GLOBAL SMOOTH SOLUTIONS OF THE NERNST-PLANCK-DARCY SYSTEM

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Abstract. The Nernst-Planck-Darcy system models ionic electrodiffusion in porous media. We consider the system for two ionic species with opposite valences and equal diffusivities. We prove that the initial value problem for the Nernst-Planck-Darcy system in two or three dimensions has global unique smooth solutions for arbitrary large data. We obtain $W^{1,p}(\mathbb{T}^d)$, for $p \geq 2$, and higher regularity bounds.

1. Introduction

The Nernst-Planck (NP) system is given by

$$\partial_t c_i + \nabla \cdot (uc_i - D_i \nabla c_i - z_i D_i c_i \nabla \Phi) = 0, \qquad i = 1, \ldots, N, \tag{1.1}$$

$$-\varepsilon \Delta \Phi = \rho, \tag{1.2}$$

$$\rho = \sum_{i=1}^{N} z_i c_i, \tag{1.3}$$

where $c_i \colon \Omega \times [0,T] \to \mathbb{R}^+$ are ionic concentrations, $z_i \in \mathbb{Z}$ are corresponding valences, $D_i > 0$ are constant diffusivities, $\Phi \colon \Omega \times [0,T] \to \mathbb{R}$ is the nondimensional electrical potential, $\rho \colon \Omega \times [0,T] \to \mathbb{R}$ is the nondimensional charge density, $u \colon \Omega \times [0,T] \to \mathbb{R}^d$ is the fluid velocity field, and $\varepsilon > 0$ is a constant proportional to the square of the Debye length [6, 20]. Here, $d = 2, 3$ is the space dimension and Ω is a d-dimensional domain. The Nernst-Planck equation (1.1) describes the evolution of ions which are carried by a fluid and interact among themselves via an electric potential and thermal fluctuations. The Poisson equation (1.2)–(1.3) relates the electrostatic potential to the ionic charge density.

The Nernst-Planck equations coupled to fluid equations are basic models of ionic electrodiffusion in fluids. The equations are supplemented by boundary conditions and initial conditions. Ionic electrodiffusion is important in many fields, including biology, chemistry and physics, and has wide applications [20]. There are extensive mathematical studies of models coupling the Nernst-Planck equations with various fluid dynamical systems. For the fluids that are described by the Navier-Stokes equations, the system is known as the Nernst-Planck-Navier-Stokes (NPNS) system. In the whole space $\mathbb{R}^d$ ($d = 2, 3$), local existence of solutions of the NPNS system is obtained in [15], and later weak solutions are proved to exist globally in time [19], with some L^2 decay if the dimension is two [26]. In bounded domains $\Omega \subset \mathbb{R}^d$, the NPNS system has global solutions under various appropriate boundary conditions. For example, with blocking boundary conditions, global weak solutions exist in both two and three dimensions [10, 16]. For ionic concentrations satisfying blocking boundary conditions while the electrical potential satisfying the Dirichlet boundary condition, global weak solutions exist in two and three dimensions if the initial data is small [21]. In the case of blocking boundary conditions for the ionic concentrations and homogeneous Neumann boundary condition for the electrical potential, weak solutions are global in two dimensions [22]. Moreover, with blocking boundary conditions for the ionic concentrations and Robin boundary condition for the electrical potential, two dimensional strong solutions are global [2]. The same result holds in three dimensions if the fluid velocity remains regular for all time [17]. When the ionic concentrations satisfy either the blocking boundary conditions or the uniformly selective boundary conditions, strong solutions are global in two dimensions [6], and in three dimensions

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provided that the initial data is a small perturbation of a steady state [8]. If both the ionic concentrations and the electrical potential obey the Dirichlet boundary conditions, global strong solutions exist in three dimensions as long as the fluid velocity is regular [9]. With periodic boundary conditions, two dimensional strong solutions exist globally in time, and long time behavior of the solutions under the influence of body forces and added electrical charges is studied in [1]. Vanishing Debye length limit ($\varepsilon \to 0$ in (1.2)) results for the NPNS system are proved in [7, 18, 23, 24]. In the limit of zero viscosity in the Navier-Stokes equations, the solutions of NPNS system in two dimensions converges to the solutions of the corresponding Nernst-Planck-Euler (NPE) system, whose solutions exist and are global [14, 25, 27]. The existence of globally smooth solutions of three dimensional Nernst-Planck equations with arbitrary large data coupled to Stokes equations driven by the Lorentz force has been obtained only recently in [9, 17].

For models of flow through porous media, the Stokes operator is replaced by Darcy's law. In the case of a flow through a porous medium of an incompressible fluid forced by macroscopic electrostatic Lorentz forces $-\rho\nabla\Phi$ due to a ionic charge density, Darcy's law and incompressibility are expressed as

$$\kappa u + \nabla p = -\rho\nabla\Phi, \tag{1.4}$$

$$\nabla \cdot u = 0, \tag{1.5}$$

where $u\colon \Omega \times [0,T] \to \mathbb{R}^d$ is the fluid velocity field, $p\colon \Omega \times [0,T] \to \mathbb{R}$ is the fluid pressure, and $\kappa > 0$ is a positive coefficient.

We refer to the equations (1.1)–(1.5) as the Nernst-Planck-Darcy (NPD) equations by analogy to the Nernst-Planck-Navier-Stokes equations. The Poisson equation is considered as being part of the Nernst-Planck system, it is a manifestation of the electrostatic approximation. The NPD system models the advection and diffusion of ions in porous media. The NPD system was studied in [11, 13] where it was shown that global weak solutions exist in bounded domains $\Omega \subset \mathbb{R}^d$, $d = 2,3$, in L^2-based Sobolev spaces by fixed point arguments. Similar results are obtained in two space dimensions for more than two ionic species [12].

The analysis of systems coupling Nernst-Planck to fluid equations presents challenges due to the boundary conditions and challenges due to the nonlinearity. In this paper, we focus on the latter by considering the spatially periodic case $\Omega = \mathbb{T}^d$, $d = 2,3$. By contrast with the Stokes operator which is an elliptic operator of order two, Darcy's law, being an operator of order zero, "looses" two differential orders, and the analysis is more difficult. In this paper we discuss the initial value problem for the NPD equations (1.1)–(1.5) in two or three space dimensions ($d = 2,3$) with two ionic species ($N = 2$) with opposite valences ($z_1 = -z_2 = 1$) and with equal diffusivities ($D_1 = D_2 = D$). The initial data of the system is

$$c_i(\cdot,0) = c_i(0), \quad i = 1,2, \tag{1.6}$$

where the ionic concentrations are nonnegative, $c_i(0) \geq 0$, and the electric charge obeys the neutrality condition

$$\int_{\mathbb{T}^d} \rho(x,0)\,\mathrm{d}x = \sum_{i=1}^{2} \int_{\mathbb{T}^d} z_i c_i(x,0)\,\mathrm{d}x = \int_{\mathbb{T}^d} c_1(x,0) - c_2(x,0)\,\mathrm{d}x = 0. \tag{1.7}$$

It follows from (1.1) that the neutrality condition (1.7) is preserved in time. This condition is necessary for the solvability of the Poisson equation (1.2) with periodic boundary conditions. The potential Φ is determined up to a constant, and Φ never enters the equations without at least one derivative being applied to it. Without loss of generality we take the spatial average of Φ to vanish, and thus Φ is uniquely determined by ρ, $\Phi = -\varepsilon^{-1}\Delta^{-1}\rho$. For regular solutions of NPNS it is shown in [6, 9] that if $c_i(0) \geq 0$, then $c_i(x,t)$ remains nonnegative for $t > 0$. Indeed, this property follows from (1.1) if c_i are known to be sufficiently regular, and the same proof and result hold for the NPD equations (see Theorem 2.8). The positivity of the concentrations is an essential ingredient in the proof of global regularity, as it confers a nonlinear dissipation mechanism (cf. (3.11)) that is a key stepping stone for high regularity.

