prove this claim, we note first that

$$(B(\widetilde{u}, u), u)_{L^2} = (\widetilde{u}\cdot\nabla c_1, c_1)_{L^2} = (\widetilde{u}\cdot\nabla c_2, c_2)_{L^2} = 0. \tag{215}$$

Since u has mean zero, an application of Ladyzhenskaya's inequality followed by Poincaré's inequality gives

$$|(B(u, \widetilde{u}), u)_{L^2}| \le \|\nabla\widetilde{u}\|_{L^2}\|u\|_{L^4}^2 \le C\|\nabla\widetilde{u}\|_{L^2}\|\nabla u\|_{L^2}^2 \le C\|\nabla\widetilde{u}\|_{L^2}E_1. \tag{216}$$

Using in addition elliptic regularity and the fact that ρ has mean zero, we obtain

$$\begin{aligned}|(\mathbb{P}(\rho\nabla\widetilde{\Phi} + \widetilde{\rho}\nabla\Phi), u)_{L^2}| &\le \|u\|_{L^4}\|\rho\|_{L^4}\|\nabla\widetilde{\Phi}\|_{L^2} + \|u\|_{L^4}\|\widetilde{\rho}\|_{L^2}\|\nabla\Phi\|_{L^4}\\ &\le C\|\nabla u\|_{L^2}(\|\nabla c_1\|_{L^2} + \|\nabla c_2\|_{L^2})(\|\nabla\widetilde{\Phi}\|_{L^2} + \|\widetilde{\rho}\|_{L^2})\\ &\le C\left(1 + \|\nabla\widetilde{\Phi}\|_{L^2}^2 + \|\widetilde{\rho}\|_{L^2}^2\right)E_1.\end{aligned} \tag{217}$$

Now, we estimate

$$|(u\cdot\nabla\widetilde{c}_1, c_1)_{L^2}| = |(u\cdot\nabla c_1, \widetilde{c}_1)_{L^2}| \le C\|\nabla u\|_{L^2}\|\nabla c_1\|_{L^2}\|\widetilde{c}_1\|_{L^4} \le C(1 + \|\widetilde{c}_1\|_{L^4}^2)E_1, \tag{218}$$

$$|(u\cdot\nabla\widetilde{c}_2, c_2)_{L^2}| \le C(1 + \|\widetilde{c}_2\|_{L^4}^2)E_1, \tag{219}$$

$$\begin{aligned}|(\nabla\cdot(c_1\nabla\widetilde{\Phi} + \widetilde{c}_1\nabla\Phi), c_1)_{L^2}| &\le C(\|c_1\|_{L^2}\|\nabla\widetilde{\Phi}\|_{L^\infty}\|\nabla c_1\|_{L^2} + \|\widetilde{c}_1\|_{L^2}\|\nabla c_1\|_{L^2}(\|\nabla c_1\|_{L^2} + \|\nabla c_2\|_{L^2}))\\ &\le C(\|\nabla\widetilde{\Phi}\|_{L^\infty}^2 + \|\widetilde{c}_1\|_{L^2} + \|\widetilde{c}_1\|_{L^2}^2 + 1)E_1 + E_0\end{aligned} \tag{220}$$

and

$$|(\nabla\cdot(c_2\nabla\widetilde{\Phi} + \widetilde{c}_2\nabla\Phi), c_2)_{L^2}| \le C(\|\nabla\widetilde{\Phi}\|_{L^\infty}^2 + \|\widetilde{c}_2\|_{L^2} + \|\widetilde{c}_2\|_{L^2}^2 + 1)E_1 + E_0 \tag{221}$$

This ends the proof of the first claim.

