LONG TIME DYNAMICS OF A MODEL OF ELECTROCONVECTION

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ABSTRACT. We study a model of electroconvection in which a two dimensional viscous fluid caries electrical charges and interacts with them. The system has global solutions, but in general the solutions do not have bounded mean. Tracking the mean, we associate to each solution a mean zero frame and show that in the mean zero frame the system has a compact, finite dimensional global attractor. If the fluid is forced only by electrical forces and no other body forces are present, then the attractor reduces to one point.

1. INTRODUCTION

We consider an electroconvection model that describes the evolution of a surface charge density interacting with a two dimensional fluid. The model was used in theoretical and numerical studies related to experiments of electroconvection in thin smectic layers of liquid crystals ([9], [13]). Analogies with Rayleigh-Bénard convection motivated the physical studies ([12]).

The surface charge density $q = q(x,t)$ is a real valued function of position x and time t. Its evolution is a continuity equation, with the current density J given by the sum of the Ohmic density σE, with E the electric field, and the advective current density qu, where u is the velocity of the fluid. Magnetic effects are neglected and the electric field E is the gradient of a potential. The restriction to a two dimensional region results in a nonlocal relation between the surface charge density and the divergence of the electrical field ([2], [12], [13]). The evolution of the surface charge density is given by

$$\partial_t q + \nabla \cdot J = 0 \tag{1}$$

where the current density J is given by

$$J = \sigma E + qu \tag{2}$$

with σ a constant conductivity, and the electric field given by

$$E = -\nabla\Phi - \nabla\Lambda^{-1}q. \tag{3}$$

Here Φ is a given smooth function which represents the restriction to the surface of the potential due to the applied voltage, and $\Lambda^{-1}q$ (with Λ the square root of the two dimensional spatially periodic Laplacian) is the restriction to the surface of the potential due to the surface density charge q. The equation is coupled to the incompressible Navier-Stokes system

$$\partial_t u + u\cdot\nabla u + \nabla p - \Delta u = qE + f, \quad \nabla\cdot u = 0, \tag{4}$$

where f are body forces in the fluid. In this paper we consider two dimensional periodic boundary conditions. The potential Φ and forces f are time independent and smooth.

The global existence of regular solutions of this system with homogeneous Dirichlet boundary conditions was established in [2]. In this work we focus on long time dynamics. The long time dynamics of dissipative partial differential equations has been investigated by many authors. The

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two dimensional forced Navier-Stokes equations are known to possess global finite dimensional attractors ([3], [4], and references therein). The long time behavior of various types of dissipative PDE has been studied extensively ([1], [7], [8], [10], [11]). Closer to the present system, the study of long time dynamics of the critical dissipative SQG system with fractional Laplacian dissipation and the existence of a finite dimensional global attractor were done in [5].

We investigate the system (1), (2), (3), (4). This has weak solutions in L^2 (Theorem 1) which, however, are not known to be unique. After any positive time, weak solutions become strong, and strong solutions exist globally and are unique (Theorem 2). Our main result is the existence of a global attractor X which is compact in a natural phase space of strong solutions and has finite fractal dimension. In order to establish the existence of the attractor we need to account for the fact that spatial averages of the velocity are time dependent, and might grow in time, driven by the integral $\int q\nabla\Phi$. This integral does not vanish in general, nor is it time integrable. The remarkable property of the system is that the spatial average of velocity can be tracked, or "moded" out, and the resulting system has a compact global attractor. In this mean zero frame, the initial value problem for the system is solved by a nonlinear semigroup $S(t)$ which has a compact absorbing ball, is Lipschitz continuous in various norms, is injective, and high dimensional volume elements carried by its flow decay in phase space.

The paper is organized as follows. In Section 2 we gather preliminaries concerning the dissipative operators. A lower bound, in the spirit of [6], Proposition 2, is proved in Appendix A. Commutator estimates (Proposition 3) and a uniform Gronwall lemma for exponential decay (Lemma 1) are also proved in this section. Section 3 is devoted to basic PDE results: existence of weak solutions, existence and uniqueness of strong solutions. Here we also prove uniform long time bounds for various norms of the solutions, which have the feature that the initial data contributions to them decay exponentially, leaving only contributions coming from the steady forces. The passage to the mean zero frame is described in Section 4. The absorbing ball for the nonlinear semigroup is described in Section 5. In Section 6 continuity properties of the semigroup are established, and Section 7 is devoted to the proof of backward uniqueness. Decay of volume elements is proved in Section 8. In Section 9 we prove the finite dimensionality of the attractor for general fluid body forces f. We also show that in the absence of body forces in the fluid, the system has a unique globally attracting steady solution in the mean zero frame. In this case, in the original variables, the fluid's spatial average velocity has a finite limit in infinite time.

2. Preliminaries

We denote functions spaces of spatially periodic functions on the torus without distinct notation for vector valued functions. We write the Fourier series for mean zero velocities u as

$$u = \sum_{j\in\mathbb{Z}^2\setminus\{0\}} u_j e^{ij\cdot x} \tag{5}$$

with $u_j \in \mathbb{C}^2$. The reality condition for the series is $\overline{u}_j = u_{-j}$. The divergence-free condition is

$$j \cdot u_j = 0. \tag{6}$$

For $s \in \mathbb{R}$, the fractional Laplacian Λ^s applied to a mean zero scalar function q is defined as a Fourier multiplier with symbol $|k|^s$, that is, for q given by

$$q = \sum_{k\in\mathbb{Z}^2\setminus\{0\}} q_k e^{ik\cdot x}, \tag{7}$$

we have that

$$\Lambda^s q = \sum_{k\in\mathbb{Z}^2\setminus\{0\}} |k|^s q_k e^{ik\cdot x}. \tag{8}$$

The Stokes operator $\mathbb{P}(-\Delta)$ is denoted by A. It is defined on Fourier series by

$$\mathbb{P}(-\Delta u) = Au = \sum_{j\in\mathbb{Z}^2\setminus\{0\}} |j|^2 \mathbb{P}_j(u_j) e^{ij\cdot x} \tag{9}$$

where

$$\mathbb{P}u = \sum_{j\in\mathbb{Z}^2\setminus\{0\}} \mathbb{P}_j(u_j) e^{ij\cdot x} \tag{10}$$

is the Leray-Hodge projector and

$$\mathbb{P}_j(v) = v - (v\cdot j)\frac{j}{|j|^2} \tag{11}$$

is the projector in $\mathbb{C}^2$ orthogonal on the unit vector $\frac{j}{|j|}$. We consider the Hilbert space $\mathcal{H}$

$$\mathcal{H} = H \oplus L^2 \tag{12}$$

where H is the Hilbert space of L^2 periodic vector fields which are mean zero and divergence-free, $H = \mathbb{P}(L^2)$. The scalar product in $\mathcal{H}$ is denoted $(\cdot;\cdot)$:

$$((u_1, q_1); (u_2, q_2)) = \int_{\mathbb{T}^2} (u_1\cdot u_2 + q_1 q_2) dx. \tag{13}$$

As all spatial integrals are on $\mathbb{T}^2$, we denote them simply by $\int$. We consider the operator $\mathcal{A}$ defined on $\mathcal{H}$ by

$$\mathcal{A}w = (Au, \Lambda q) \tag{14}$$

where $w = (u, q)$. The domain of definition of $\mathcal{A}$ is

$$\mathcal{D}(\mathcal{A}) = (H^2 \cap H) \oplus H^1. \tag{15}$$

The spaces H^s for mean zero functions or vectors are the same as the homogeneous Sobolev spaces $\dot{H}^s(\mathbb{T}^2)$. They are the closure of the space of zero-average functions in $C^\infty(\mathbb{T}^2)$ under the norm

$$\|\phi\|_{H^s} = \|\Lambda^s\phi\|_{L^2}. \tag{16}$$

The operator

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H} \tag{17}$$

is positive and selfadjoint. There is an orthonormal basis of the Hilbert space $\mathcal{H}$ formed by a sequence w_k of eigenvectors,

$$\mathcal{A}w_k = \mu_k w_k. \tag{18}$$

The set of eigenvalues is precisely the union of the eigenvalues of A and those of Λ, counted with their multiplicities. The multiplicity of an eigenvalue λ of A is the same as the multiplicity of the same eigenvalue λ considered as an eigenvalue of the scalar Laplacian with periodic boundary conditions on $[0, 2\pi]\times[0, 2\pi]$. This follows from the fact that in two dimensions we can uniquely associate a stream function to each eigenfunction of the Stokes operator A. It can be shown that the eigenvalues μ_k obey $0 < \mu_1 \le \ldots \mu_k \le \ldots$ and that there exists a constant C_0 such that

$$\mu_k \ge C_0\mu_1\sqrt{k} \tag{19}$$

holds for all $k \ge 1$. If we denote the eigenvalues of A counted with multiplicity by $0 < \lambda_1 \le \lambda_2 \dots \le \lambda_j \le \dots$ and those of Λ, counted also with multiplicity as $0 < r_1 \le r_2 \dots \le r_j \le \dots$ then we have $j \le c_1 \lambda_j$ and $k \le c_2 r_k^2$ with c_1, c_2 positive constants. Assuming that

$$\{\mu_i \mid \; i = 1, \dots N\} = \{\lambda_i \mid \; i = 1, \dots, j\} \cup \{r_i \mid \; i = 1, \dots, k\}$$

if $\mu_N = \lambda_j$ it follows that $j \le c_1 \mu_N$ and if $\mu_N = r_k$ it follows that $k \le c_2 \mu_N^2$. Because $N = j + k$ it follows that $N \le c_1 \mu_N + c_2 \mu_N^2 \le (c_1 + c_2) \mu_N^2$ because $\mu_N \ge 1$, and thus (19) follows.

For $p \in [1, \infty]$ we denote by $W^{s,p} = W^{s,p}(\mathbb{T}^2)$ the space of mean-zero $L^p(\mathbb{T}^2)$ functions ϕ, which can be written as $\phi = \Lambda^{-s}\psi$, with mean zero $\psi \in L^p$. This is normed by $\|\phi\|_{W^{s,p}} = \|\Lambda^s \phi\|_{L^p}$. The spaces H^s are the same as $W^{s,2}$. We recall that the Riesz transforms $R = (R_1, R_2)$ for periodic functions are defined as multipliers

$$(R_j q)_k = i \frac{k_j}{|k|} q_k, \quad k \in \mathbb{Z}^2 \smallsetminus \{0\}, \quad j = 1, 2, \tag{20}$$

and they are bounded operators in L^p, $1 < p < \infty$.

The fractional Laplacian has certain lower bounds in L^p spaces which we are going to use. A Poincaré inequality in L^p spaces is given in [5] in the following proposition

Proposition 1. *Let $p = 2m$, $m \ge 1$, $0 \le \alpha \le 2$, and let $q \in C^\infty$ have zero mean on $\mathbb{T}^2$. Then*

$$\int_{\mathbb{T}^2} q^{p-1}(x) \Lambda^\alpha q(x) dx \ge \frac{1}{p} \|\Lambda^{\alpha/2}(q^{p/2})\|_{L^2}^2 + \lambda \|q\|_{L^p}^p \tag{21}$$

holds, with an explicit constant $\lambda > 0$, which is independent of p.

Proposition 2. *The inequality*

$$\int \nabla q \cdot \Lambda \nabla q dx \ge c \|q\|_{L^4}^{-\frac{2}{3}} \|\nabla q\|_{L^{\frac{8}{3}}}^{\frac{8}{3}} \tag{22}$$

holds for $q \in H^{\frac{3}{2}}$.

This inequality is based on [6], [5]. For completeness, the proof is given in Appendix A.

The following commutator estimates are needed in the sequel.