We denote $\rho = c_1 - c_2$, $\sigma = c_1 + c_2$. Using (1.5), the Nernst-Planck system (1.1) is equivalent to the equations

$$\partial_t \rho = -u \cdot \nabla \rho + D(\Delta \rho + \nabla \sigma \cdot \nabla \Phi + \sigma \Delta \Phi), \tag{1.8}$$
$$\partial_t \sigma = -u \cdot \nabla \sigma + D(\Delta \sigma + \nabla \rho \cdot \nabla \Phi + \rho \Delta \Phi). \tag{1.9}$$

We have from (1.2) that

$$-\varepsilon \Delta \Phi = \rho, \tag{1.10}$$

and from (1.4)–(1.5) that

$$\kappa u + \nabla p = -\rho \nabla \Phi, \tag{1.11}$$
$$\nabla \cdot u = 0. \tag{1.12}$$

The system (1.8)–(1.12) has initial data from (1.6),

$$\begin{aligned} \rho(\cdot, 0) &= \rho(0) = c_1(0) - c_2(0), \\ \sigma(\cdot, 0) &= \sigma(0) = c_1(0) + c_2(0), \end{aligned} \tag{1.13}$$

and

$$c_1 = \frac{\sigma + \rho}{2} \quad \text{and} \quad c_2 = \frac{\sigma - \rho}{2}$$

solve the original Nernst-Planck-Darcy system (1.1)–(1.5).

We fix the parameters $\varepsilon > 0$ and $D > 0$ in (1.8)–(1.10) and $\kappa > 0$ in (1.11) for simplicity of exposition. We will not make explicit the dependence of various constants on ε, D and κ in the rest of the paper.

In this paper, we establish the global existence and uniqueness of smooth solutions for arbitrary large data. Once the existence of strong solutions is established it follows that the solutions are C^∞ smooth.

The main theorems are the following.

Theorem 1.1. *Let $d = 2, 3$ and $r \geq 2$. Let $c_1(0), c_2(0) \in W^{1,r}(\mathbb{T}^d)$ be nonnegative functions satisfying (1.7). Then for any $T > 0$, there exist unique $c_1(x,t) \geq 0$, $c_2(x,t) \geq 0$ and $u(x,t)$, such that $c_1 - c_2 = \rho$ and $c_1 + c_2 = \sigma$ obey $\rho, \sigma \in L^\infty([0,T]; W^{1,r}(\mathbb{T}^d)) \cap L^2(0,T; H^2(\mathbb{T}^d))$, $u(x,t)$ is divergence-free and obeys $u \in L^\infty([0,T]; W^{1,r}(\mathbb{T}^d))$, and (ρ, σ, Φ, u) solve the initial value problem (1.8)–(1.13) in $L^2(0,T; L^2(\mathbb{T}^d))$. The charge density ρ and total concentration σ satisfy the following bounds*

$$\begin{aligned} &(i) \quad \|\rho(t)\|_{L^p} + \left\|\sigma(t) - \bar{\sigma}\right\|_{L^p} \leq C_p e^{-C't}, \qquad \forall p \geq 2, \\ &(ii) \quad \|\nabla \Phi(t)\|_{L^\infty} \leq C e^{-C't}, \\ &(iii) \quad \|\nabla \rho(t)\|_{L^2}^2 + \|\nabla \sigma(t)\|_{L^2}^2 + \int_0^t \|\Delta \rho(\tau)\|_{L^2}^2 + \|\Delta \sigma(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \leq C, \\ &(iv) \quad \left\|\nabla \rho(t)\right\|_{L^r} + \left\|\nabla \sigma(t)\right\|_{L^r} \leq C e^{Ct}, \end{aligned} \tag{1.14}$$

with constants $C, C_p > 0$ depending on D, ε, κ, p, r, and the initial data $\|\rho(0)\|_{W^{1,r}}$ and $\|\sigma(0)\|_{W^{1,r}}$ and with $C' > 0$ bounded from below independently of p. Moreover, the fluid velocity u satisfies the bound

$$\|\nabla u(t)\|_{L^r} \leq C e^{C''t}, \tag{1.15}$$

where $C'' \in \mathbb{R}$ is a constant depending on $\|\rho(0)\|_{W^{1,r}}$ and $\|\sigma(0)\|_{W^{1,r}}$.

Remark 1.2. In view of the a priori bounds (1.14) and (1.15), the right hand sides of equations (1.8)–(1.12) belong to $L^2(0,T; L^2(\mathbb{T}^d))$. See Lemma 2.3 below.

Theorem 1.3. *Let $d = 2, 3$. Let $c_1(0), c_2(0) \in H^3(\mathbb{T}^d)$ be nonnegative functions satisfying (1.7). Then for any $T > 0$, there exists a unique solution $\rho, \sigma \in L^\infty([0,T]; H^3(\mathbb{T}^d)) \cap L^2(0,T; H^4(\mathbb{T}^d))$, $\Phi \in L^\infty([0,T]; H^5(\mathbb{T}^d))$,* *and* $u \in L^\infty([0,T];H^3(\mathbb{T}^d))$ *of the initial value problem* (1.8)–(1.13)*. In addition to the bounds* (1.14) *for the ionic concentrations, we also have for any* $t > 0$*,*

$$\begin{aligned} &\|\Delta\rho(t)\|_{L^2} + \|\Delta\sigma(t)\|_{L^2} + \int_0^t \|\nabla\Delta\rho(\tau)\|_{L^2}^2 + \|\nabla\Delta\sigma(\tau)\|_{L^2}^2 \le C, \\ &\|\nabla\Delta\rho(t)\|_{L^2} + \|\nabla\Delta\sigma(t)\|_{L^2} + \int_0^t \|\Delta^2\rho(\tau)\|_{L^2}^2 + \|\Delta^2\sigma(\tau)\|_{L^2}^2 \le C, \end{aligned} \tag{1.16}$$

where $C > 0$ *depends only on* ε*,* D*,* κ *and the initial data. For the fluid velocity* u*, in addition to the estimates* (1.15)*, we also have for any* $t > 0$

$$\|u(t)\|_{H^3} \le C.$$

In this paper we use the nonlinear structure of the NP equations coupled to Darcy's law, in order to prove the global regularity. The construction of solutions is achieved by the following procedure. We first prove local existence and uniqueness of strong solutions (Theorem 2.4), which are solutions whose concentrations belong to $L^\infty(0,T;H^1(\mathbb{T}^d)) \cap L^2(0,T;H^2(\mathbb{T}^d))$. The system (1.8)–(1.12) is semilinear mixed elliptic parabolic. The local in time existence of smooth solutions can be obtained by many methods: semigroup (Picard iteration), or Galerkin (approximation in eigenfunction expansions) or other approximation procedures. We choose Galerkin approximations for simplicity. The positivity of concentrations is essential for establishing global existence of solutions. By Theorem 2.8, we have that as long as the solutions are strong, the concentrations remain nonnegative, if they are initially so. We show that strong solutions can be uniquely extended for all time provided certain quantitative information is obtained (Theorem 2.7). In order to prove global existence therefore it is enough to obtain uniform, time independent a priori estimates. These a priori estimates are the heart of the matter. Using positivity, the global a priori L^p-estimate (1.14)(i) is a consequence of the special nonlinear structure of (1.8)–(1.9), and is the basis for higher derivative estimates.