Next, we claim that

$$\|L(\widetilde{w})w\|_{\mathcal{H}}^2 \le B_1(t)E_1 + B_0(t)E_0 \tag{222}$$

with

$$\int_0^T (B_0(t) + B_1(t))dt < \infty. \tag{223}$$

Since u and ρ have mean zero, then elliptic regularity together with an application of Hölder, Ladyzhenskaya, Poincaré and Young inequalities gives

$$\|B(\widetilde{u},u)+B(u,\widetilde{u})\|_{L^2}^2 \le C(\|\widetilde{u}\|_{L^\infty}^2\|\nabla u\|_{L^2}^2+\|\nabla u\|_{L^2}^2\|\nabla\widetilde{u}\|_{L^4}^2) \le C(\|\widetilde{u}\|_{L^\infty}^2+\|\nabla\widetilde{u}\|_{L^4}^2)E_1, \tag{224}$$

$$\|\mathbb{P}(\rho\nabla\widetilde{\Phi}+\widetilde{\rho}\nabla\Phi)\|_{L^2}^2 \le C(\|\nabla\rho\|_{L^2}^2\|\nabla\widetilde{\Phi}\|_{L^\infty}^2+\|\widetilde{\rho}\|_{L^2}^2\|\nabla\rho\|_{L^2}^2) \le C(\|\nabla\widetilde{\Phi}\|_{L^\infty}^2+\|\widetilde{\rho}\|_{L^2}^2)E_1, \tag{225}$$

$$\|u\cdot\nabla\widetilde{c}_1+\widetilde{u}\cdot\nabla c_1\|_{L^2}^2 \le C(\|\nabla u\|_{L^2}^2\|\nabla\widetilde{c}_1\|_{L^4}^2+\|\widetilde{u}\|_{L^\infty}^2\|\nabla c_1\|_{L^2}^2) \le C(\|\nabla\widetilde{c}_1\|_{L^4}^2+\|\widetilde{u}\|_{L^\infty}^2)E_1, \tag{226}$$

$$\|u\cdot\nabla\widetilde{c}_2+\widetilde{u}\cdot\nabla c_2\|_{L^2}^2 \le C(\|\nabla\widetilde{c}_2\|_{L^4}^2+\|\widetilde{u}\|_{L^\infty}^2)E_1, \tag{227}$$

$$\begin{aligned}
\|\nabla\cdot(c_1\nabla\widetilde{\Phi}+\widetilde{c}_1\nabla\Phi)\|_{L^2}^2 &= \|c_1\Delta\widetilde{\Phi}+\nabla c_1\nabla\widetilde{\Phi}+\widetilde{c}_1\Delta\Phi+\nabla\widetilde{c}_1\nabla\Phi\|_{L^2}^2\\
&\le C(\|c_1\|_{L^2}\|\nabla c_1\|_{L^2}\|\widetilde{\rho}\|_{L^2}\|\nabla\widetilde{\rho}\|_{L^2}+\|\nabla c_1\|_{L^2}^2\|\nabla\widetilde{\Phi}\|_{L^\infty}^2)\\
&+C(\|\widetilde{c}_1\|_{L^4}^2\|\rho\|_{L^2}\|\nabla\rho\|_{L^2}+\|\nabla\widetilde{c}_1\|_{L^2}^2\|\nabla\rho\|_{L^2}^2)\\
&\le C(\|\widetilde{\rho}\|_{L^2}^2\|\nabla\widetilde{\rho}\|_{L^2}^2+\|\nabla\widetilde{\Phi}\|_{L^\infty}^2+\|\widetilde{c}_1\|_{L^4}^2+\|\nabla\widetilde{c}_1\|_{L^2}^2)E_1+E_0,
\end{aligned} \tag{228}$$

and

$$\|\nabla\cdot(c_2\nabla\widetilde{\Phi}+\widetilde{c}_2\nabla\Phi)\|_{L^2}^2 \le C(\|\widetilde{\rho}\|_{L^2}^2\|\nabla\widetilde{\rho}\|_{L^2}^2+\|\nabla\widetilde{\Phi}\|_{L^\infty}^2+\|\widetilde{c}_2\|_{L^4}^2+\|\nabla\widetilde{c}_2\|_{L^2}^2)E_1+E_0. \tag{229}$$

Thus, the second claim is proved.