Proposition 3. *Let $u \in H^2 \cap H$ and $q \in H^{s+\alpha}$. Let $s \in (-1, 1)$ and let $0 \le \alpha \le 1$ with $s + \alpha \le 1$. Then the commutator $[\Lambda^s, u \cdot \nabla]$ obeys the inequality*

$$\|[\Lambda^s, u \cdot \nabla] q\|_{L^2} \le C_s [u]_{1-\alpha} \|\Lambda^{s+\alpha} q\|_{L^2} \tag{23}$$

where

$$[u]_{1-\alpha} = \sum_{j \in \mathbb{Z} \smallsetminus \{0\}} |j|^{1-\alpha} |u_j|. \tag{24}$$

Proof. The function $\phi = [\Lambda^s, u \cdot \nabla] q$ has the Fourier expansion

$$\phi_l = i \sum_{j+k=l} (u_j \cdot k) q_k (|l|^s - |k|^s). \tag{25}$$

In view of the fact that $u_j \cdot j = 0$ we have $u_j \cdot k = -u_j \cdot l$ and therefore

$$|u_j \cdot k| \le |u_j| \min\{|l|, |k|\}.$$

If s is negative then we write

$$|l|^{-r} - |k|^{-r} = \frac{|k|^r - |l|^r}{|l|^r |k|^r}$$

with $r = |s|$, and we estimate for positive numbers $m \le M$ and exponent $0 \le r \le 1$ using the conjugate powers:

$$(M^r - m^r)(M^{1-r} + m^{1-r}) = M - m + M^r m^{1-r} - m^r M^{1-r} \le 2(M - m).$$

We denote by $M = \max\{|l|, |k|\}$, $m = \min\{|l|, |k|\}$. For $s < 0$, using the triangle inequality $M - m \le |j| \le 2M$, we obtain that

$$\frac{2m|j|^\alpha}{M^r m^r (M^{1-r} + m^{1-r})} \le \frac{2^{1+\alpha} m^{1-r} M^\alpha}{M} = 2^{1+\alpha} m^{-r+\alpha} \left(\frac{m}{M}\right)^{1-\alpha} = 2^{1+\alpha} M^{-r+\alpha} \left(\frac{m}{M}\right)^{(1-r)} \le 2^{1+\alpha} |k|^{-r+\alpha}$$

and therefore

$$|(u_j \cdot k) q_k (|l|^s - |k|^s)| \le \frac{2m|j|}{M^r m^r (M^{1-r} + m^{1-r})} |u_j||q_k| \le 2^{1+\alpha} |j|^{1-\alpha} |k|^{s+\alpha} |u_j||q_k|.$$

Similarly for $s > 0$ we obtain with $s = r$

$$\frac{2m|j|^\alpha}{M^{1-r} + m^{1-r}} \le 2^{1+\alpha} m M^{\alpha + r - 1} = 2^{1+\alpha} m^{r+\alpha} \left(\frac{m}{M}\right)^{1-r-\alpha} \le 2^{1+\alpha} |k|^{s+\alpha}$$

and thus

$$|(u_j \cdot k) q_k (|l|^s - |k|^s)| \le \frac{2m|j|}{M^{1-r} + m^{1-r}} |u_j||q_k| \le 2^{1+\alpha} |j|^{1-\alpha} |k|^{s+\alpha} |u_j||q_k|.$$

The proof is concluded by noting that the $\ell_2(\mathbb{Z}^2)$ norm of the sequence ϕ_l is bounded by the product of the $\ell_1(\mathbb{Z}^2)$ norm of the sequence $|j|^{1-\alpha}|u_j|$ and the $\ell_2(\mathbb{Z}^2)$ norm of the sequence $|k|^{s+\alpha}|q_k|$.

We need also a uniform Gronwall lemma.

Lemma 1. *Let $y(t) \ge 0$ obey a differential inequality*

$$\frac{d}{dt} y + c_1 y \le F_1 + F(t) \tag{26}$$

with initial datum $y(0) = y_0$, with F_1 a positive constant and $F(t) \ge 0$ obeying

$$\int_t^{t+1} F(s) ds \le g_0 e^{-c_2 t} + F_2 \tag{27}$$

where c_1, c_2, g_0, F_2 are positive constants. Then

$$y(t) \le y_0 e^{-c_1 t} + g_0 e^{c_1 + c}(t+1) e^{-ct} + \frac{1}{c_1} F_1 + \frac{e^{c_1}}{1 - e^{-c_1}} F_2 \tag{28}$$

holds with $c = \min\{c_1, c_2\}$.

The main point of the lemma is that the constants y_0 and g_0 are multiplied by exponentially decaying factors.

Proof. Integrating, we have

$$y(t) \le y_0 e^{-c_1 t} + \frac{1}{c_1} F_1 + \int_0^t e^{-c_1(t-s)} F(s) ds, \tag{29}$$

and, taking N to be the integer part of t, i.e. $t \in [N, N+1)$, we have

$$\begin{aligned}
\int_0^t e^{-c_1(t-s)} F(s) ds &\le \sum_{k=0}^{N} e^{-c_1(t-k-1)} \int_k^{k+1} F(s) ds \\
&\le e^{c_1} \sum_{k=0}^{N} e^{-c_1(N-k)} (g_0 e^{-c_2 k} + F_2) \\
&\le e^{c_1}(N+1) e^{-\min\{c_1, c_2\} N} g_0 + \frac{e^{c_1}}{1 - e^{-c_1}} F_2.
\end{aligned} \tag{30}$$

Note that

$$e^{c_1}(N+1)e^{-\min\{c_1,c_2\}N} \le e^{c_1+c}(t+1)e^{-ct} \le C_\gamma e^{-\gamma t}$$

for $\gamma < c = \min\{c_1, c_2\}$.

3. PDE: Existence and uniqueness of solutions

We consider the system

$$\begin{cases} \partial_t q + u\cdot\nabla q + \Lambda q = \Delta\Phi \\ \partial_t u + u\cdot\nabla u - \Delta u + \nabla p = -qRq - q\nabla\Phi + f \\ \nabla\cdot u = 0. \end{cases} \tag{31}$$

The unknowns u, q are periodic in space. We consider smooth, mean zero, divergence-free body forces f, and smooth potential Φ. The body forces and the potential are time independent. We discuss first a class of weak solutions. The equations (31) are meant in distribution sense, assuming that $q \in L^\infty(0,T;L^2)$ and u is divergence-free and belongs to $L^\infty(0,T;L^2)$.

Theorem 1. Weak solutions. *Let $u_0 \in L^2$ be divergence-free, let $q_0 \in L^2$ with $\int q_0 = 0$, and let $T > 0$ be arbitrary. There exists a weak solution (u,q) of the system (31) satisfying $u \in L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$ and $q \in L^\infty(0,T;L^2)\cap L^2(0,T;H^{\frac12})$. Moreover the following inequalities hold a.e. in $0 \le t \le T$,*

$$\|q(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac12} q\|_{L^2}^2 \le \|q_0\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac32}\Phi\|_{L^2}^2, \tag{32}$$

$$\|q(t)\|_{L^2} \le \|q_0\|_{L^2} e^{-\lambda t} + \frac{1}{\lambda}\|\Delta\Phi\|_{L^2}, \tag{33}$$

and

$$\begin{aligned} &\|u(t)\|_{L^2}^2 + \|\Lambda^{-\frac12}(q(t)-Q)\|_{L^2}^2 + \int_0^t \left(\|q(s)-Q\|_{L^2}^2 + \|\nabla u(s)\|_{L^2}^2\right) ds \\ &\le \|u_0\|_{L^2}^2 + \|\Lambda^{-\frac12}(q_0-Q)\|_{L^2}^2 + \int_0^t \|\Lambda^{-1} f\|_{L^2}^2 dt, \end{aligned} \tag{34}$$

where Q is defined by

$$Q = -\Lambda\Phi. \tag{35}$$

Furthermore,

$$t\|q(t)\|_{L^4}^2 \le C\|q_0\|_{L^2}^2 + C\int_0^T \|\Lambda^{\frac32}\Phi\|_{L^2}^2 ds + \frac{1}{\lambda}\int_0^T s\|\Delta\Phi\|_{L^4}^2 ds \tag{36}$$

and

$$\begin{aligned} t\|\nabla u(t)\|_{L^2}^2 &\le \|u_0\|_{L^2}^2 + \|\Lambda^{-\frac12}(q_0-Q)\|_{L^2}^2 + \int_0^T \|\Lambda^{-1}f\|_{L^2}^2 ds + C\int_0^T s(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4) ds \\ &+ C\left[\|q_0\|_{L^2}^2 + \int_0^T \|\Lambda^{\frac32}\Phi\|_{L^2}^2 ds + \frac{1}{\lambda}\int_0^T s\|\Delta\Phi\|_{L^4}^2 ds\right]\left(\|q_0\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac32}\Phi\|_{L^2}^2 ds\right) \end{aligned} \tag{37}$$

hold t-a.e. in $[0,T]$.

Proof. We consider a viscous approximation of the system with smoothed out initial data. For $0 < \epsilon \le 1$, we let J_ϵ be a standard mollifier operator, and we consider the system

$$\begin{cases} \partial_t q^\epsilon + u^\epsilon\cdot\nabla q^\epsilon + \Lambda q^\epsilon - \epsilon\Delta q^\epsilon = \Delta\Phi \\ \partial_t u^\epsilon + u^\epsilon\cdot\nabla u^\epsilon - \Delta u^\epsilon + \nabla p^\epsilon = -q^\epsilon R q^\epsilon - q^\epsilon\nabla\Phi + f, \\ \nabla\cdot u^\epsilon = 0 \end{cases} \tag{38}$$

with $q_0^\epsilon = J_\epsilon q_0, u_0^\epsilon = J_\epsilon u_0$. For fixed positive ϵ this system has global smooth solutions for $t > 0$, a fact that can be proved using a number of different methods. We provide a priori bounds and pass to the limit $\epsilon \to 0$.

We note that the mean of q^ϵ is zero, and therefore we can use the Poincaré inequality (21). Multiplying the first equation of system (38) by $(q^\epsilon)^{p-1}$, with $p \ge 2$ even, and integrating, we obtain, by using u^ϵ is divergence-free, the non-negativity of the integral involving the Laplacian, (21), and a Hölder inequality that

$$\frac{1}{p}\partial_t \|q^\epsilon\|_{L^p}^p + \lambda \|q^\epsilon\|_{L^p}^p \le \left| \int \Delta\Phi (q^\epsilon)^{p-1} dx \right| \le \|q^\epsilon\|_{L^p}^{p-1} \|\Delta\Phi\|_{L^p}. \tag{39}$$

Thus the L^p norms of q^ϵ obey diffferential inequalities

$$\partial_t \|q^\epsilon\|_{L^p} + \lambda \|q^\epsilon\|_{L^p} \le \|\Delta\Phi\|_{L^p}. \tag{40}$$

The $L^2(0,T;H^{\frac{1}{2}})$ norm of q^ϵ is bounded using

$$\frac{1}{2}\frac{d}{dt}\|q^\epsilon\|_{L^2}^2 + \int q^\epsilon \Lambda q^\epsilon \le \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2} \|\Lambda^{\frac{1}{2}} q^\epsilon\|_{L^2}$$

and integrating in time, leading to

$$\|q^\epsilon(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{1}{2}} q^\epsilon\|_{L^2}^2 \le \|q_0\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2. \tag{41}$$

A cancellation is used to obtain bounds for u^ϵ in L^2. We take the scalar product in L^2 with u^ϵ in the second equation, and in the first equation we mutiply by $\Lambda^{-1}(q^\epsilon - Q)$ and integrate. We obtain

$$\frac{1}{2}\frac{d}{dt}\|u^\epsilon\|_{L^2}^2 + \|\nabla u^\epsilon\|_{L^2}^2 \le \int f\cdot u^\epsilon - \int q^\epsilon u^\epsilon \cdot R(q^\epsilon - Q)$$

and

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{-\frac{1}{2}}(q^\epsilon - Q)\|_{L^2}^2 + \|(q^\epsilon - Q)\|_{L^2}^2 \le \int q^\epsilon u^\epsilon \cdot R(q^\epsilon - Q) + \epsilon \int q^\epsilon \Lambda Q.$$