The proof of Theorem 1.1 is in Section 3 and the proof of Theorem 1.3 is in Section 4.

2. Local existence of strong solutions

We consider real-valued periodic functions

$$f(x) = \sum_{k\in\mathbb{Z}^d} f_k e^{ik\cdot x}$$

with Fourier coefficients $f_k \in \mathbb{C}$ satisfying the requirement $\bar{f}_k = f_{-k}$, and Sobolev spaces $H^s(\mathbb{T}^d)$ defined by

$$H^s(\mathbb{T}^d) = \{f \mid \sum_{k\in\mathbb{Z}^d} |f_k|^2|k|^{2s} < \infty\}.$$

The velocity spaces are similar,

$$u(x) = \sum_{k\in\mathbb{Z}^d} u_k e^{ik\cdot x}$$

with $u_k \in \mathbb{C}^d$, the reality condition $\bar{u}_k = u_{-k}$ imposed component-wise and the divergence-free condition $k \cdot u_k = 0$. Subspaces of finite dimension $H^s_m(\mathbb{T}^d)$ are obtained by restricting the range of wave numbers k to $|k| \le m$, and corresponding projectors P_m,

$$P_m : H^s(\mathbb{T}^d) \to P_m(H^s(\mathbb{T}^d)) = H^s_m(\mathbb{T}^d),$$

are obtained by mapping $f(x) = \sum_{k\in\mathbb{Z}^d} f_k e^{ik\cdot x}$ to

$$(P_m f)(x) = \sum_{|k|\le m} f_k e^{ik\cdot x}.$$

Given a function ρ with mean zero,

$$\rho(x) = \sum_{k\neq 0} \rho_k e^{ik\cdot x},$$

in this paper we always consider the unique mean zero solution of (1.10) given by

$$\Phi = \varepsilon^{-1}(-\Delta)^{-1}\rho \tag{2.1}$$

with

$$(-\Delta)^{-1}\rho(x) = \sum_{k\neq 0} |k|^{-2}\rho_k e^{ik\cdot x}.$$

Similarly, given ρ and Φ as above, the unique solution of (1.11)–(1.12) is given by

$$u = -\kappa^{-1}\mathbb{P}(\rho\nabla\Phi) \tag{2.2}$$

where $\mathbb{P}$ is the Leray projector on divergence free functions, which at the level of Fourier coefficients acts by mapping $v_k \in \mathbb{C}^d$ to $u_k = v_k - |k|^{-2}(v_k\cdot k)k$ for $k \neq 0$ and setting $u_0 = 0$.

In the sequel we omit the integration domain $\mathbb{T}^d$ and write $\int f = \int_{\mathbb{T}^d} f(x)\,\mathrm{d}x$. We denote

$$\bar{f} = \frac{1}{|\mathbb{T}^d|}\int f,$$

the average of a function f over the torus $\mathbb{T}^d$. (This notation should not be confused with the complex conjugate. We will not use the complex conjugate notation in the rest of the paper.) In inequalities, C and C' denote constants which may change from line to line. Throughout this paper we take $d = 2, 3$. The embedding inequalities are quoted for $d = 3$, but they are also valid for $d = 2$.

We start by defining the notion of strong solution.

Definition 2.1. We say that (ρ, σ, u, Φ) is a strong solution of (1.8)–(1.12) on $[0, T]$ if ρ and σ belong to $L^\infty(0, T; H^1(\mathbb{T}^d)) \cap L^2(0, T; H^2(\mathbb{T}^d))$, Φ is given by (2.1), u is given by (2.2) and the equations (1.8)–(1.9) are satisfied in $L^2((0, T) \times \mathbb{T}^d)$.

Remark 2.2. The fact that the right hand sides of (1.8) and (1.9) belong to L^2 follows by Sobolev embedding inequalities. More precisely, we have the following lemma.

Lemma 2.3. *Let $\rho, \sigma \in L^\infty(0, T; H^1(\mathbb{T}^d)) \cap L^2(0, T; H^2(\mathbb{T}^d))$, then*

$$\Phi \in L^\infty(0, T; W^{2,6}(\mathbb{T}^d)) \tag{2.3}$$

and

$$u \in L^\infty(0, T; H^1(\mathbb{T}^d)) \cap L^2(0, T; H^2(\mathbb{T}^d)). \tag{2.4}$$

Consequently, each of the terms $u\cdot\nabla\rho$, $u\cdot\nabla\sigma$, $\mathrm{div}(\nabla\rho + \sigma\nabla\Phi)$, $\mathrm{div}(\nabla\sigma + \rho\nabla\Phi)$ belongs to $L^2(0, T; L^2(\mathbb{T}^d))$. This shows that the equation can be tested with any function in $L^2(0, T; L^2(\mathbb{T}^d))$.

Proof. The bound (2.3) follows directly from the Poisson equation (1.10) and the fact that $\rho \in L^\infty(0, T; L^6(\mathbb{T}^d))$ by the Sobolev embedding $H^1(\mathbb{T}^d) \hookrightarrow L^6(\mathbb{T}^d)$. Note that, in particular we have that $\Phi \in L^\infty(0, T; W^{1,\infty}(\mathbb{T}^d))$.

The bounds in (2.4) follow from the relation (2.2), the fact that $\mathbb{P}$ is bounded in L^p spaces, $1 < p < \infty$, (2.3) and Sobolev embeddings. □

Theorem 2.4. *Let $\rho(0) \in H^1(\mathbb{T}^d)$ with mean zero, $\bar{\rho}(0) = 0$, and $\sigma(0) \in H^1(\mathbb{T}^d)$ be given functions. There exists $T_0 > 0$ depending only on the norms $\|\rho(0)\|_{H^1(\mathbb{T}^d)}$ and $\|\sigma(0)\|_{H^1(\mathbb{T}^d)}$ and a unique strong solution (ρ, σ, u, Φ) of (1.8)–(1.12) on $[0, T_0]$ with initial data $\rho(0)$, $\sigma(0)$.*

Moreover, if $\rho(0), \sigma(0)$ belong to $H^s(\mathbb{T}^d)$ with $s > 1$, then $\rho(t), \sigma(t)$ belong to $H^s(\mathbb{T}^d)$ on $[0, T_0]$.