As a consequence of the above claims and Schwarz inequality, we deduce the differential inequalities

$$\frac{d}{dt}\frac{E_1}{E_0} \le 2B_1(t)\frac{E_1}{E_0}+2B_0(t) \tag{230}$$

and

$$\frac{d}{dt}Y(t) \le (2A_1(t)+1)\frac{E_1}{E_0}+2A_0(t) \tag{231}$$

which imply that $Y(t)\in L^\infty(0,T)$. This ends the proof.

8. Appendix B

We give the proof of the exponential decay of volume elements.

For each t, choose an orthonormal basis $b_i=(v_i,r_i^1,r_i^2)$ of the linear span of $w_1,\dots,w_n$. Then

$$\mathrm{Trace}((\mathcal{A}+L(\widetilde{w}))Q_n)=\sum_{i=1}^n(\mathcal{A}b_i,b_i)_{\mathcal{H}}+\sum_{i=1}^n(L(\widetilde{w})b_i,b_i)_{\mathcal{H}}. \tag{232}$$

We note that

$$\mathrm{Trace}(\mathcal{A}Q_n)=\sum_{i=1}^n(\mathcal{A}b_i,b_i)_{\mathcal{H}}=\sum_{i=1}^n\left[(\nu Av_i,v_i)_{\mathcal{H}}+(-D\Delta r_i^1,r_i^1)_{L^2}+(-D\Delta r_i^2,r_i^2)_{L^2}\right]\ge\mu_1+\dots+\mu_n \tag{233}$$

where μ_i are eigenvalues of $\mathcal{A}$ in $\mathcal{H}$. By Proposition 1, there exists a constant C such that $\mu_k\ge Ck$ for all $k\ge1$. It follows that $\mathrm{Trace}(\mathcal{A}Q_n)\ge C_0n^2$ for some positive constant C_0.

Let $\rho_i=r_i^1-r_i^2$ and $\Phi_i=\frac{1}{\epsilon}\Lambda^{-2}\rho_i$. In view of Hölder's inequality, Ladyzhenskaya's inequality, elliptic regularity and the fact that $\|b_i\|_{\mathcal{H}}=1$ for all i, we have the bounds

$$\left|\sum_{i=1}^n(B(v_i,\widetilde{u}),v_i)_{L^2}\right|\le\sum_{i=1}^n\|v_i\|_{L^4}^2\|\nabla\widetilde{u}\|_{L^2}\le C\|\nabla\widetilde{u}\|_{L^2}n^{1/2}\left(\sum_{i=1}^n\|\nabla v_i\|_{L^2}^2\right)^{1/2}, \tag{234}$$

$$
\begin{aligned}
\left|\sum_{i=1}^{n}(\mathbb{P}(\rho_i\nabla\widetilde{\Phi}+\widetilde{\rho}\nabla\Phi_i),b_i)_{L^2}\right| &\le \sum_{i=1}^{n}\left(\|\nabla\widetilde{\Phi}\|_{L^\infty}\|\rho_i\|_{L^2}\|b_i\|_{L^2}+\|\nabla\Phi_i\|_{L^\infty}\|\widetilde{\rho}\|_{L^2}\|b_i\|_{L^2}\right)\\
&\le \sum_{i=1}^{n}\left(2\|\nabla\widetilde{\Phi}\|_{L^\infty}+C\|\widetilde{\rho}\|_{L^2}(\|r_i^1\|_{L^4}+\|r_i^2\|_{L^4})\right)\\
&\le \sum_{i=1}^{n}\left(2\|\nabla\widetilde{\Phi}\|_{L^\infty}+C\|\widetilde{\rho}\|_{L^2}\|\nabla r_i^1\|_{L^2}^{1/2}+C\|\widetilde{\rho}\|_{L^2}\|\nabla r_i^2\|_{L^2}^{1/2}\right)\\
&\le 2\|\nabla\widetilde{\Phi}\|_{L^\infty}n+C\|\widetilde{\rho}\|_{L^2}n^{3/4}\left[\left(\sum_{i=1}^{n}\|\nabla r_i^1\|_{L^2}^2\right)^{1/4}+\left(\sum_{i=1}^{n}\|\nabla r_i^2\|_{L^2}^2\right)^{1/4}\right],
\end{aligned}
\tag{235}
$$