Adding we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|u^\epsilon\|_{L^2}^2 + \|\Lambda^{-\frac{1}{2}}(q^\epsilon - Q)\|_{L^2}^2\right] + \|\nabla u^\epsilon\|_{L^2}^2 + \|(q^\epsilon - Q)\|_{L^2}^2 \\
&\quad \le \|\Lambda^{-1} f\|_{L^2}\|\nabla u^\epsilon\|_{L^2} + \epsilon \int q^\epsilon \Lambda Q
\end{aligned} \tag{42}$$

and consequently

$$\begin{aligned}
&\|u^\epsilon(t)\|_{L^2}^2 + \|\Lambda^{-\frac{1}{2}}(q^\epsilon(t) - Q)\|_{L^2}^2 + \int_0^t \left(\|\nabla u^\epsilon\|_{L^2}^2 + \|(q^\epsilon - Q)\|_{L^2}^2\right) ds \\
&\quad \le \|u_0\|_{L^2}^2 + \|\Lambda^{-\frac{1}{2}}(q_0 - Q)\|_{L^2}^2 + \int_0^t \left(\|\Lambda^{-1} f\|_{L^2}^2 + \epsilon^2 \|\Lambda Q\|_{L^2}^2 + 2\epsilon \|\Lambda^{\frac{1}{2}} Q\|_{L^2}^2\right) ds.
\end{aligned} \tag{43}$$

Now, from (40) we deduce

$$\frac{d}{dt} t\|q^\epsilon(t)\|_{L^4}^2 + \lambda t \|q^\epsilon\|_{L^4}^2 \le t\frac{1}{\lambda}\|\Delta\Phi\|_{L^4}^2 + \|q^\epsilon(t)\|_{L^4}^2, \tag{44}$$

and in view of the embedding $H^{\frac{1}{2}} \subset L^4$ and (41) we deduce

$$t\|q^\epsilon(t)\|_{L^4}^2 \le C\|q_0\|_{L^2}^2 + C\int_0^t \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2 ds + \frac{1}{\lambda}\int_0^t s e^{-\lambda(t-s)} \|\Delta\Phi\|_{L^4}^2 ds. \tag{45}$$

We take the second equation of (38), multiply by $-\Delta u^\epsilon$ and integrate in space. We use the identity

$$Tr(M^T M^2) = 0,$$

valid for any two-by-two traceless matrix M, which follows because M^2 is a multiple of the identity matrix. We use this identity in our case for a matrix M with entries $M_{ij} = \frac{\partial u_i^\epsilon}{\partial x_j}$, and obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u^\epsilon\|_{L^2}^2 + \|\Delta u^\epsilon\|_{L^2}^2 = \int [f - q^\epsilon R(q^\epsilon - Q)]\cdot(-\Delta u^\epsilon) \tag{46}$$

and thus

$$\frac{d}{dt}\|\nabla u^\epsilon\|_{L^2}^2 + \|\Delta u^\epsilon\|_{L^2}^2 \le \|f - q^\epsilon R(q^\epsilon - Q)\|_{L^2}^2. \tag{47}$$

We multiply by t and integrate in time

$$t\|\nabla u^\epsilon(t)\|_{L^2}^2 + \int_0^t s\|\Delta u^\epsilon\|^2 ds \le \int_0^t \|\nabla u^\epsilon(s)\|_{L^2}^2 ds + C\int_0^t s\left(\|f\|_{L^2}^2 + \|q^\epsilon(s)\|_{L^4}^4 + \|Q\|_{L^4}^4\right) ds. \tag{48}$$

In view of (41), (43) and (45) we obtain

$$\begin{aligned} t\|\nabla u^\epsilon(t)\|_{L^2}^2 &+ \int_0^t s\|\Delta u^\epsilon\|^2 ds \le \|u_0\|_{L^2}^2 + \|\Lambda^{-\frac{1}{2}}(q_0 - Q)\|_{L^2}^2 \\ &+ \int_0^t \left(\|\Lambda^{-1} f\|_{L^2}^2 + \epsilon^2\|\Lambda Q\|_{L^2}^2 + 2\epsilon\|\Lambda^{\frac{1}{2}} Q\|_{L^2}^2\right) ds + C\int_0^t s(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4) ds \\ &+ C\left[\|q_0\|_{L^2}^2 + \int_0^T \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2 ds + \int_0^T s e^{-\lambda(T-s)}\frac{1}{\lambda}\|\Delta\Phi\|_{L^4}^2 ds\right]\left(\|q_0\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2 ds\right). \end{aligned} \tag{49}$$

These inequalities are used to pass to the limit. From (41) and (43) it folllows that q^ϵ is bounded in $L^2(0,T;H^{\frac{1}{2}})$ and u^ϵ is bounded in $L^2(0,T;H^1)$ on any sequence $\epsilon \to 0$. The equation (38) and the Aubin-Lions lemma imply that there exist $q \in L^2(0,T;H^{\frac{1}{2}})$ and $u \in L^2(0,T;H^1)$ such that

$$\lim_{\epsilon\to 0}\int_0^T \left(\|u^\epsilon(t) - u(t)\|_{L^2}^2 + \|q^\epsilon(t) - q(t)\|_{L^2}^2\right) dt = 0, \tag{50}$$

and, without loss of generality,

$$\lim_{\epsilon\to 0}\left(\|u^\epsilon(t) - u(t)\|_{L^2}^2 + \|q^\epsilon(t) - q(t)\|_{L^2}^2\right) = 0, \quad t - \text{a.e. in } [0,T]. \tag{51}$$

At each t where $q^\epsilon(t) \to q(t)$ strongly in L^2 it follows that $q^\epsilon(t)$ converges weakly to $q(t)$ in L^4, and therefore, by the weak lower semicontinuity of the L^4 norm, we have

$$\|q(t)\|_{L^4} \le \liminf_{\epsilon\to 0}\|q^\epsilon(t)\|_{L^4}, \quad t - \text{a.e. in } [0,T]. \tag{52}$$

Similarly, at any t where $u^\epsilon(t)$ converges strongly in L^2 to $u^\epsilon(t)$, the gradient $\nabla u^\epsilon(t)$ converges weakly in L^2 to ∇u. Therefore, by the weak lower semicontinuity of the L^2 norm

$$\|\nabla u(t)\|_{L^2}^2 \le \liminf_{\epsilon\to 0}\|\nabla u^\epsilon(t)\|_{L^2}, \quad t - \text{a.e. in } [0,T]. \tag{53}$$

The inequalities (45) and (49) thus yield (36) and (37) in the limit $\epsilon \to 0$. The fact that q and u obtained in the limit solve weakly the system (31) follows by testing the system (31) by test functions and passing to the limit. The proof of Theorem 1 is complete.

Remark 1. *Weak solutions are not known to be unique. The inequalities (36) and (37) show that for any $t_0 > 0$ the weak solutions become more regular, $u(t_0) \in H^1$, $q(t_0) \in L^4$ with quantitative bounds. This level of regularity generates strong solutions which are unique, as shown in the next theorem.*

Theorem 2. Strong solutions. *Let $u_0 \in H^1$ be divergence-free, let $q_0 \in L^4$ have mean zero, and let T be arbitrary. There exists a unique solution (u,q) of the system (31) with initial data (u_0,q_0) such that $u \in L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$ is divergence-free and $q \in L^\infty(0,T;L^4)\cap L^2(0,T;H^{\frac{1}{2}})$. Moreover,*

$$\|q(t)\|_{L^4} \le \|q(0)\|_{L^4}e^{-\lambda t} + \frac{1}{\lambda}\|\Delta\Phi\|_{L^4}, \tag{54}$$

$$\|\nabla u(t)\|_{L^2}^2 \le \|\nabla u_0\|_{L^2}^2 e^{-t} + C_\gamma\|q_0\|_{L^4}^4 e^{-\gamma t} + C_\lambda\left(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4 + \|\Delta\Phi\|_{L^4}^4\right), \tag{55}$$

with $0<\gamma<\min\{1,\ 4\lambda\}$, and

$$\int_0^T \|\Delta u\|_{L^2}^2 \le \|\nabla u_0\|_{L^2}^2 + C_\gamma\|q_0\|_{L^4}^4 + C_\lambda T\left(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4 + \|\Delta\Phi\|_{L^4}^4\right). \tag{56}$$

hold.

Proof. We provide a priori bounds directly on the equations of (31). Their justification can be done using a viscous approximation of the q equation. The differential inequality

$$\partial_t\|q(t)\|_{L^4} + \lambda\|q(t)\|_{L^4} \le \|\Delta\Phi\|_{L^4} \tag{57}$$

is obtained as (40) above, and yields

$$\|q(t)\|_{L^4} \le \|q(0)\|_{L^4}e^{-\lambda t} + \frac{1}{\lambda}\|\Delta\Phi\|_{L^4}. \tag{58}$$

The differential inequality

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \le \|f - qR(q-Q)\|_{L^2}^2 \le C\left(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4 + \|q\|_{L^4}^4\right) \tag{59}$$

is obtained like the inequality (47) above. Because the gradient has mean zero, we have a Poincaré inequality for the gradient

$$\|\Delta u\|_{L^2}^2 \ge \|\nabla u\|_{L^2}^2 \tag{60}$$

and, using it, we obtain

$$\|\nabla u(t)\|_{L^2}^2 \le \|\nabla u_0\|_{L^2}^2 e^{-t} + C_\gamma\|q_0\|_{L^4}^4 e^{-\gamma t} + C_\lambda\left(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4 + \|\Delta\Phi\|_{L^4}^4\right), \tag{61}$$

with $0<\gamma<\min\{1,4\lambda\}$. This follows from (58) because

$$\int_0^t e^{-(t-s)}\left(e^{-4\lambda s}\|q_0\|_{L^4}^4 + \lambda^{-4}\|\Delta\Phi\|_{L^4}^4\right)ds \le \|q_0\|_{L^4}^4 e^{-t}\int_0^t e^{(1-4\lambda)s}ds + \lambda^{-4}\|\Delta\Phi\|_{L^4}^4.$$

Returning to (59) we deduce

$$\int_0^T \|\Delta u\|_{L^2}^2 dt \le \|\nabla u_0\|_{L^2}^2 + C_\gamma\|q_0\|_{L^4}^4 + CT\left(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4 + \|\Delta\Phi\|_{L^4}^4\right). \tag{62}$$

For the proof of uniqueness we take two solutions (u_1,q_1) and (u_2,q_2) of (31) and we write $q = q_2 - q_1$, $u = u_2 - u_1$. The differences obey the equations

$$\partial_t q + \Lambda q + u_1\cdot\nabla q + u\cdot\nabla q + u\cdot\nabla q_1 = 0, \tag{63}$$

and

$$\partial_t u + u_2\cdot\nabla u + u\cdot\nabla u_1 + \nabla p - \Delta u + q_1 Rq + qRq + qRq_1 - qRQ = 0. \tag{64}$$

We multiply (63) by $\Lambda^{-1}q$, (64) by u and integrate. The cubic terms cancel

$$\int (u\cdot\nabla q)\Lambda^{-1}q + \int qRq\cdot u = 0$$

and the q_1 terms cancel as well

$$\int (u\cdot\nabla q_1)\Lambda^{-1}q + \int q_1 Rq\cdot u = 0,$$

and we are left with

$$\frac{1}{2}\frac{d}{dt}\left(\|\Lambda^{-\frac{1}{2}}q\|_{L^2}^2 + \|u\|_{L^2}^2\right) + \|\nabla u\|_{L^2}^2 + \|q\|_{L^2}^2 \tag{65}$$

$$= \int qu_1\cdot Rq - \int u\cdot\nabla u_1\cdot u + \int q(R(Q-q_1)\cdot u. \tag{66}$$

We estimate

$$\left|\int u\cdot\nabla u_1\cdot u\right| \le C\|u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u_1\|_{L^2} \tag{67}$$

and

$$\left|\int q(R(Q-q_1)\cdot u\right| \le C\|q\|_{L^2}\|Q-q_1\|_{L^4}\|u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}} \tag{68}$$

using L^4 bounds for u and the Ladyzhenskaya interpolation inequality. The first term in the right hand side of (66) can be written adding and subtracting zero as

$$\left|\int qu_1\cdot Rq\right| = \left|\int \left(\left[\Lambda^{-\frac{1}{2}}, u_1\cdot\nabla\right]q\right)\Lambda^{-\frac{1}{2}}q\right| \tag{69}$$

and using Proposition 3 with $s = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$ we obtain

$$\left|\int qu_1\cdot Rq\right| \le C[u_1]_{\frac{1}{2}}\|q\|_{L^2}\|\Lambda^{-\frac{1}{2}}q\|_{L^2} \tag{70}$$

Using Young inequalities in (67), (68) and (70) we obtain from (66),

$$\frac{d}{dt}\left[\|\Lambda^{-\frac{1}{2}}q\|_{L^2}^2 + \|u\|_{L^2}^2\right] \le C\left(\|\nabla u_1\|_{L^2}^2 + \|Q-q_1\|_{L^4}^4\right)\|u\|_{L^2}^2 + C[u_1]_{\frac{1}{2}}^2\|\Lambda^{-\frac{1}{2}}q\|_{L^2}^2. \tag{71}$$

Using the bound

$$[u_1]_{\frac{1}{2}} \le C\|\Delta u_1\|_{L^2} \tag{72}$$

for u_1 we obtain uniqueness from the fact that

$$\int_0^T (\|\Delta u_1\|_{L^2}^2 + \|q_1\|_{L^4}^4)dt < \infty \tag{73}$$

This concludes the proof of Theorem 2.