Proof. We only sketch the proof of existence. We consider Galerkin approximations $\rho_m, \sigma_m \in H^1_m(\mathbb{T}^d)$, with potential $\Phi_m = \varepsilon^{-1}(-\Delta)^{-1}\rho_m$ and velocity

$$u_m = -\kappa^{-1}\mathbb{P}_m(\rho_m\nabla\Phi_m)$$

where $\mathbb{P}_m$ is the Leray projector applied after applying P_m in each component, $\mathbb{P}_m = \mathbb{P}P_m$. We solve the system of ODEs

$$\partial_t\rho_m = P_m\left(-u_m\cdot\nabla\rho_m + D\nabla\cdot(\nabla\rho_m + \sigma_m\nabla\Phi_m)\right), \tag{2.5}$$

$$\partial_t \sigma_m = P_m \left(-u_m \cdot \nabla \sigma_m + D \nabla \cdot (\nabla \sigma_m + \rho_m \nabla \Phi_m) \right) \tag{2.6}$$

with initial data $\rho_m(0) = P_m(\rho(0))$, $\sigma_m(0) = P_m(\sigma(0))$. This system has a local existence time T_0 that is uniform in m. This follows from nonlinear inequalities

$$\frac{d}{dt}(\|\rho_m\|^2_{H^1(\mathbb{T}^d)} + \|\sigma_m\|^2_{H^1(\mathbb{T}^d)}) + (\|\rho_m\|^2_{H^2(\mathbb{T}^d)} + \|\sigma_m\|^2_{H^2(\mathbb{T}^d)}) \le C(\|\rho_m\|^2_{H^1(\mathbb{T}^d)} + \|\sigma_m\|^2_{H^1(\mathbb{T}^d)})^2$$

where C does not depend on m. Then passing to the limit of $m \to \infty$ using the Aubin-Lions lemma yields the strong solution. The preservation of higher regularity is obtained by energy estimates as well.

Now, we show that strong solutions are unique. Let $(\rho_i, \sigma_i, u_i, \Phi_i)$, $i = 1, 2$ be two strong solutions of (1.8)–(1.12) with the same initial data $\rho(0), \sigma(0)$ satisfying $\rho(0), \sigma(0) \in H^1(\mathbb{T}^d)$. Denoting by $\rho = \rho_1 - \rho_2$, $\sigma = \sigma_1 - \sigma_2$, $\Phi = \Phi_1 - \Phi_2$, $u = u_1 - u_2$ and by $\widetilde{\rho} = \frac{1}{2}(\rho_1 + \rho_2)$, $\widetilde{\sigma} = \frac{1}{2}(\sigma_1 + \sigma_2)$, $\widetilde{\Phi} = \frac{1}{2}(\Phi_1 + \Phi_2)$, $\widetilde{u} = \frac{1}{2}(u_1 + u_2)$, the equations become

$$(\partial_t + \widetilde{u} \cdot \nabla)\rho - D\Delta\rho = -u \cdot \nabla\widetilde{\rho} + D\nabla \cdot (\widetilde{\sigma}\nabla\Phi + \sigma\nabla\widetilde{\Phi}), \tag{2.7}$$

$$(\partial_t + \widetilde{u} \cdot \nabla)\sigma - D\Delta\sigma = -u \cdot \nabla\widetilde{\sigma} + D\nabla \cdot (\widetilde{\rho}\nabla\Phi + \rho\nabla\widetilde{\Phi}) \tag{2.8}$$

together with

$$u = -\kappa^{-1}\mathbb{P}(\rho\nabla\widetilde{\Phi} + \widetilde{\rho}\nabla\Phi) \tag{2.9}$$

where $\mathbb{P}$ is the Leray projector and

$$-\varepsilon\Delta\Phi = \rho. \tag{2.10}$$

Multiplying (2.7) by ρ and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\rho\|^2_{L^2} + D\|\nabla\rho\|^2 = \int \widetilde{\rho} u \cdot \nabla\rho \, dx - D\int (\widetilde{\sigma}\nabla\Phi + \sigma\nabla\widetilde{\Phi}) \cdot \nabla\rho \, dx. \tag{2.11}$$

Similarly, multiplying (2.8) by σ and integrating by parts gives

$$\frac{1}{2}\frac{d}{dt}\|\sigma\|^2_{L^2} + D\|\nabla\sigma\|^2 = \int \widetilde{\sigma} u \cdot \nabla\sigma \, dx - D\int (\widetilde{\rho}\nabla\Phi + \rho\nabla\widetilde{\Phi}) \cdot \nabla\sigma \, dx. \tag{2.12}$$

Now we use the fact that $\widetilde{\rho}$ and $\widetilde{\sigma}$ are in L^∞ due to the embedding $H^2(\mathbb{T}^d) \hookrightarrow L^\infty$ to estimate

$$\left| \int \widetilde{\rho} u \cdot \nabla\rho \, dx \right| \le \frac{D}{16}\|\nabla\rho\|^2 + (4D)^{-1}\|\widetilde{\rho}\|^2_{L^\infty}\|u\|^2_{L^2} \tag{2.13}$$

and similarly

$$\left| \int \widetilde{\sigma} u \cdot \nabla\sigma \, dx \right| \le \frac{D}{16}\|\nabla\sigma\|^2 + (4D)^{-1}\|\widetilde{\sigma}\|^2_{L^\infty}\|u\|^2_{L^2}. \tag{2.14}$$

We note that, in view of (2.9), we have

$$\|u\|^2_{L^2} \le C\left(\|\nabla\widetilde{\Phi}\|^2_{L^\infty} + \|\widetilde{\rho}\|^2_{L^3} \right)\|\rho\|^2_{L^2} \tag{2.15}$$

where we used the estimate

$$\|\nabla\Phi\|_{L^6} \le C\|\rho\|_{L^2}. \tag{2.16}$$

We obtain

$$\left| D\int (\widetilde{\sigma}\nabla\Phi + \sigma\nabla\widetilde{\Phi}) \cdot \nabla\rho \, dx \right| \le CD\left(\|\widetilde{\sigma}\|_{L^3} + \|\nabla\widetilde{\Phi}\|_{L^\infty} \right)(\|\rho\|_{L^2} + \|\sigma\|_{L^2})\|\nabla\rho\|_{L^2} \tag{2.17}$$

and

$$\left| D\int (\widetilde{\rho}\nabla\Phi + \rho\nabla\widetilde{\Phi}) \cdot \nabla\sigma \, dx \right| \le CD\left(\|\widetilde{\rho}\|_{L^3} + \|\nabla\widetilde{\Phi}\|_{L^\infty} \right)\|\rho\|_{L^2}\|\nabla\sigma\|_{L^2} \tag{2.18}$$

by using (2.16). From the inequalities (2.11)–(2.18), we arrive at

$$\frac{d}{dt}(\|\rho\|^2_{L^2} + \|\sigma\|^2_{L^2}) \le C\left(\|\widetilde{\rho}\|^2_{L^\infty} + \|\widetilde{\sigma}\|^2_{L^\infty} + 1 \right)(\|\nabla\widetilde{\Phi}\|^2_{L^\infty} + \|\widetilde{\rho}\|^2_{L^3})\left(\|\rho\|^2_{L^2} + \|\sigma\|^2_{L^2} \right). \tag{2.19}$$

In view of the fact that

$$\|\nabla\widetilde{\Phi}\|_{L^\infty} \le C\|\widetilde{\rho}\|_{L^4}, \tag{2.20}$$

the embedding $H^1 \hookrightarrow L^3$, and the embedding $H^2 \hookrightarrow L^\infty$, we deduce that the function

$$\left(\|\widetilde{\rho}\|_{L^\infty}^2 + \|\widetilde{\sigma}\|_{L^\infty}^2 + 1\right)(\|\nabla\widetilde{\Phi}\|_{L^\infty}^2 + \|\widetilde{\rho}\|_{L^3}^2) \in L^1(0,T)$$

is integrable in time because $\widetilde{\rho}$ and $\widetilde{\sigma}$ belong to $L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$. From the ordinary differential inequality (2.19) we conclude that ρ and σ must vanish as they start from 0. Then the inequality (2.9) implies that u must vanish, and the inequality (2.16) implies that Φ must vanish as well. □

Remark 2.5. In Theorem 2.4 the initial data are attained strongly in L^2, that is

$$\lim_{t\to 0}\|\sigma(t) - \sigma(0)\|_{L^2(\mathbb{T}^d)} = 0, \quad \lim_{t\to 0}\|\rho(t) - \rho(0)\|_{L^2(\mathbb{T}^d)} = 0.$$

Indeed, σ and ρ belong to $C([0,T];L^2(\mathbb{T}^d))$ because $\partial_t\rho$ and $\partial_t\sigma$ belong to $L^2(0,T,L^2(\mathbb{T}^d))$.