$$
\begin{aligned}
\left|\sum_{i=1}^{n}(v_i\cdot\nabla\widetilde{c}_1,r_i^1)_{L^2}\right| &\le \sum_{i=1}^{n}\|v_i\|_{L^4}\|r_i^1\|_{L^4}\|\nabla\widetilde{c}_1\|_{L^2}\le\sum_{i=1}^{n}C\|\nabla v_i\|_{L^2}^{1/2}\|\nabla r_i^1\|_{L^2}^{1/2}\|\nabla\widetilde{c}_1\|_{L^2}\\
&\le C\|\nabla\widetilde{c}_1\|_{L^2}n^{1/2}\left(\sum_{i=1}^{n}\|\nabla v_i\|_{L^2}^2\right)^{1/4}\left(\sum_{i=1}^{n}\|\nabla r_i^1\|_{L^2}^2\right)^{1/4}
\end{aligned}
\tag{236}
$$

and

$$
\left|\sum_{i=1}^{n}(v_i\cdot\nabla\widetilde{c}_2,r_i^2)_{L^2}\right|\le C\|\nabla\widetilde{c}_2\|_{L^2}n^{1/2}\left(\sum_{i=1}^{n}\|\nabla v_i\|_{L^2}^2\right)^{1/4}\left(\sum_{i=1}^{n}\|\nabla r_i^2\|_{L^2}^2\right)^{1/4}. \tag{237}
$$

Now, using the triangle inequality, we have

$$
\begin{aligned}
&\left|\sum_{i=1}^{n}\left[-(\nabla\cdot(r_i^1\nabla\widetilde{\Phi}+\widetilde{c}_1\nabla\Phi_i),r_i^1)_{L^2}+(\nabla\cdot(r_i^2\nabla\widetilde{\Phi}+\widetilde{c}_2\nabla\Phi_i),r_i^2)_{L^2}\right]\right|\\
&\le\left|\sum_{i=1}^{n}\left[(r_i^1\nabla\widetilde{\Phi},\nabla r_i^1)_{L^2}-(r_i^2\nabla\widetilde{\Phi},\nabla r_i^2)_{L^2}\right]\right|\\
&+\left|\sum_{i=1}^{n}\left[((\widetilde{c}_1-\bar{\widetilde{c}}_1)\nabla\Phi_i,\nabla r_i^1)_{L^2}-((\widetilde{c}_2-\bar{\widetilde{c}}_2)\nabla\Phi_i,\nabla r_i^2)_{L^2}+(\bar{c}\nabla\Phi_i,\nabla(r_i^1-r_i^2))_{L^2}\right]\right|
\end{aligned}
\tag{238}
$$

where $\bar{c}=\bar{\widetilde{c}}_1=\bar{\widetilde{c}}_2$, and using the same inequalities as above, we obtain

$$
\begin{aligned}
\left|\sum_{i=1}^{n}[(r_i^1\nabla\widetilde{\Phi},\nabla r_i^1)_{L^2}-(r_i^2\nabla\widetilde{\Phi},\nabla r_i^2)_{L^2}]\right| &\le\sum_{i=1}^{n}\left[\|\nabla\widetilde{\Phi}\|_{L^\infty}\|\nabla r_i^1\|_{L^2}+\|\nabla\widetilde{\Phi}\|_{L^\infty}\|\nabla r_i^2\|_{L^2}\right]\\
&\le\|\nabla\widetilde{\Phi}\|_{L^\infty}n^{1/2}\left[\left(\sum_{i=1}^{n}\|\nabla r_i^1\|_{L^2}^2\right)^{1/2}+\left(\sum_{i=1}^{n}\|\nabla r_i^2\|_{L^2}^2\right)^{1/2}\right]
\end{aligned}
\tag{239}
$$