Remark 2. *The proof of uniqueness shows that we have weak-strong uniqueness: Strong solutions are unique among the larger class of weak solutions.*

Remark 3. *We have*

$$\int_t^{t+T}\|\Delta u(s)\|_{L^2}^2 ds \le \|\nabla u_0\|_{L^2}^2 e^{-t} + C_\gamma\|q_0\|_{L^4}^4 e^{-\gamma t} + C(1+T)\left(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4 + \|\Delta\Phi\|_{L^4}^4\right). \tag{74}$$

This is obtained by applying (56) on the interval $[t, t+T]$ and using the bounds (54) and (55) for the terms involving the "initial" time t.

Proposition 4. *The $H^{\frac{1}{2}}$ norm of the q component of strong solutions is locally uniformly bounded and their H^1 norm is locally uniformly square integrable in time. Moreover, for any $2 \le p \le \infty$, p even,*

$$\|q(t)\|_{L^p} \le \|q_0\|_{L^p}e^{-\lambda t} + \frac{1}{\lambda}\|\Delta\Phi\|_{L^p} \tag{75}$$

holds for all t.

Proof. The bound (32) holds for strong solutions. In view of it, for $t \ge t_0 > 0$ we consider the evolution of $\|\Lambda^{\frac{1}{2}} q\|_{L^2}$. We have

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{\frac{1}{2}} q\|_{L^2}^2 + \|\Lambda q\|_{L^2}^2 = \int \Lambda Q \Lambda q - \int \left(\left[\Lambda^{\frac{1}{2}}, u\cdot\nabla\right] q\right)\Lambda^{\frac{1}{2}} q. \tag{76}$$

We use Proposition 3 with $s = \frac{1}{2}$ and $\alpha = \frac{1}{2}$ and (72) for u, and deduce, after using a Young inequality that

$$\frac{d}{dt}\|\Lambda^{\frac{1}{2}} q\|_{L^2}^2 + \|\Lambda q\|_{L^2}^2 \le \|\Lambda Q\|_{L^2}^2 + C\|\Delta u\|_{L^2}^2\|\Lambda^{\frac{1}{2}} q\|_{L^2}^2. \tag{77}$$

Therefore the bound (62) implies

$$\|\Lambda^{\frac{1}{2}} q\|_{L^2}^2 \le C\left[T\|\Lambda Q\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} q(t_0)\|_{L^2}^2\right]\exp K \tag{78}$$

with K given by

$$K = \|\nabla u_0\|_{L^2}^2 + C_\lambda\|q_0\|_{L^4}^4 + CT\left(\|f\|_{L^2}^2 + \|Q\|_{L^4}^4 + \|\Delta\Phi\|_{L^4}^4\right) \tag{79}$$

and consequently

$$\int_{t_0}^t \|\Lambda q\|_{L^2}^2 \le T\|\Lambda Q\|_{L^2}^2 + C\left[T\|\Lambda Q\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} q(t_0)\|_{L^2}^2\right] K\exp K + \|\Lambda^{\frac{1}{2}} q(t_0)\|_{L^2}^2 \tag{80}$$

hold for $0 < t_0 \le t \le T$.

The L^p bound (75) follows from the uniform Poincaré inequality (21) and the fact that u is divergence-free.

Remark 4. *The quantitative bound (80) shows that there exists $t_1 \in [t_0, t_0+T]$ such that $q(t_1) \in H^1$, with a quantitative bound on its H^1 norm.*

Proposition 5. *Let $u_0 \in H^1$ be divergence-free and $q_0 \in H^1$ have mean zero. Then $\|\nabla q(t)\|_{L^2}$ can be bounded as*

$$\|\nabla q(t)\|_{L^2} \le C\left[1 + \|\nabla q_0\|_{L^2} + \|q_0\|_{L^4} + \|\nabla u_0\|_{L^2}\right]^8 e^{-c_1 t} + R_1(\Phi, f) \tag{81}$$

where $c_1 > 0$ is an explicit positive number and $R_1(\Phi, f)$ is an explicit function of norms of Φ and f. Moreover

$$\int_t^{t+T} \|\Lambda^{\frac{3}{2}} q(s)\|_{L^2}^2 ds \le C\left[1 + \|\nabla q_0\|_{L^2} + \|q_0\|_{L^4} + \|\nabla u_0\|_{L^2}\right]^{16} e^{-c_2 t} + R_2(\Phi, f) \tag{82}$$

with $c_2 > 0$ and $R_2(\Phi, f)$ an explicit function of the norms of Φ and f. Moreover, if $u_0 \in H^2$ we have

$$\|\Delta u(t)\|_{L^2} \le C\left[1 + \|\nabla q_0\|_{L^2} + \|q_0\|_{L^4} + \|\Delta u_0\|_{L^2}\right]^{16} e^{-c_3 t} + R_3(\Phi, f) \tag{83}$$

with $c_3 > 0$ and $R_3(\Phi, f)$ and explicit function of the norms of Φ and f.

Proof. We take the first equation of (31) obeyed by q, multiply by $-\Delta q$ and integrate. We obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla q(t)\|_{L^2}^2 + \int (\Lambda\nabla q)\nabla q = \int \Lambda Q(-\Delta q) - \int (\nabla u \nabla q)\nabla q \tag{84}$$

We bound

$$\left|\int \Lambda Q(-\Delta q)\right| \le \|\Delta Q\|_{L^2}\|\Lambda^{\frac{3}{2}} q\|_{L^2} \tag{85}$$

and we bound

$$\left|\int (\nabla u\nabla q)\nabla q\right| \le \|\nabla u\|_{L^4}\|\nabla q\|_{L^{\frac{8}{3}}}^2. \tag{86}$$

Using (22) and a Young inequality we deduce

$$\frac{d}{dt}\|\nabla q\|_{L^2}^2 + c\|\Lambda^{\frac{3}{2}} q\|_{L^2}^2 \le C\|\Delta Q\|_{L^2}^2 + C\|q\|_{L^4}^2\|\nabla u\|_{L^4}^4. \tag{87}$$

In view of the Ladyzhenskaya inequality

$$\|\nabla u\|_{L^4}^4 \le C\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}^2, \tag{88}$$

and the inequalities (55), (74), (75) it follows that the function

$$F(t) = \|q(t)\|_{L^4}^2\|\nabla u(t)\|_{L^4}^4$$

obeys the assumptions of the uniform Gronwall lemma, Lemma 1. The result (81) then follows using Lemma 1 for $y(t) = \|\nabla q\|_{L^2}^2$. The inequality (82) follows then by integrating in time (87).

For the bound (83) we apply $-\Delta$ to the equation obeyed by u. We obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \|\nabla\Delta u\|_{L^2}^2 = -\int \Delta(u\cdot\nabla u)\Delta u + \int \nabla(q(R(q-Q)-f)\nabla\Delta u. \tag{89}$$

After a cancellation due to the divergence-free condition, we have

$$\left|\int \Delta(u\cdot\nabla u)\Delta u\right| \le C\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\|\nabla\Delta u\|_{L^2}. \tag{90}$$

Here we also used L^4 norms of the second order derivatives of u and Ladyzhenskaya interpolation inequality. We have also

$$\begin{aligned}&\left|\int \nabla(q(R(q-Q)-f)\nabla\Delta u\right| \\ &\quad\le C\left(\|\nabla q\|_{L^4}\left(\|q\|_{L^4}+\|Q\|_{L^4}\right) + \|\nabla Q\|_{L^4}\|q\|_{L^4} + \|\nabla f\|_{L^2}\right)\|\nabla\Delta u\|_{L^2}.\end{aligned} \tag{91}$$

Using the embedding $H^{\frac{1}{2}} \subset L^4$ for ∇q, we obtain

$$\begin{aligned}&\frac{d}{dt}\|\Delta u\|_{L^2}^2 + \|\nabla\Delta u\|_{L^2}^2 \\ &\quad\le C\left[\|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}^2 + \left(\|\Lambda^{\frac{3}{2}}q\|_{L^2}^2 + \|\nabla Q\|_{L^4}^2\right)\|q\|_{L^4}^2 + \|\Lambda^{\frac{3}{2}}q\|_{L^2}^2\|Q\|_{L^4}^2 + \|\nabla f\|_{L^2}^2\right].\end{aligned} \tag{92}$$

In view of (55), (74), (75), (82) we have that the function

$$F(t) = \|\nabla u\|_{L^2}^2\|\Delta u\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}}q\|_{L^2}^2\left(\|q\|_{L^4}^2 + \|Q\|_{L^4}^2\right) + \|\nabla Q\|_{L^4}^2\|q\|_{L^4}^2 + \|\nabla f\|_{L^2}^2$$

obeys the assumptions of Lemma 1. The inequality (83) then follows from this lemma applied to $y(t) = \|\Delta u\|_{L^2}^2$.

4. THE MEAN ZERO FRAME

The second equation in (31) does not maintain a bounded average velocity u. Decomposing

$$u = v + u'(t) \tag{93}$$

where $v = v(t) \in \mathbb{R}^2$ is the average of $u(t)$, i.e.

$$u = v + \sum_{j\in\mathbb{Z}^2\setminus\{0\}} u_j e^{ij\cdot x} \tag{94}$$

we can rewrite the system (31) as

$$\begin{cases} \frac{d}{dt}v = -(2\pi)^{-2}\int q\nabla\Phi, \\ \partial_t q + (v+u')\cdot\nabla q + \Lambda q = \Delta\Phi \\ \partial_t u' + (v+u')\cdot\nabla u' - \Delta u' + \nabla p = -qRq - q\nabla\Phi + (2\pi)^{-2}\int(q\nabla\Phi) + f \\ \nabla\cdot u' = 0 \end{cases} \tag{95}$$

where we used the fact that R is antisymmetric and f has mean zero. Given a solution of (95), we compute the displacement

$$\ell(t) = \int_0^t v(s)ds \tag{96}$$

and define the change of variables

$$X(x,t) = x + \int_0^t v(s) = x + \ell(t) \tag{97}$$

with inverse

$$Y(y,t) = y - \ell(t) \tag{98}$$

and note that

$$\frac{d}{dt}F(x+\ell(t),t) = (\partial_t + v(t)\cdot\nabla)F\circ X(t). \tag{99}$$

Introducing the variables

$$\widetilde{u}(y,t) = u'(Y(y,t),t) \tag{100}$$

and

$$\widetilde{q}(y,t) = q(Y(y,t),t) \tag{101}$$

i.e.

$$u'(x,t) = \widetilde{u}(x+\ell(t),t) = \widetilde{u}\circ X, \quad q(x,t) = \widetilde{q}(x+\ell(t),t) = \widetilde{q}\circ X \tag{102}$$

we obtain the equations

$$\partial_t\widetilde{q} + \widetilde{u}\cdot\nabla\widetilde{q} + \Lambda\widetilde{q} = \Delta\widetilde{\Phi} \tag{103}$$

and

$$\partial_t\widetilde{u} + \widetilde{u}\cdot\nabla\widetilde{u} - \Delta\widetilde{u} + \nabla\widetilde{p} = -\widetilde{q}Rq - \widetilde{q}\nabla\widetilde{\Phi} + (2\pi)^{-2}\int\widetilde{q}\nabla\widetilde{\Phi} + \widetilde{f} \tag{104}$$

together with the divergence-free condition $\nabla\cdot\widetilde{u} = 0$. We used the translation invariance of the operators involved, and we used the notation

$$\widetilde{F}(y,t) = F(Y(y,t),t) \tag{105}$$

The new variables are still periodic in space with period 2π in each direction. The average of $\widetilde{u}$ is zero.