Remark 2.6. Note that in Theorem 2.4 no assumption of positivity of concentrations is needed.

Theorem 2.7. *Let $T_1 > 0$ and let (ρ,σ,u,Φ) be a strong solution of* (1.8)–(1.12) *on* $[0,T_1]$*. Let*

$$\int_0^{T_1}(\|\rho\|_{H^1(\mathbb{T}^d)}^2 + \|\sigma\|_{H^1(\mathbb{T}^d)}^2)dt = A(T_1) \tag{2.21}$$

Then the solution obeys

$$\sup_{0\le t\le T_1}(\|\rho\|_{H^1(\mathbb{T}^d)}^2 + \|\sigma\|_{H^1(\mathbb{T}^d)}^2) \le C\exp\left(T_1\exp CA(T_1)\right) \tag{2.22}$$

and there exists $T_2 > T_1$ such that the solution can be uniquely extended to a strong solution on $[0,T_2]$*.*

The proof of (2.22) follows from the fact that strong solutions obey nonlinear inequalities

$$\frac{d}{dt}(\|\rho\|_{H^1(\mathbb{T}^d)}^2 + \|\sigma\|_{H^1(\mathbb{T}^d)}^2) + (\|\rho\|_{H^2(\mathbb{T}^d)}^2 + \|\sigma\|_{H^2(\mathbb{T}^d)}^2) \le C(\|\rho\|_{H^1(\mathbb{T}^d)}^2 + \|\sigma\|_{H^1(\mathbb{T}^d)}^2)^2,$$

and the Grönwall lemma. The unique extension then follows from Theorem 2.4.

We also have a preservation of positivity result for the concentrations $c_1 = \frac{\rho+\sigma}{2}$ and $c_2 = \frac{\sigma-\rho}{2}$.

Theorem 2.8. *Let (ρ,σ,u,Φ) be a strong solution of* (1.8)–(1.12) *on* $[0,T]$*. Assume that $c_1(0)$ and $c_2(0)$ are almost everywhere nonnegative. Then $c_1(t)$ and $c_2(t)$ are almost everywhere nonnegative on* $[0,T]$*.*

The proof of this theorem is the same as in [9] and is omitted.

3. A priori bounds in $W^{1,p}$

In this section, we present the a priori estimates of Theorem 1.1. We split the proof of the a priori estimates into several lemmas.

The first lemma concerns a priori dissipative bounds for the potential Φ in the system (1.8)–(1.12).

Lemma 3.1. *Let $r \ge 2$. Let $c_1(0), c_2(0) \in W^{1,r}(\mathbb{T}^d)$ be nonnegative functions satisfying* (1.7)*. Suppose (ρ,σ,Φ,u) solves* (1.8)–(1.12) *with initial data* (1.13) *on the interval* $[0,T]$*. Then for any $t \in [0,T]$, we have*

$$\|\nabla\Phi(t)\|_{L^2}^2 + \frac{2}{\varepsilon}\int_0^t \|u(\tau)\|_{L^2}^2\,\mathrm{d}\tau + \frac{2D}{\varepsilon}\int_0^t \|\rho(\tau)\|_{L^2}^2\,\mathrm{d}\tau \le 2\|\nabla\Phi(0)\|_{L^2}^2. \tag{3.1}$$

Proof. Using (1.8) and (1.10), we have an evolutionary equation

$$\partial_t(-\Delta\Phi) = -\frac{1}{\varepsilon}u\cdot\nabla\rho + \frac{D}{\varepsilon}(\Delta\rho + \nabla\sigma\cdot\nabla\Phi + \sigma\Delta\Phi).$$

Testing this equation with Φ and integrating by parts, we find

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Phi\|_{L^2}^2 &= -\frac{1}{\varepsilon}\int u\cdot\nabla\rho\Phi + \frac{D}{\varepsilon}\int\Delta\rho\Phi + \frac{D}{\varepsilon}\int\nabla\cdot(\sigma\nabla\Phi)\Phi\\ &= \frac{1}{\varepsilon}\int u\cdot(\rho\nabla\Phi) + \frac{D}{\varepsilon}\int\rho\Delta\Phi - \frac{D}{\varepsilon}\int\sigma|\nabla\Phi|^2.\end{aligned}$$

By equations (1.11) and (1.10), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Phi\|_{L^2}^2 + \frac{1}{\varepsilon}\|u\|_{L^2}^2 + \frac{D}{\varepsilon^2}\|\rho\|_{L^2}^2 + \frac{D}{\varepsilon}\int\sigma|\nabla\Phi|^2 = 0.$$

Integrating in time and discarding the last term on the left hand side (notice that $\sigma = c_1 + c_2 \geq 0$) give the inequality (3.1). □

In the following lemma we establish useful bounds for the fluid velocity field u.

Lemma 3.2. *Let (ρ, u) satisfy (1.11)–(1.12). Then*

$$\|u\|_{L^p} \leq C\|\rho\nabla\Phi\|_{L^p} \leq C\|\rho\|_{L^p}\|\nabla\Phi\|_{L^\infty}, \tag{3.2}$$

$$\|\nabla u\|_{L^r} \leq C\|\nabla\rho\|_{L^r}\|\nabla\Phi\|_{L^\infty}, \tag{3.3}$$

for all $p, r \in (1,\infty)$.

Proof. Applying the Leray projector to (1.11) and noting that the Leray projector is bounded on L^p when $p \in (1,\infty)$ (see, e.g., [5]), we obtain (3.2). As for (3.3), we take the curl of (1.11) and have

$$\nabla^\perp\cdot u = -\kappa^{-1}\nabla^\perp\rho\cdot\nabla\Phi \quad \text{if } d = 2,$$

$$\operatorname{curl} u = -\kappa^{-1}\nabla\rho\times\nabla\Phi \quad \text{if } d = 3.$$

Then the proof is completed by invoking the well-known estimate for divergence free functions,

$$\|\nabla u\|_{L^r} \leq C\|\operatorname{curl} u\|_{L^r}$$

for all $r \in (1,\infty)$. □

A key step of proving global a priori bounds for the weak solutions is to obtain (1.14)(ii) and thus (1.14)(i). In either two or three dimensions, in view of (1.10), elliptic estimates, and Sobolev embeddings $H^1(\mathbb{T}^d) \hookrightarrow L^6(\mathbb{T}^d)$ and $W^{1,4}(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ for $d = 2, 3$, we have

$$\|\nabla\Phi\|_{L^6} \leq C\|\rho\|_{L^2}, \tag{3.4}$$

$$\|\nabla\Phi\|_{L^\infty} \leq C\|\rho\|_{L^4}. \tag{3.5}$$

The following lemma states the pointwise exponential decay of $\|\rho\|_{L^p}$ and $\|\nabla\Phi\|_{L^\infty}$.