and

$$
\begin{aligned}
&\left|\sum_{i=1}^{n}((\widetilde{c}_1-\bar{\widetilde{c}}_1)\nabla\Phi_i,\nabla r_i^1)_{L^2}\right|\le\sum_{i=1}^{n}\|\nabla r_i^1\|_{L^2}\|\nabla\Phi_i\|_{L^\infty}\|\widetilde{c}_1-\bar{\widetilde{c}}_1\|_{L^2}\\
&\le\sum_{i=1}^{n}C\|\nabla r_i^1\|_{L^2}\|\nabla\rho_i\|_{L^2}^{1/2}\|\widetilde{c}_1-\bar{\widetilde{c}}_1\|_{L^2}\le\sum_{i=1}^{n}C\|\nabla r_i^1\|_{L^2}\left(\|\nabla r_i^1\|_{L^2}^{1/2}+\|\nabla r_i^2\|_{L^2}^{1/2}\right)\|\widetilde{c}_1-\bar{\widetilde{c}}_1\|_{L^2}\\
&\le Cn^{1/4}\|\widetilde{c}_1-\bar{\widetilde{c}}_1\|_{L^2}\left(\sum_{i=1}^{n}\|\nabla r_i^1\|_{L^2}^2\right)^{3/4}+Cn^{1/4}\|\widetilde{c}_1-\bar{\widetilde{c}}_1\|_{L^2}\left(\sum_{i=1}^{n}\|\nabla r_i^2\|^2\right)^{1/4}\left(\sum_{i=1}^{n}\|\nabla r_i^1\|_{L^2}^2\right)^{1/2},
\end{aligned}
\tag{240}
$$

$$\left|\sum_{i=1}^{n}((\widetilde{c}_2 - \overline{\widetilde{c}_2})\nabla\Phi_i, \nabla r_i^2)_{L^2}\right| \le Cn^{1/4}\|\widetilde{c}_2 - \overline{\widetilde{c}}_2\|_{L^2}\left(\sum_{i=1}^{n}\|\nabla r_i^2\|_{L^2}^2\right)^{3/4} + Cn^{1/4}\|\widetilde{c}_2 - \overline{\widetilde{c}}_2\|_{L^2}\left(\sum_{i=1}^{n}\|\nabla r_i^1\|^2\right)^{1/4}\left(\sum_{i=1}^{n}\|\nabla r_i^2\|_{L^2}^2\right)^{1/2} \tag{241}$$

and

$$\left|\sum_{i=1}^{n}(\bar{c}\nabla\Phi_i, \nabla(r_i^1 - r_i^2))_{L^2}\right| = \sum_{i=1}^{n}\frac{\bar{c}}{\epsilon}\|\nabla\Lambda^{-1}(r_i^1 - r_i^2)\|_{L^2}^2 \le \sum_{i=1}^{n}C\bar{c}\|r_i^1 - r_i^2\|_{L^2}^2 \le 4C\bar{c}n. \tag{242}$$

Since $\widetilde{w}_0 \in \mathcal{B}_{R_1}^M$, there exists t_0 depending on R_1 such that $\widetilde{w}(t) = \mathcal{S}(t)\widetilde{w}_0 \in \mathcal{B}_{R_1}^M$ for all $t \ge t_0$.

Combining the bounds (234)–(242) and applying Young's inequality give

$$\frac{1}{t}\int_0^t \mathrm{Trace}((\mathcal{A} + L(\widetilde{w}))Q_n)ds \ge \frac{1}{4}\mathrm{Trace}(\mathcal{A}Q_n) - C_1\bar{c}n - C_2C(R_1)n \ge n\left(\frac{1}{4}C_0n - C_1\bar{c} - C_2C(R_1)\right) \tag{243}$$

for all $t \ge t_0$. Here, C_1, C_2 are universal positive constants, $C(R_1)$ is a constant depending on R_1, and $0 \le \bar{c} \le M$. Thus, choosing

$$n \ge \frac{4}{C_0}(1 + C_1M + C_2C(R_1)) \tag{244}$$

ends the proof.

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