We note also that we can recover the solution (u,q) from the solution $(\widetilde{u},\widetilde{q})$ with the same initial data by the change of variables (102) and (96) where $v(t)$ is computed as

$$\frac{d}{dt}v(t) = -(2\pi)^{-2}\int\widetilde{q}\nabla\widetilde{\Phi}. \tag{106}$$

The two systems are equivalent, solution by solution. Dropping tildes we consider the system

$$\begin{cases} \partial_t q + u\cdot\nabla q + \Lambda q = \Delta\Phi \\ \partial_t u + u\cdot\nabla u - \Delta u + \nabla p = -qRq - q\nabla\Phi + (2\pi)^{-2}\int(q\nabla\Phi) + f \\ \nabla\cdot u = 0 \end{cases} \tag{107}$$

in which both u and q have mean zero. This is the system for which we can show that solutions have a finite dimensional attractor.

5. Long time dynamics

We are concerned with the long time behavior of solutions of (31) in the mean zero frame (107). Summarizing the results of Section 3 we know that solutions $(u(x,t), q(x,t))$ of the system (107) with initial data in L^2 exist globally, and they become strong at positive times. Strong solutions are unique, and have additional properties. We consider the subset $\mathcal{V} \subset \mathcal{H}$ where $\mathcal{H}$ is defined in (12)

$$\mathcal{V} = H^1 \cap H \oplus L^4 \tag{108}$$

and study the evolution of solutions $(u(t), q(t))$ of (107) with initial data $w_0 = (u_0, q_0) \in \mathcal{V}$. The solution map

$$S(t)(u_0, q_0) = (u(t), q(t)) \tag{109}$$

is a semigroup

$$S(t) : \mathcal{V} \mapsto \mathcal{H}, \tag{110}$$

$$S(t+s)w_0 = S(t)(S(s)w_0) \tag{111}$$

for $t, s \geq 0$. The abstract formulation of the system (107) is

$$\begin{cases} \partial_t u + Au + B(u,u) + \mathbb{P}(qR(q-Q)) = f, \\ \partial_t q + \Lambda q + u \cdot \nabla q = \Lambda Q \end{cases} \tag{112}$$

where

$$B(u,v) = \mathbb{P}(u \cdot \nabla v), \tag{113}$$

and $Q = -\Lambda\Phi$, as before. Note that, in view of

$$u = \mathbb{P}u \tag{114}$$

and the fact that $-\Delta$ commutes with $\mathbb{P}$ in the periodic case, we have

$$Au = -\Delta u. \tag{115}$$

Theorem 1 implies that there exist weak solutions of (112) with initial data in $\mathcal{H}$. If the initial data are in $\mathcal{V}$ the solutions are strong, unique and have additional properties.

Proposition 6. *There exists a constant R_0 depending on Φ and f, such that for any $w_0 = (u_0, q_0) \in \mathcal{V}$, there exists t_0 depending only on $\|u_0\|_{H^1}$ and $\|q_0\|_{L^4}$ such that the strong solution $(u(t), q(t)) = S(t)w_0$ of (107) with initial data $w_0 = (u_0, q_0)$ satisfies*

$$\|u(t)\|_{H^1} + \|q(t)\|_{L^4} \leq R_0 \tag{116}$$

for all $t \geq t_0$

Proof. Because u has mean zero we have the Poincaré inequality

$$\|u(t)\|_{L^2}^2 \leq \|\nabla u(t)\|_{L^2}^2. \tag{117}$$

The result (116) follows from (55) and (75) because of the translation invariance of norms

$$\|\nabla u\|_{L^2} = \|\nabla(u \circ X)\|_{L^2}, \quad \|q\|_{L^p} = \|q \circ X\|_{L^p}. \tag{118}$$

Proposition 7. *There exists* R_1 *depending only on* Φ *and* f*, and* $t_1 > 0$ *depending only on* R_0 *and* R_1 *such that for any* $w_0 = (u_0, q_0) \in \mathcal{V}$ *satsfying*

$$\|u_0\|_{H^1} + \|q_0\|_{L^4} \le R_0 \tag{119}$$

we have

$$\|\Lambda^{\frac{1}{2}} q(t_1)\|_{L^2} \le R_1 \tag{120}$$

and

$$\frac{1}{T}\int_{t_1}^{t_1+T} \left(\|\Delta u\|_{L^2}^2 + \|\Lambda q\|_{L^2}^2\right) dt \le R_1^2 \tag{121}$$

for any $T > 0$*. There exists* $t_2 > t_1$*, depending on* R_1 *such that*

$$\|\Delta u(t_2)\|_{L^2}^2 + \|\Lambda q(t_2)\|_{L^2}^2 \le R_1 \tag{122}$$

holds.

Proof. The bound on $\|\Lambda^{\frac{1}{2}} q(t_1)\|_{L^2}$ follows from

$$\int_0^t \|\Lambda^{\frac{1}{2}} q(s)\|_{L^2}^2 ds \le \|q_0\|_{L^2}^2 + t\|\Lambda^{\frac{3}{2}}\Phi\|_{L^2}^2 \tag{123}$$

(see (41)) and the Chebyshev inequality. The inequality (121) follows from (56) and (80). The existence of t_2 for which (122) is true follows from (121).

Theorem 3. Absorbing ball. *There exists* R_2 *depending only on* Φ *an* f *such that, for any initial data* $w_0 = (u_0, q_0) \in \mathcal{V}$*, there exists* $t_3 > 0$ *depending only on the norms* $\|u_0\|_{H^1}$*,* $\|q_0\|_{L^4}$ *and on* R_2 *such that, for any* $t \ge t_3$

$$\|u(t)\|_{H^2} + \|q\|_{H^1} \le R_2 \tag{124}$$

holds for $t \ge t_3$*, i.e.*

$$S(t)w_0 \in K_{R_2} = \{w \in \mathcal{V} \mid \ \|u\|_{H^2} + \|q\|_{H^1} \le R_2\}. \tag{125}$$

holds for $t \ge t_3$*.*

Proof. By Proposition 6 and Proposition 7 above there exists R_1 depending on f and Φ and $t_2 > 0$ depending on the norms $\|u_0\|_{H^1}$ and $\|q_0\|_{L^4}$ such that $\|u(t_2)\|_{H^2} + \|q(t_2)\|_{H^1} \le R_1$. Then the result follows from Proposition 5

6. Continuity properties of the solution map

In addition to the topology of $\mathcal{H}$ with norm

$$\|w\|_{\mathcal{H}}^2 = \|u\|_{L^2}^2 + \|q\|_{L^2}^2 \tag{126}$$

we consider the natural topology of $\mathcal{V}$ which is a Banach space on its own, with norm

$$\|w\|_{\mathcal{V}}^2 = \|u\|_{H^1}^2 + \|q\|_{L^4}^2 \tag{127}$$

Theorem 4. Continuity. *Let* $w_1^0 = (u_1^0, q_1^0) \in \mathcal{V}$ *and* $w_2^0 = (u_2^0, q_2^0) \in \mathcal{V}$*. Let* $t > 0$*. There exist constants* $C(t)$*,* $C_1(t)$*, and* $C_2(t)$ *locally uniformly bounded above as functions of* $t \ge 0$ *and locally bounded as initial data* w_1^0, w_2^0 *are varied in* $\mathcal{V}$*, such that* $S(t)$ *is Lipschitz continuous in* $\mathcal{H}$*, obeying*

$$\|S(t)w_1^0 - S(t)w_2^0\|_{\mathcal{H}} \le C(t)\|w_1^0 - w_2^0\|_{\mathcal{H}}, \tag{128}$$

$S(t)$ *is Lipschitz continuous in* $\mathcal{V}$*, obeying*

$$\|S(t)w_1^0 - S(t)w_2^0\|_{\mathcal{V}} \le C_1(t)\|w_1^0 - w_2^0\|_{\mathcal{V}}, \tag{129}$$

and $S(t)$ is Lipschitz continuous for $t > 0$ from $\mathcal{H}$ to $\mathcal{V}$, obeying

$$\sqrt{t}\|S(t)w_1^0 - S(t)w_2^0\|_{\mathcal{V}} \le C_2(t)\|w_1^0 - w_2^0\|_{\mathcal{H}}. \tag{130}$$

Proof. We take the two solutions of (112) $w_1 = S(t)w_1^0 = (u_1(t), q_1(t))$ and $w_2 = S(t)w_2^0 = (u_2(t), q_2(t))$ and denote $w(t) = S(t)w_2^0 - S(t)w_1^0 = (u_2(t) - u_1(t), q_2(t) - q_1(t))$ and $\overline{w} = (\overline{u}, \overline{q}) = \frac{1}{2}(S(t)w_1^0 + S(t)w_2^0)$. Then $w(t)$ satisfies the system

$$\begin{cases} \partial_t u + Au + B(u, \overline{u}) + B(\overline{u}, u) + \mathbb{P}(qR(\overline{q} - Q) + \overline{q}Rq) = 0, \\ \partial_t q + \Lambda q + \overline{u} \cdot \nabla q + u \cdot \nabla \overline{q} = 0. \end{cases} \tag{131}$$

We obtain

$$\frac{d}{dt}\|w(t)\|_{\mathcal{H}}^2 + \|\nabla u\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}} q\|_{L^2}^2 \le C(\|\nabla \overline{u}\|_{L^2}^2 + \|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2 + \|\nabla \overline{q}\|_{L^2}^4)\|w(t)\|_{\mathcal{H}}^2 \tag{132}$$

by using estimates

$$\left|\int u \cdot \nabla \overline{q} q\right| \le \|\nabla \overline{q}\|_{L^2}\|q\|_{L^4}\|u\|_{L^4}$$

and interpolation. Thus (128) holds with

$$C(t) = \exp\left\{C\int_0^t \left(\|\nabla \overline{u}\|_{L^2}^2 + \|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2 + \|\nabla \overline{q}\|_{L^2}^4\right) ds\right\} \tag{133}$$

which is a locally uniformly bounded function of time and initial data $w_1^0, w_2^0 \in \mathcal{V}$.