Lemma 3.3. *Let $r \geq 2$. Let $c_1(0), c_2(0) \in W^{1,r}(\mathbb{T}^d)$ be nonnegative functions satisfying (1.7). Suppose (ρ, σ, Φ, u) solves (1.8)–(1.12) with initial data (1.13) on the interval $[0, T]$. Then for any $t \in [0, T]$, we have*

$$\|\rho(t)\|_{L^p} \leq C_p e^{-C't}, \qquad \forall p \geq 2, \tag{3.6}$$

$$\|\nabla\Phi(t)\|_{L^\infty} \leq Ce^{-C't}, \tag{3.7}$$

for some constants $C_p, C' > 0$, with C' independent of p.

Proof. We first observe that (1.9) is equivalent to

$$\partial_t(\sigma - \bar\sigma) = -u\cdot\nabla(\sigma - \bar\sigma) + D\left(\Delta(\sigma - \bar\sigma) + \nabla\rho\cdot\nabla\Phi + \rho\Delta\Phi\right). \tag{3.8}$$

Here $\bar\sigma \geq 0$ since $c_1, c_2 \geq 0$ when $t = 0$ and $\bar\sigma$ is conserved in time due to (1.9). The time dependent function σ will converge in the long time limit to the time independent average $\bar\sigma$. Moreover, by referring to the departure from average we reveal the dissipative nature of the system formed by $\sigma-\bar\sigma$ and ρ. Indeed, let $p\ge 2$. We multiply (1.8) by $\frac{1}{p-1}\rho|\rho|^{p-2}$ and (3.8) by $\frac{1}{p-1}(\sigma-\bar\sigma)|\sigma-\bar\sigma|^{p-2}$, and then integrate by parts,

$$\frac{1}{p(p-1)}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^p}^p=-D\int|\rho|^{p-2}|\nabla\rho|^2-D\int|\rho|^{p-2}(\sigma-\bar\sigma)\nabla\rho\cdot\nabla\Phi-D\int|\rho|^{p-2}\bar\sigma\nabla\rho\cdot\nabla\Phi, \tag{3.9}$$

$$\frac{1}{p(p-1)}\frac{\mathrm{d}}{\mathrm{d}t}\|\sigma-\bar\sigma\|_{L^p}^p=-D\int|\sigma-\bar\sigma|^{p-2}|\nabla(\sigma-\bar\sigma)|^2-D\int|\sigma-\bar\sigma|^{p-2}\rho\nabla(\sigma-\bar\sigma)\cdot\nabla\Phi. \tag{3.10}$$

Taking $p=2$, summing (3.9) and (3.10), and using (1.10), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\rho\|_{L^2}^2+\|\sigma-\bar\sigma\|_{L^2}^2\right)+D\left(\|\nabla\rho\|_{L^2}^2+\|\nabla(\sigma-\bar\sigma)\|_{L^2}^2\right)+\frac{D}{\varepsilon}\int\sigma\rho^2=0.$$

Recall that the ionic concentrations $c_1,c_2\ge 0$, so that $\sigma=c_1+c_2\ge 0$, and thus, the last term on the left hand side is nonnegative. Furthermore, since $|\rho|=|c_1-c_2|\le c_1+c_2=\sigma$, we have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\rho\|_{L^2}^2+\|\sigma-\bar\sigma\|_{L^2}^2\right)+D\left(\|\nabla\rho\|_{L^2}^2+\|\nabla(\sigma-\bar\sigma)\|_{L^2}^2\right)+\frac{D}{\varepsilon}\|\rho\|_{L^3}^3\le 0. \tag{3.11}$$

By the Poincaré inequality and Grönwall's inequality, we deduce the following exponential pointwise decay

$$\|\rho(t)\|_{L^2}^2+\|\sigma(t)-\bar\sigma\|_{L^2}^2\le\left(\|\rho(0)\|_{L^2}^2+\|\sigma(0)-\bar\sigma\|_{L^2}^2\right)e^{-C't}, \tag{3.12}$$

and the bounds

$$2D\int_0^t\|\nabla\rho(\tau)\|_{L^2}^2+\|\nabla\sigma(\tau)\|_{L^2}^2\,\mathrm{d}\tau+\frac{2D}{\varepsilon}\int_0^t\|\rho(\tau)\|_{L^3}^3\,\mathrm{d}\tau\le\|\rho(0)\|_{L^2}^2+\|\sigma(0)-\bar\sigma\|_{L^2}^2. \tag{3.13}$$

We obtain from (3.4) and (3.12) that

$$\|\nabla\Phi(t)\|_{L^6}\le C\|\rho(t)\|_{L^2}\le Ce^{-C't}. \tag{3.14}$$

For $p\ge 4$, we have from (1.8)–(1.10) that

$$\frac{1}{p(p-1)}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^p}^p=-D\int|\rho|^{p-2}|\nabla\rho|^2-\frac{D\bar\sigma}{(p-1)\varepsilon}\int|\rho|^p-D\int|\rho|^{p-2}(\sigma-\bar\sigma)\nabla\rho\cdot\nabla\Phi, \tag{3.15}$$

$$\frac{1}{p(p-1)}\frac{\mathrm{d}}{\mathrm{d}t}\|\sigma-\bar\sigma\|_{L^p}^p=-D\int|\sigma-\bar\sigma|^{p-2}|\nabla(\sigma-\bar\sigma)|^2-D\int|\sigma-\bar\sigma|^{p-2}\rho\nabla(\sigma-\bar\sigma)\cdot\nabla\Phi. \tag{3.16}$$

Notice that by Hölder's inequalities with exponents $6,3,2$ and $\frac{p}{3},\frac{p}{p-3}$, the Gagliardo–Nirenberg interpolation inequality

$$\|f\|_{L^{\frac{3(p-2)}{p-3}}}\le C\|\nabla f\|_{L^2}^{\frac{dp}{6(p-2)}}\|f\|_{L^2}^{1-\frac{dp}{6(p-2)}}+\|f\|_{L^2},$$

and Young's inequality with exponents $\frac{12}{6+d},\frac{12}{6-d}$ and $2,2$, we have

$$\begin{aligned}
&D\int|\rho|^{p-2}(\sigma-\bar\sigma)\nabla\rho\cdot\nabla\Phi\\
&\le\frac{2D}{p}\|\nabla\Phi\|_{L^6}\|\nabla|\rho|^{\frac{p}{2}}\|_{L^2}\||\rho|^{\frac{p-2}{2}}(\sigma-\bar\sigma)\|_{L^3}\\
&\le\frac{2D}{p}\|\nabla\Phi\|_{L^6}\|\nabla|\rho|^{\frac{p}{2}}\|_{L^2}\||\rho|^{\frac{p}{2}}\|_{L^{\frac{3(p-2)}{p-3}}}^{\frac{p-2}{p}}\|\sigma-\bar\sigma\|_{L^p}\\
&\le C\|\nabla\Phi\|_{L^6}\|\nabla|\rho|^{\frac{p}{2}}\|_{L^2}^{1+\frac{d}{6}}\|\rho\|_{L^p}^{\frac{(6-d)p}{12}-1}\|\sigma-\bar\sigma\|_{L^p}+C\|\nabla\Phi\|_{L^6}\|\nabla|\rho|^{\frac{p}{2}}\|_{L^2}\|\rho\|_{L^p}^{\frac{p-2}{2}}\|\sigma-\bar\sigma\|_{L^p}\\
&\le\frac{D}{2}\int|\rho|^{p-2}|\nabla\rho|^2+C\|\nabla\Phi\|_{L^6}^{\frac{12}{6-d}}\|\rho\|_{L^p}^{p-\frac{12}{6-d}}\|\sigma-\bar\sigma\|_{L^p}^{\frac{12}{6-d}}+C\|\nabla\Phi\|_{L^6}^2\|\rho\|_{L^p}^{p-2}\|\sigma-\bar\sigma\|_{L^p}^2.
\end{aligned} \tag{3.17}$$

Similarly, we have

$$
\begin{aligned}
&D\int|\sigma-\bar\sigma|^{p-2}\rho\nabla\sigma\cdot\nabla\Phi\\
&\quad\le \frac{D}{2}\int|\sigma-\bar\sigma|^{p-2}|\nabla\sigma|^2+C\|\nabla\Phi\|_{L^6}^{\frac{12}{6-d}}\|\sigma-\bar\sigma\|_{L^p}^{p-\frac{12}{6-d}}\|\rho\|_{L^p}^{\frac{12}{6-d}}+C\|\nabla\Phi\|_{L^6}^2\|\sigma-\bar\sigma\|_{L^p}^{p-2}\|\rho\|_{L^p}^2.
\end{aligned}
\tag{3.18}
$$

Using (3.17)–(3.18) for the nonlinear terms in (3.15)–(3.16), absorbing and neglecting the terms involving $\int|\rho|^{p-2}|\nabla\rho|^2$ and $\int|\sigma-\bar\sigma|^{p-2}|\nabla\sigma|^2$ yield

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\rho\|_{L^p}+\|\sigma-\bar\sigma\|_{L^p}\right)+\frac{D\bar\sigma}{\varepsilon}\|\rho\|_{L^p}\le C\left(\|\nabla\Phi\|_{L^6}^2+\|\nabla\Phi\|_{L^6}^{\frac{12}{6-d}}\right)\left(\|\rho\|_{L^p}+\|\sigma-\bar\sigma\|_{L^p}\right).
$$

Dropping the dissipation term and applying Grönwall's inequality give

$$
\|\rho(t)\|_{L^p}+\|\sigma(t)-\bar\sigma\|_{L^p}\le\left(\|\rho(0)\|_{L^p}+\|\sigma(0)-\bar\sigma\|_{L^p}\right)e^{C\int_0^t\|\nabla\Phi(\tau)\|_{L^6}^2+\|\nabla\Phi(\tau)\|_{L^6}^{\frac{12}{6-d}}\,\mathrm{d}\tau}.
$$

In view of (3.14), $\int_0^t\|\nabla\Phi(\tau)\|_{L^6}^2+\|\nabla\Phi(\tau)\|_{L^6}^{\frac{12}{6-d}}\,\mathrm{d}\tau$ is uniformly bounded for all $t>0$. Thus, we obtain

$$
\|\rho(t)\|_{L^p}+\|\sigma(t)-\bar\sigma\|_{L^p}\le C,
\tag{3.19}
$$

where $C>0$ is a constant depending only on p, the initial data, and the parameters of the problem.

Now we use (3.17) again in (3.15), and update with the new estimate (3.19) to derive

$$
\begin{aligned}
\frac{1}{p(p-1)}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_{L^p}^p+\frac{D\bar\sigma}{(p-1)\varepsilon}\|\rho\|_{L^p}^p&\le C\|\nabla\Phi\|_{L^6}^{\frac{12}{6-d}}\|\rho\|_{L^p}^{p-\frac{12}{6-d}}\|\sigma-\bar\sigma\|_{L^p}^{\frac{12}{6-d}}+C\|\nabla\Phi\|_{L^6}^2\|\rho\|_{L^p}^{p-2}\|\sigma-\bar\sigma\|_{L^p}^2\\
&\le C\left(\|\nabla\Phi\|_{L^6}^2+\|\nabla\Phi\|_{L^6}^{\frac{12}{6-d}}\right).
\end{aligned}
$$

Integrating in time and applying (3.14) then yield

$$
\|\rho(t)\|_{L^p}^p\le e^{-\frac{D\bar\sigma p}{\varepsilon}t}\left(\|\rho(0)\|_{L^p}^p+C\int_0^t\|\nabla\Phi(\tau)\|_{L^6}^2+\|\nabla\Phi(\tau)\|_{L^6}^{\frac{12}{6-d}}\,\mathrm{d}\tau\right)\le Ce^{-\frac{D\bar\sigma p}{\varepsilon}t},
$$

which leads to (3.6). Finally, in view of (3.5) and (3.6), we conclude (3.7). □

The following lemma establishes the pointwise decay of $\|\sigma-\bar\sigma\|_{L^p}$, whose proof is based on a Moser's type iteration argument as in [2, 3, 8, 14]. The purpose of this is to obtain by induction, from properties of the dissipative factors, L^p bounds for higher values of p.

Lemma 3.4. *Let $r\ge 2$. Let $c_1(0),c_2(0)\in W^{1,r}(\mathbb{T}^d)$ be nonnegative functions satisfying* (1.7)*. Suppose (ρ,σ,Φ,u) solves* (1.8)*–*(1.12) *with initial data* (1.13) *on the interval $[0,T]$. Then for any $t\in[0,T]$, we have*

$$
\|\sigma(t)-\bar\sigma\|_{L^p}\le C_pe^{-C't},\qquad\forall p\ge 2,
\tag{3.20}
$$

for some constants $C_p,C'>0$ with C' independent of p.

Proof. From (3.10), we have

$$
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\left\||\sigma-\bar\sigma|^{\frac{p}{2}}\right\|_{L^2}^2+D(p-1)\int|\sigma-\bar\sigma|^{p-2}|\nabla(\sigma-\bar\sigma)|^2=-D(p-1)\int|\sigma-\bar\sigma|^{p-2}\rho\nabla(\sigma-\bar\sigma)\cdot\nabla\Phi.
$$

We use the bounds

$$
D(p-1)\int|\sigma-\bar\sigma|^{p-2}|\nabla(\sigma-\bar\sigma)|^2=\frac{4D(p-1)}{p^2}\int\left|\nabla|\sigma-\bar\sigma|^{\frac{p}{2}}\right|^2\ge\frac{2D}{p}\int\left|\nabla|\sigma-\bar\sigma|^{\frac{p}{2}}\right|^2
$$

and

$$
D(p-1)\int|\sigma-\bar\sigma|^{p-2}\rho\nabla(\sigma-\bar\sigma)\cdot\nabla\Phi\le 2D\|\rho\|_{L^p}\|\nabla\Phi\|_{L^\infty}\left\||\sigma-\bar\sigma|^{\frac{p}{2}}\right\|_{L^2}^{\frac{p-2}{p}}\left\|\nabla|\sigma-\bar\sigma|^{\frac{p}{2}}\right\|_{L^2}
$$

to deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\||\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^2}^2+2D\left\|\nabla|\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^2}^2\leq 2Dp\|\rho\|_{L^p}\|\nabla\Phi\|_{L^\infty}\left\||\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^2}^{\frac{p-2}{p}}\left\|\nabla|\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^2}.$$

By Young's inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\||\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^2}^2+D\left\|\nabla|\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^2}^2\leq Dp^2\|\rho\|_{L^p}^2\|\nabla\Phi\|_{L^\infty}^2\left\||\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^2}^{2-\frac{4}{p}}. \tag{3.21}$$

The Gagliardo-Nirenberg interpolation inequality and Young's inequality imply that

$$\begin{aligned}\left\||\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^2}^2&\leq M\left\|\nabla|\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^2}^{\frac{2d}{2+d}}\left\||\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^1}^{\frac{4}{2+d}}+M\left\||\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^1}^2\\&\leq\delta\left\|\nabla|\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^2}^2+\frac{2d^{\frac{d}{2}}}{(2+d)^{1+\frac{d}{2}}}\frac{M^{\frac{2+d}{2}}+M\delta^{\frac{d}{2}}}{\delta^{\frac{d}{2}}}\left\||\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^1}^2\\&\leq\delta\left\|\nabla|\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^2}^2+\frac{M^{\frac{2+d}{2}}+M\delta^{\frac{d}{2}}}{\delta^{\frac{d}{2}}}\left\||\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^1}^2,\end{aligned} \tag{3.22}$$

where $M>0$ is the constant from the interpolation inequality and δ is a number to be chosen later.