The evolution of the norm the H^1 norm of u is obtained from the identity ([4])

$$(B(\overline{u}, u) + B(u, \overline{u}), Au)_H = -(B(u,u), A\overline{u})_H \tag{134}$$

which yields

$$\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}u\|_H^2 + \|Au\|_H^2 = (B(u,u), A\overline{u})_H - (\mathbb{P}(qR(\overline{q} - Q) + \overline{q}Rq), Au)_H \tag{135}$$

and results in

$$\frac{d}{dt}\|A^{\frac{1}{2}}u\|_H^2 + \|Au\|_H^2 \le C\|A\overline{u}\|_H^{\frac{4}{3}}\|u\|_H^{\frac{2}{3}}\|A^{\frac{1}{2}}u\|_H^{\frac{4}{3}} + C\left[\|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2\right]\|q\|_{L^4}^2. \tag{136}$$

The L^4 norm of q evolves according to

$$\frac{1}{4}\frac{d}{dt}\|q\|_{L^4}^4 + \int q^3 \Lambda q + \int q^3(u\nabla \overline{q}) = 0 \tag{137}$$

The inequality 21 and the embedding $H^{\frac{1}{2}} \subset L^4$ results in

$$\int q^3 \Lambda q \ge c\|q\|_{L^8}^4 \tag{138}$$

and using the embedding $H^1 \subset L^8$ we deduce

$$\left|\int q^3(u\nabla \overline{q})\right| \le \|q\|_{L^8}^3\|u\|_{L^8}\|\nabla \overline{q}\|_{L^2} \le C\|q\|_{L^8}^3\|A^{\frac{1}{2}}u\|_H\|\nabla \overline{q}\|_{L^2}, \tag{139}$$

and therefore,

$$\frac{d}{dt}\|q\|_{L^4}^4 \le C\|A^{\frac{1}{2}}u\|_H^4\|\nabla \overline{q}\|_{L^2}^4. \tag{140}$$

Putting these together we obtain

$$\frac{d}{dt}\left[\|A^{\frac{1}{2}}u\|_H^4 + \|q\|_{L^4}^4\right] \le C\left(\|A\overline{u}\|_H^{\frac{4}{3}} + \|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2 + \|\nabla \overline{q}\|_{L^2}^4\right)\left[\|A^{\frac{1}{2}}u\|_{L^2}^4 + \|q\|_{L^4}^4\right] \tag{141}$$

Thus (129) holds with

$$C_1(t) = \exp\left\{C\int_0^t\left(\|A\overline{u}\|_H^{\frac{4}{3}} + \|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2 + \|\nabla\overline{q}\|_{L^2}^4\right)ds\right\} \tag{142}$$

which is a locally uniformly bounded function of $t > 0$ and initial data w_1^0, w_2^0 in $\mathcal{V}$.

For the Lipschitz continuity from $\mathcal{H}$ to $\mathcal{V}$, we estimate slightly differently in (135),

$$\frac{d}{dt}\|A^{\frac{1}{2}}u\|_H^2 + \|Au\|_H^2 \le C\|A\overline{u}\|_H^{\frac{4}{3}}\|u\|_H^{\frac{2}{3}}\|A^{\frac{1}{2}}u\|_H^{\frac{4}{3}} + C\left[\|\overline{q}\|_{L^\infty}^2 + \|R\overline{q}\|_{L^\infty}^2 + \|RQ\|_{L^\infty}^2\right]\|q\|_{L^2}^2. \tag{143}$$

Using the inequality $\|Au\|_H\|u\|_H \ge \|A^{\frac{1}{2}}\|_H^2$ and a Young inequality, we obtain

$$\frac{d}{dt}\|A^{\frac{1}{2}}u\|_H^2 + \frac{1}{2}\|Au\|_H^2 \le C\|A\overline{u}\|_H^2\|u\|_H^2 + C\left[\|\overline{q}\|_{L^\infty}^2 + \|R\overline{q}\|_{L^\infty}^2 + \|RQ\|_{L^\infty}^2\right]\|q\|_{L^2}^2. \tag{144}$$

Integrating in time in (132) and using (128) we have

$$\int_0^t\left(\|A^{\frac{1}{2}}u(s)\|_H^2 + \|\Lambda^{\frac{1}{2}}q(s)\|_{L^2}^2\right)ds \le \widetilde{C}(t)\|w_0\|_{\mathcal{H}}^2 \tag{145}$$

with

$$\widetilde{C} = 1 + C\int_0^t C(s)(\|\nabla\overline{u}\|_{L^2}^2 + \|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2 + \|\nabla\overline{q}\|_{L^2}^4)ds \tag{146}$$

Multiplying (144) by t, using (145) and (128) we obtain

$$t\|A^{\frac{1}{2}}u(t)\|_H^2 \le C_3(t)\|w_0\|_{\mathcal{H}}^2 \tag{147}$$

with $C_3(t)$ an explicit function of time which is locally uniformly bounded for $t \ge 0$, and locally bounded as initial data w_1^0, w_2^0 vary in $\mathcal{V}$. Returning to (139) but estimating differently, using the Hölder inequality with exponents $2, 4, 4$ and then interpolation, we obatin

$$\left|\int q^3(u\nabla\overline{q})\right| \le C\|q\|_{L^8}^2\|q\|_{L^4}\|u\|_H^{\frac{1}{2}}\|A^{\frac{1}{2}}u\|_H^{\frac{1}{2}}\|\nabla\overline{q}\|_{L^4} \tag{148}$$

and therefore, from (137) we obtain after a Young inequality and use of (138),

$$\frac{d}{dt}\|q\|_{L^4}^2 \le C\|u\|_H\|A^{\frac{1}{2}}u\|_H\|\nabla\overline{q}\|_{L^4}^2. \tag{149}$$

Multiplying (149) by t, integrating in time, and using (145), the embedding $H^{\frac{1}{2}} \subset L^4$ and (147) we obtain

$$t\|q(t)\|_{L^4}^2 \le C_4(t)\|w_0\|_H^2 \tag{150}$$

with $C_4(t)$ an explicit function of time which locally uniformly bounded for $t \ge 0$, and locally bounded as initial data w_1^0, w_2^0 vary in $\mathcal{V}$. From (147) and (150) we obtain (130).

7. Backward uniqueness

Theorem 5. Backward uniqueness. *Let w_1^0, w_2^0 be two initial data in $\mathcal{V}$. For any $T > 0$, if $S(T)w_1^0 = S(T)w_2^0$, then $w_1^0 = w_2^0$.*

Proof. We use the notation of the proof of Theorem 4. The difference $w(t)$ obeys (131). We can write this abstractly as

$$\partial_t w + \mathcal{A}w + L(\overline{w})w = 0 \tag{151}$$

where $w = (u, q)$, $\overline{w} = (\overline{u}, \overline{q})$, and

$$\begin{aligned} &L(\overline{w})w = (L_1(\overline{w})w,\ L_2(\overline{w})w),\ \text{with}\\ &L_1(\overline{w})w = B(u,\overline{u}) + B(\overline{u},u) + \mathbb{P}(qR(\overline{q}-Q) + \overline{q}Rq),\ \text{and}\\ &L_2(\overline{w})w = \overline{u}\cdot\nabla q + u\cdot\nabla\overline{q}\end{aligned} \tag{152}$$

Let us consider the evolution of the norm

$$E_0 = \|u\|_{L^2}^2 + \|q\|_{H^{-\frac{1}{2}}}^2 \tag{153}$$

obtained by taking the scalar product in $\mathcal{H}$ of the equation (151) with $(u, \Lambda^{-1}q) = (\mathbb{I} \oplus \Lambda^{-1})w = \mathcal{B}w$. The operator

$$\mathcal{B} = \mathbb{I} \oplus \Lambda^{-1} \tag{154}$$

is selfadjoint and commutes with $\mathcal{A}$. We obtain

$$\frac{1}{2}\frac{d}{dt}E_0 + E_1 + (L(\overline{w})w, \mathcal{B}w)_{\mathcal{H}} = 0 \tag{155}$$

where

$$E_1 = \|A^{\frac{1}{2}}u\|_H^2 + \|q\|_{L^2}^2 = (w, \mathcal{A}\mathcal{B}w)_{\mathcal{H}}. \tag{156}$$

Now we denote by

$$\mu = \frac{E_1}{E_0} \tag{157}$$

and observe that

$$\frac{1}{2}\frac{d}{dt}\log\left(\frac{1}{E_0}\right) = \mu + (L(\overline{w})\phi, \mathcal{B}\phi)_H \tag{158}$$

where

$$\phi = E_0^{-\frac{1}{2}}w. \tag{159}$$

Let us consider the function

$$Y(t) = \log\left(\frac{1}{E_0}\right), \tag{160}$$

and so we have

$$\frac{1}{2}\frac{d}{dt}Y(t) = \mu + (L(\overline{w})\phi, \mathcal{B}\phi)_{\mathcal{H}}. \tag{161}$$

The aim is to show that $Y(t)$ cannot reach the value $+\infty$ in finite time. To this end we take the derivative of μ and note

$$\frac{d}{dt}\mu = E_0^{-1}\frac{d}{dt}E_1 - \mu\frac{d}{dt}\log E_0 = E_0^{-1}\frac{d}{dt}E_1 + \mu\frac{d}{dt}Y. \tag{162}$$

We have

$$\frac{1}{2}\frac{d}{dt}E_1 + (w, \mathcal{A}^2\mathcal{B}w)_{\mathcal{H}} + (L(\overline{w})w, \mathcal{A}\mathcal{B}w)_{\mathcal{H}} = 0. \tag{163}$$

which implies that

$$E_0^{-1}\frac{d}{dt}E_1 = -2(\phi, \mathcal{A}^2\mathcal{B}\phi)_{\mathcal{H}} - 2(L(\overline{w})\phi, \mathcal{A}\mathcal{B}\phi)_{\mathcal{H}} \tag{164}$$

and therefore

$$\frac{1}{2}\frac{d}{dt}\mu = -(\phi, \mathcal{A}^2\mathcal{B}\phi)_{\mathcal{H}} - (L(\overline{w})\phi, \mathcal{A}\mathcal{B}\phi)_{\mathcal{H}} + \mu\left(\mu + (L(\overline{w})\phi, \mathcal{B}\phi)_H\right). \tag{165}$$

Let us note that

$$\mu = (\mathcal{A}\phi, \mathcal{B}\phi)_{\mathcal{H}} \tag{166}$$

and if we introduce the scalar product in $\mathcal{H}$ defined by

$$(a, b)_{\mathcal{B}} = (a, \mathcal{B}b)_{\mathcal{H}} \tag{167}$$

then we see that

$$\|\phi\|_{\mathcal{B}}^2 = 1 \tag{168}$$

and

$$(\mathcal{A}^2\phi,\phi)_{\mathcal{B}} - \mu^2 = \|(\mathcal{A}-\mu)\phi\|^2_{\mathcal{B}} \tag{169}$$

hold. The equation (165) becomes

$$\frac{1}{2}\frac{d}{dt}\mu = -\|(\mathcal{A}-\mu)\phi\|^2_{\mathcal{B}} - (L(\overline{w})\phi,(\mathcal{A}-\mu)\phi)_{\mathcal{B}}. \tag{170}$$

Let us note also that (161) can be written as

$$\frac{1}{2}\frac{d}{dt}Y(t) = \mu + (L(\overline{w})\phi,\phi)_{\mathcal{B}}. \tag{171}$$

This is a general structure, we could have used any postive selfadjoint operator $\mathcal{B}$ which commutes with $\mathcal{A}$, and it did really not matter what $L(\overline{w})$ or $\mathcal{A}$ were. Our choice is of course motivated by the properties of the latter, but some general features already can be taken advantage of.

We compute in our case

$$(L(\overline{w})\phi,\phi)_{\mathcal{B}} = \frac{1}{E_0}\left[(B(u,\overline{u}),u)_H + \int (qR(\overline{q}-Q)\cdot u - q\overline{u}\cdot Rq)\,dx\right] \tag{172}$$

where we used the cancellation of the terms involving $\overline{q}u\cdot Rq$ and $(u\cdot\nabla\overline{q})\Lambda^{-1}q$. The estimate

$$(L(\overline{w})\phi,\phi)_{\mathcal{B}} \le K_0(t)\mu \tag{173}$$

with

$$K_0(t) = C\left[\|A\overline{u}\|_H + \|R(\overline{q}-Q)\|_{L^\infty}\right] \tag{174}$$

holds, and

$$\int_0^T K_0(t)dt < \infty \tag{175}$$

holds as well (see (74) and (75)). If we decompose

$$L(\overline{w})\phi = T_1\phi + T_2\phi \tag{176}$$

where

$$\|T_1\phi\|^2_{\mathcal{B}} \le K^2(t)\|\mathcal{A}^{\frac{1}{2}}\phi\|^2_{\mathcal{B}} \tag{177}$$

then the contribution coming from T_1 can be estimated using the Schwartz inequality in the term $(T_1\phi,(\mathcal{A}-\mu)\phi)_{\mathcal{B}}$, and we obtain that

$$\frac{d}{dt}\mu \le -\|(\mathcal{A}-\mu)\phi\|^2_{\mathcal{B}} + K^2(t)\mu - 2(T_2\phi,(\mathcal{A}-\mu)\phi)_{\mathcal{B}}. \tag{178}$$

The bound (177) means that the velocity component of T_1w is bounded from $H^1\times L^2$ to L^2 and the second component is bounded from $H^1\times L^2$ to $H^{-\frac{1}{2}}$. The requirement (177) is satisfied in our case by

$$T_1w = (L_1(\overline{w})w, u\cdot\nabla\overline{q}). \tag{179}$$

Indeed, (177) holds, i.e.

$$\|L_1(\overline{w})w\|^2_{L^2} + \|\Lambda^{-\frac{1}{2}}(u\cdot\nabla\overline{q})\|^2_{L^2} \le K^2(t)\left[\|A^{\frac{1}{2}}u\|^2_H + \|q\|^2_{L^2}\right] \tag{180}$$

with

$$K(t) = C\left[\|A\overline{u}\|_H + \|R(\overline{q}-Q)\|_{L^\infty} + \|\overline{q}\|_{L^\infty} + \|\nabla\overline{q}\|^{\frac{1}{2}}_{L^2}\|\nabla\overline{q}\|^{\frac{1}{2}}_{L^\infty}\right]. \tag{181}$$

It remains to examine what happens to T_2,

$$T_2w = (0,\overline{u}\cdot\nabla q) \tag{182}$$

which does not satisfy (177). Its contribution to the evolution of μ in (178) is

$$2(T_2\phi, (\mathcal{A}-\mu)\phi)_{\mathcal{B}} = 2E_0^{-1}\int (\overline{u}\cdot\nabla q)(\Lambda-\mu)\Lambda^{-1}q = -2E_0^{-1}\mu\int(\overline{u}\cdot\nabla q)\Lambda^{-1}q. \tag{183}$$

In view of the fact that

$$\int(\overline{u}\cdot\nabla q)\Lambda^{-1}q = -\int\Lambda^{-\frac{1}{2}}q\left[\overline{u}\cdot\nabla,\Lambda^{-\frac{1}{2}}\right]q \tag{184}$$

and Proposition 3 with $s=-\frac{1}{2}$ and $\alpha=0$, we have

$$-2(T_2\phi,(\mathcal{A}-\mu)\phi)_{\mathcal{B}} \le C[\overline{u}]_1\mu. \tag{185}$$

Thus, putting together the bounds (178) and (185) we obtain

$$\frac{d}{dt}\mu \le C(K^2(t)+[\overline{u}]_1)\mu \tag{186}$$

and because

$$\int_0^T (K^2(t)+[\overline{u}]_1)dt < \infty \tag{187}$$

it follows that $\mu(t)$ is locally bounded in time. From the bounds (173) and (175) it follows that $Y(t)$ is locally bounded.

8. Decay of volume elements

We consider a solution $\overline{w}=S(t)\overline{w}_0$ of (112) with initial data in the absorbing ball $\overline{w}_0\in K_{R_2}=\{w\in\mathcal{V}\mid \|u\|_{H^2}+\|q\|_{H^1}\le R_2\}$. We consider the linearization of $S(t)$ along $\overline{w}(t)$,

$$w_0\mapsto w(t)=S'(t,\overline{w})w_0 \tag{188}$$

viewed as an operator in $\mathcal{H}$. The function $w(t)$ solves

$$\partial_t w+\mathcal{A}w+L(\overline{w})w=0 \tag{189}$$

with initial data w_0. We denote $w=(u,q)$, $\overline{w}=(\overline{u},\overline{q})$, and

$$\begin{aligned} &L(\overline{w})w=(L_1(\overline{w})w,\ L_2(\overline{w})w),\ \text{with}\\ &L_1(\overline{w})w=B(u,\overline{u})+B(\overline{u},u)+\mathbb{P}(qR(\overline{q}-Q)+\overline{q}Rq),\ \text{and}\\ &L_2(\overline{w})w=\overline{u}\cdot\nabla q+u\cdot\nabla\overline{q}.\end{aligned} \tag{190}$$

The volume elements associated to it are the norms in $\bigwedge^N\mathcal{H}$. The scalar product in $\bigwedge^N\mathcal{H}$ is

$$(w_1\wedge\cdots\wedge w_N; y_1\wedge\cdots\wedge y_N)_{\bigwedge^N\mathcal{H}}=\det(w_i,y_j)_{\mathcal{H}} \tag{191}$$

and the volume elements are norms

$$V_N(t)=\|w_1(t)\wedge\ldots w_N(t)\|_{\bigwedge^N\mathcal{H}} \tag{192}$$

where

$$w_i(t)=S'(t,\overline{w})w_i(0) \tag{193}$$

are the images under the linearization of N linearly independent vectors. The monomial $w_1(t)\wedge\cdots\wedge w_N(t)$ evolves according to

$$\partial_t(w_1(t)\wedge\cdots\wedge w_N(t))+(\mathcal{A}+L(\overline{w}))_N(w_1(t)\wedge\cdots\wedge w_N(t))=0 \tag{194}$$

with

$$(\mathcal{A}+L(\overline{w}))_N(w_1(t)\wedge\cdots\wedge w_N(t))=(\mathcal{A}+L(\overline{w}))w_1\wedge\cdots\wedge w_N+\cdots+w_1\wedge\cdots\wedge(\mathcal{A}+L(\overline{w}))w_N \tag{195}$$

and, as a consequence, the volume element evolves according to

$$\frac{d}{dt}V_N(t) + \text{Trace}\left((\mathcal{A} + L(\overline{w}))Q_N\right)V_N(t) = 0 \tag{196}$$

where Q_N is orthogonal projection in $\mathcal{H}$ onto the linear subspace spanned by the vectors w_i, $1 \le i \le N$. These are calculations which parallel well known calculations for the Navier-Stokes equations ([3], [4]).

The volume element $V_N(t)$ decays if N is large enough, as specified in the following theorem.

Theorem 6. *There exists a constant M depending on R_2 and norms of Φ and of f such that, for any initial data $\overline{w}_0$ in the absorbing ball K_{R_2}, for any $N \ge M$, and any initial data $w_1(0)$, $w_2(0)$, $\dots w_N(0)$ in $\mathcal{H}$, we have that*

$$\|S'(t,\overline{w})w_1(0) \wedge \dots \wedge S'(t,\overline{w})w_N(0)\|_{\bigwedge^N \mathcal{H}} \le V_N(0)e^{-cN^{\frac{3}{2}}t} \tag{197}$$

holds for $t \ge t_0$, with t_0 depending on R_2.

Proof. The trace in (196) is computed as follows. At each instant of time t we choose an orthonormal basis $\phi_i = (v_i, r_i)$ of the linear span of $w_1, \dots w_N$. Then

$$\text{Trace}\left((\mathcal{A} + L(\overline{w}))Q_N\right) = \sum_{i=1}^N (\mathcal{A}\phi_i, \phi_i)_{\mathcal{H}} + \sum_{i=1}^N (L(\overline{w})\phi_i, \phi_i)_{\mathcal{H}}. \tag{198}$$

Now

$$\text{Trace}(\mathcal{A}Q_N) = \sum_{i=1}^N (\mathcal{A}\phi_i, \phi_i)_{\mathcal{H}} = \sum_{i=1}^N \left[(Av_i, v_i)_H + (\Lambda r_i, r_i)_{L^2}\right] \ge \mu_1 + \dots \mu_N, \tag{199}$$

and

$$\sum_{i=1}^N (L(\overline{w})\phi_i, \phi_i)_{\mathcal{H}} = \sum_{i=1}^N \left[(B(v_i, \overline{u}), v_i)_H + (\mathbb{P}(r_i R(\overline{q} - Q) + \overline{q}Rr_i), v_i)_H + \int (v_i \cdot \nabla\overline{q})r_i\right]. \tag{200}$$

On one hand we have a lower bound

$$\sum_{i=1}^N (\mathcal{A}\phi_i, \phi_i)_{\mathcal{H}} \ge \sum_{i=1}^N \left[\|A^{\frac{1}{2}}v_i\|_H^2 + c\|r_i\|_{L^4}^2\right], \tag{201}$$

and on the other hand we have the upper bound

$$\left|\sum_{i=1}^N (L(\overline{w})\phi_i, \phi_i)_{\mathcal{H}}\right| \tag{202}$$

$$\le C\sum_{i=1}^N \left[\|\nabla\overline{u}\|_{L^2}\|v_i\|_H\|A^{\frac{1}{2}}v_i\|_H + (\|\overline{q}\|_{L^4} + \|Q\|_{L^4})\|r_i\|_{L^4}\|v_i\|_{L^2} + \|\nabla\overline{q}\|_{L^2}\|v_i\|_H^{\frac{1}{2}}\|A^{\frac{1}{2}}v_i\|_H^{\frac{1}{2}}\|r_i\|_{L^4}\right].$$

Applying Schwartz inequalities in the first two terms in the right hand side of (202), and a Hölder inequality in $\mathbb{R}^N$ with exponents $4, 4, 2$ in the last term, followed by Young inequalities, we deduce after taking advantage of (201) that

$$\left|\sum_{i=1}^N (L(\overline{w})\phi_i, \phi_i)_{\mathcal{H}}\right| \le \frac{1}{2}\text{Trace}(\mathcal{A}Q_N) + C\left(\|\nabla\overline{u}\|_{L^2}^2 + \|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2 + \|\nabla\overline{q}\|_{L^2}^4\right)\sum_{i=1}^N \|v_i\|_H^2. \tag{203}$$

Because of the normalization $\|v_i\|_H^2 + \|r_i\|_{L^2}^2 = \|\phi_i\|_{\mathcal{H}}^2 = 1$ we obtain

$$\left|\sum_{i=1}^N (L(\overline{w})\phi_i, \phi_i)_{\mathcal{H}}\right| \le \frac{1}{2}\text{Trace}(\mathcal{A}Q_N) + CN\left(\|\nabla\overline{u}\|_{L^2}^2 + \|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2 + \|\nabla\overline{q}\|_{L^2}^4\right). \tag{204}$$

Let us note that, in view of the fact that K_{R_2} is an absorbing ball, we have

$$\sup_{T\ge 0} \frac{1}{T}\int_0^T \left(\|\nabla \overline{u}\|_{L^2}^2 + \|\overline{q}\|_{L^4}^2 + \|Q\|_{L^4}^2 + \|\nabla \overline{q}\|_{L^2}^4\right) dt \le C(R_2) \tag{205}$$

with $C(R_2)$ a nondecreasing function of R_2. From (19) we have

$$\mu_1 + \cdots + \mu_N \ge cN^{\frac{3}{2}}, \tag{206}$$

and, in view of (196), (198), (199) and (204) we see that if

$$N^{\frac{1}{2}} \ge 8c^{-1}CC(R_2) \tag{207}$$

then $V_N(t)$ decays exponentially,

$$V_N(t) \le V_N(0)e^{-cN^{\frac{3}{2}}t} \tag{208}$$

for $t \ge t_0$ with t_0 depending on R_2. Therefore the proof is complete.

9. Global Attractor

The properties of $S(t)$ of existence of a compact absorbing ball K_{R_2} (Theorem 3), continuity in $\mathcal{H}$ (Theorem 4), backward uniqueness (Theorem 5) imply the existence of a global attractor.

Theorem 7. *Let*

$$X = \bigcap_{t>0} S(t)K_{R_2} \tag{209}$$

where $S(t)$ is the semigroup solving (112) and K_{R_2} is the absorbing ball (125). Then:

(i) *X is compact in $\mathcal{H}$.*
(ii) *$S(t)X = X$ for all $t \ge 0$.*
(iii) *If Z is bounded in $\mathcal{V}$ in the norm of of $\mathcal{V}$, and $S(t)Z = Z$ for all $t \ge 0$, then $Z \subset X$.*
(iv) *For every $w_0 \in \mathcal{V}$, $\lim_{t\to\infty} dist_{\mathcal{H}}(S(t)w_0, X) = 0$.*
(v) *X is connected.*

The proof of this result follows verbatim the proof of the analogous result in [4]. If the body forces vanish, then the attractor is particularly simple, it is a singleton.

Theorem 8. *Let $f = 0$. Then the attractor is a singleton, formed with the unique, globally attracting steady solution $w_Q = (0, Q)$,*

$$X = \{w_Q\}. \tag{210}$$

Proof. We take the scalar product in H of the first equation of (112) with u, we take the scalar product in L^2 of the second equation with $\Lambda^{-1}(q - Q)$ and add. The terms

$$(\mathbb{P}(qR(q-Q)), u)_H + (u\cdot\nabla q, \Lambda^{-1}(q-Q))_{L^2} = 0 \tag{211}$$

cancel, and we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_H^2 + \|\Lambda^{-\frac{1}{2}}(q-Q)\|_{L^2}^2\right) + \|A^{\frac{1}{2}}u\|_H^2 + \|q-Q\|_{L^2}^2 = 0. \tag{212}$$

Because of the Poincaré inequality we obtain exponential decay of the distance to w_Q, first in $H \times H^{-\frac{1}{2}}$ and then in $\mathcal{H}$. The latter follows because

$$\frac{1}{2}\frac{d}{dt}\|q-Q\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}}(q-Q)\|_{L^2}^2 = -\int qu\cdot\nabla Q dx \le \|u\|_H\|q\|_{L^2}\|\nabla Q\|_{L^\infty} \tag{213}$$

and $\|q\|_{L^2}$ is bounded in time, while $\|u\|_H$ decays exponentially by (212), and therefore, from (213) we obtain the exponential convergence of w to w_Q in $\mathcal{H}$. This concludes the proof.

Remark 5. *When $f = 0$, returning to the nonzero mean velocity frame we see that the average velocity converges in time. Indeed, its time derivative, given in (106), obeys*

$$\left|\frac{d}{dt}v(t)\right| = \left|-(2\pi)^{-2}\int (q-Q)\nabla\Phi dx\right| \tag{214}$$

because $\int Q\nabla\Phi dx = 0$. The right hand side of (214) *belongs to $L^1(0,\infty)$ by* (212).

Employing methods initiated in [3] and used in many subsequent works, Theorem 6 implies

Theorem 9. *The global attractor X has finite fractal dimension*

$$D_{\mathcal{H}}(X) \le M \tag{215}$$

where M depends only on norms of f and Φ.

The fractal dimension is defined as

$$\limsup_{r\to 0}\frac{\log N_{\mathcal{H}}(r)}{\log\left(\frac{1}{r}\right)} \tag{216}$$

where $N_{\mathcal{H}}(r)$ is the minimal number of balls in $\mathcal{H}$ of radii r needed to cover X.

Theorem 10. *The global attractor X has finite fractal dimension*

$$D_{\mathcal{V}}(X) = D_{\mathcal{H}}(X). \tag{217}$$

Proof. If $B_i \subset \mathcal{H}$ are a family of balls in $\mathcal{H}$ of radii ρ and centers w_i that cover X, then, because of the invariance $S(t)X = X$, the sets $S(t)B_i$ cover X. Now because of the continuity (130), the sets $S(t)B_i$ are included in balls in $\mathcal{V}$ of radii $t^{-\frac{1}{2}}C_2(t)\rho = r$. Therefore

$$N_{\mathcal{H}}(r) \le N_{\mathcal{V}}(r) \le N_{\mathcal{H}}(\sqrt{t}C_2(t)^{-1}r). \tag{218}$$

Fixing $t > 0$ we obtain

$$\limsup_{r\to 0}\frac{\log N_{\mathcal{V}}(r)}{\log\left(\frac{1}{r}\right)} = \limsup_{r\to 0}\frac{\log N_{\mathcal{H}}(r)}{\log\left(\frac{1}{r}\right)}. \tag{219}$$

10. Appendix A

We give here the proof of Proposition 2. We recall the pointwise identity ([5])

$$\nabla q(x)\cdot\Lambda\nabla q(x) = \frac{1}{2}\Lambda(|\nabla q|^2)(x) + \frac{1}{2}D[q](x) \tag{220}$$

where

$$D[q](x) = cP.V.\int_{\mathbb{R}^2}\frac{|\nabla q(x)-\nabla q(x+y)|^2}{|y|^3}dy, \tag{221}$$

with c a universal constant. We abused notation and wrote q for the periodic extension of q, as a function defined on all $\mathbb{R}^2$.

We consider a cutoff function $\Psi : [0,\infty) \to [0,\infty)$, which is smooth, non-decreasing, identically 1 on $[2,\infty)$, vanishes on $[0,1]$ and obeys $|\Psi'| \le 3$.

For $l > 0$ to be determined, we have

$$
\begin{aligned}
D[q](x) &\geq c \int\limits_{\mathbb{R}^2} \frac{|\nabla q(x) - \nabla q(x+y)|^2}{|y|^3} \Psi\left(\frac{|y|}{l}\right) dy \\
&\geq c \int\limits_{\mathbb{R}^2} \frac{|\nabla q(x)|^2 - 2\nabla q(x) \cdot \nabla q(x+y)}{|y|^3} \Psi\left(\frac{|y|}{l}\right) dy \\
&\geq c|\nabla q(x)|^2 \int\limits_{|y|\geq l} \frac{1}{|y|^3} dy - 2c \sum_{j=1}^{2} \left| \int\limits_{\mathbb{R}^2} \frac{\partial_j q(x) \partial_j q(x+y)}{|y|^3} \Psi\left(\frac{|y|}{l}\right) dy \right| \\
&\geq c_1 \frac{|\nabla q(x)|^2}{l} - c_2 |\nabla q(x)| \sum_{j=1}^{2} \int\limits_{\mathbb{R}^2} |q(x+y)| \left| \nabla \left(\frac{1}{|y|^3} \Psi\left(\frac{|y|}{l}\right)\right)\right| dy.
\end{aligned}
$$

Now

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^2} |q(x+y)| \left|\nabla \left(\tfrac{1}{|y|^3} \Psi\left(\tfrac{|y|}{l}\right)\right)\right| dy \\
&\quad = \textstyle\sum_{j\in\mathbb{Z}^2} \int\limits_{Q_0 + 2\pi j} |q(x+y)| \left|\nabla \left(\tfrac{1}{|y|^3} \Psi\left(\tfrac{|y|}{l}\right)\right)\right| dy \\
&\quad = \textstyle\sum_{j\in\mathbb{Z}^2} \int\limits_{Q_0} |q(x+y)| \left|\nabla \left(\tfrac{1}{|y-2\pi j|^3} \Psi\left(\tfrac{|y-2\pi j|}{l}\right)\right)\right| dy \\
&\quad \leq k(l) \|q\|_{L^4},
\end{aligned}
\tag{222}
$$

where $Q_0 = [-\pi, \pi] \times [-\pi, \pi]$ and

$$
k(l) = \sum_{j\in\mathbb{Z}^2} \left[\int_{Q_0} \left| \nabla \left(\frac{1}{|y-2\pi j|^3} \Psi\left(\frac{|y-2\pi j|}{l}\right)\right)\right|^{\frac{4}{3}} dy \right]^{\frac{3}{4}}. \tag{223}
$$

The contribution of the term corresponding to $j = 0$ in the sum is of the order $l^{-\frac{5}{2}}$ for small l and $\Psi = 1$ for $j \neq 0$ and $l < \frac{\pi}{4}$. We obtain

$$
k(l) \leq C(l^{-\frac{5}{2}} + 1), \tag{224}
$$

for all $0 < l < \frac{1}{4}$, and hence have from (222)

$$
D[q](x) \geq |\nabla q(x)| (c_1 l^{-1} |\nabla q(x)| - c_2 k(l)) \|q\|_{L^4}). \tag{225}
$$

We may choose

$$
l = \min \left\{ \left(2 \frac{C c_2 \|q\|_{L^4}}{c_1 |\nabla q(x)|} \right)^{2/3}, \frac{\pi}{4} \right\} \tag{226}
$$

and deduce the pointwise inequality

$$
D[q](x) \geq \frac{C_1}{4} \|q\|_{L^4}^{-\frac{2}{3}} |\nabla q(x)|^{\frac{8}{3}} - c_3 \|q\|_{L^4}^2 \tag{227}
$$

with c_3 a positive absolute constant. Indeed, if $2\frac{Cc_2\|q\|_{L^4}}{c_1|\nabla q(x)|} \leq \frac{\pi}{4}$ then the inequality follows directly because $|\nabla q(x)|$ is bounded by a constant multiple of the $\|q\|_{L^4}$ norm, and if the opposite inequality holds, we obtain

$$
D[q](x) \geq \frac{C_1}{4} \|q\|_{L^4}^{-\frac{2}{3}} |\nabla q(x)|^{\frac{8}{3}} - C|\nabla q(x)| \|q\|_{L^4} \geq \frac{C_1}{4} \|q\|_{L^4}^{-\frac{2}{3}} |\nabla q(x)|^{\frac{8}{3}} - c_3 \|q\|_{L^4}^2
$$

because $\frac{|\nabla q(x)|}{\|q\|_{L^4}}$ is bounded below. Integrating we obtain

$$c_4\|q\|_{L^4}^2 + \int \nabla q \Lambda \nabla q \ge \frac{C_1}{4}\|q\|_{L^4}^{-\frac{2}{3}}\|\nabla q\|_{L^{\frac{8}{3}}}^{\frac{8}{3}}. \tag{228}$$

We also know that

$$\int \nabla q \Lambda \nabla q \ge c_5\|q\|_{L^4}^2 \tag{229}$$

and therefore

$$\int \nabla q \Lambda \nabla q \ge c_6\|q\|_{L^4}^{-\frac{2}{3}}\|\nabla q\|_{L^{\frac{8}{3}}}^{\frac{8}{3}} \tag{230}$$

follows with $c_6 = \frac{C_1}{4(1+\frac{c_4}{c_5})}$, and thus (22) holds.

REFERENCES

[1] C. Cao, E.S. Titi, Global well-posedness and finite dimensional global attractor for a 3 - D planetary geostrophic viscous model, Communications on Pure and Applied Mathematics **56** (2003), 198–233.

[2] P. Constantin, T. Elgindi, M. Ignatova, V. Vicol, On some electroconvection models, Journal of Nonlinear Science **27** (2017), 197–211.

[3] P. Constantin, C. Foias. Global Lyapunov exponents, Kaplan-Yorke formulas and the dimension of the attractors for 2D Navier-Stokes equations. Comm. Pure Appl. Math., **38** (1985) no. 1, 1–27.

[4] P. Consatntin, C. Foias, Navier-Stokes equations, Chicago University Press, Chicago (1988).

[5] P. Constantin, A. Tarfulea, V. Vicol, Long time dynamics of forced critical SQG, Communications in Mathematical Physics **335** (2015), no. 1, 93–141.

[6] P. Constantin, V. Vicol, Nonlinear maximum principles for dissipative linear nonlocal operators and applications, GAFA **22** (2012), 1289–1321.

[7] M. Conti, M. Coti Zelati, Attractors for the Cahn-Hilliard equation with memory in 2D, Nonlinear Analysis, Theory, Methods and Applications,**72** (2010), 1668–1682.

[8] M. Coti Zelati, P. Kalita P, Smooth attractors for weak solutions of the SQG equation with critical dissipation, Discrete and Continuous Dynamical Systems - B, **22**, (2017) 1857–1873.

[9] Z.A. Daya, V.B. Deyirmenjian, S.W. Morris, J.R. de Bruyn, Annular electroconvection with shear, Phys. Rev. Lett. **80** (1998), 964–967.

[10] I. Kukavica, Fourier parameterization of attractors for dissipative equations in one space dimension. Special issue dedicated to Victor A. Pliss on the occasion of his 70th birthday, J. Dynam. Differential Equations **15** (2003), no. 2-3, 473–484.

[11] M. Oliver, E.S. Titi, Analyticity of the global attractor and the number of determining nodes for a weakly damped driven nonlinear Schrödinger equation, Indiana University Mathematics Journal, **47** (1998), 49–73.

[12] P. Tsai, Z. Daya, S. Morris, Charge transport scaling in turbulent electroconvection, Phys. Rev E **72** (2005), 046311-1-12.

[13] P. Tsai, Z.A. Daya, V.B. Deyirmenjian, S.W. Morris, Direct numerical simulation of supercritical annular electroconvection, Phys. Rev E **76** (2007), 1–11.

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