Multiplying (3.22) by $\frac{D}{\delta}$, we get

$$D\left\|\nabla|\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^2}^2\geq\frac{D}{\delta}\left\||\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^2}^2-D\frac{M^{\frac{2+d}{2}}+M\delta^{\frac{d}{2}}}{\delta^{1+\frac{d}{2}}}\left\||\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^1}^2. \tag{3.23}$$

Thus, using (3.23) in (3.21) yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\||\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^2}^2+\frac{D}{\delta}\left\||\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^2}^2\\&\leq Dp^2\|\rho\|_{L^p}^2\|\nabla\Phi\|_{L^\infty}^2\left\||\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^2}^{2-\frac{4}{p}}+D\frac{M^{\frac{2+d}{2}}+M\delta^{\frac{d}{2}}}{\delta^{1+\frac{d}{2}}}\left\||\sigma-\bar{\sigma}|^{\frac{p}{2}}\right\|_{L^1}^2.\end{aligned} \tag{3.24}$$

Now we choose $\delta=\frac{1}{2p}$ and use a Young inequality with exponents $\frac{p}{p-2}$, $\frac{p}{2}$ to deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sigma-\bar{\sigma}\|_{L^p}^p+pD\|\sigma-\bar{\sigma}\|_{L^p}^p\leq C_p\left(\|\rho\|_{L^p}^2\|\nabla\Phi\|_{L^\infty}^2\right)^{\frac{p}{2}}+C_p\|\sigma-\bar{\sigma}\|_{L^{\frac{p}{2}}}^p. \tag{3.25}$$

Applying Grönwall's inequality then leads to

$$\begin{aligned}\left\|\sigma(t)-\bar{\sigma}\right\|_{L^p}^p\leq e^{-pDt}\bigg[&\left\|\sigma(0)-\bar{\sigma}\right\|_{L^p}^p+C_p\int_0^t e^{pD\tau}\|\rho(\tau)\|_{L^p}^p\|\nabla\Phi(\tau)\|_{L^\infty}^p\,\mathrm{d}\tau\\&+C_p\int_0^t e^{pD\tau}\left\|\sigma(\tau)-\bar{\sigma}\right\|_{L^{\frac{p}{2}}}^p\,\mathrm{d}\tau\bigg].\end{aligned} \tag{3.26}$$

From (3.6)–(3.7) it follows that

$$\int_0^t e^{-pD(t-\tau)}\|\rho(\tau)\|_{L^p}^p\|\nabla\Phi(\tau)\|_{L^\infty}^p\,\mathrm{d}\tau\leq Ce^{-pDt}\int_0^t e^{p(D-C')\tau}\,\mathrm{d}\tau\leq Ce^{-pC't}+Ce^{-pDt}\leq Ce^{-pC't}$$

holds with C' bounded from below independently of p. We estimate the last integral in (3.26) by induction. We first recall that $\|\sigma(t)-\bar{\sigma}\|_{L^2}$ decays exponentially in time (see (3.12)).

We take $p=2^{j+1}$ for $j\in\mathbb{N}$ and assume by induction that

$$\left\|\sigma(t)-\bar{\sigma}\right\|_{L^p}^p\leq Ce^{-pc_pt} \tag{3.27}$$

with $c_p>\epsilon>0$ bounded from below independently of p. We take, without loss of generality $\epsilon\leq\frac{D}{2}$. We deduce from (3.26) that

$$\left\|\sigma(t)-\bar{\sigma}\right\|_{L^p}^p\leq Ce^{-pDt}\left(1+e^{-pC't}+\int_0^t e^{p\left(D-2c_{\frac{p}{2}}\right)\tau}\,\mathrm{d}\tau\right)\leq Ce^{-pC't}+Ce^{-pkt} \tag{3.28}$$

with $k = \min\{D; 2c_{\frac{p}{2}}\} \geq \min\{D; 2\epsilon\} = 2\epsilon$. Thus c_p is bounded from below by $\epsilon > 0$ which is uniform in $p \to \infty$.

Therefore, we deduced from (3.26) that $\|\sigma(t) - \bar{\sigma}\|_{L^p}$ decays exponentially for each fixed $p \geq 2$ of the form $p = 2^j$ ($j \in \mathbb{N}$) at a rate bounded from below uniformly as $p \to \infty$. Then by interpolation, we obtain that $\|\sigma(t) - \bar{\sigma}\|_{L^p}$ decays exponentially for all $p \geq 2$, at a rate bounded from below uniformly as $p \to \infty$, which is (3.20). □

The next lemma concerns the $W^{1,r}$ norms of the solutions, which finishes the a priori estimates in Theorem 1.1.

Lemma 3.5. *Let* $r \geq 2$. *Let* $c_1(0), c_2(0) \in W^{1,r}(\mathbb{T}^d)$ *be nonnegative functions satisfying* (1.7). *Suppose* (ρ, σ, Φ, u) *solves* (1.8)–(1.12) *with initial data* (1.13) *on the interval* $[0, T]$. *Then for any* $t \in [0, T]$, *we have*

$$\|\nabla\rho(t)\|_{L^2}^2 + \|\nabla\sigma(t)\|_{L^2}^2 + \int_0^t \|\Delta\rho(\tau)\|_{L^2}^2 + \|\Delta\sigma(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \leq C, \tag{3.29}$$

$$\left\|\nabla\rho(t)\right\|_{L^r} + \left\|\nabla\sigma(t)\right\|_{L^r} \leq Ce^{Ct}, \tag{3.30}$$

$$\|\nabla u(t)\|_{L^r} \leq Ce^{C''t}, \tag{3.31}$$

for some constants $C > 0$ *and* $C'' \in \mathbb{R}$.

Proof. Testing (1.8) and (1.9) with $-\Delta\rho$ and $-\Delta\sigma$ respectively, summing the resulting equations, and using (1.10), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2\right) + D\left(\|\Delta\rho\|_{L^2}^2 + \|\Delta\sigma\|_{L^2}^2\right) + \frac{D}{\varepsilon}\int \sigma|\nabla\rho|^2 \\
&\quad = \int u\cdot\nabla\rho\Delta\rho + \int u\cdot\nabla\sigma\Delta\sigma - D\int \Delta\rho(\nabla\sigma\cdot\nabla\Phi) - D\int\Delta\sigma(\nabla\rho\cdot\nabla\Phi) \\
&\qquad - \frac{D}{\varepsilon}\int \rho\nabla\rho\cdot\nabla\sigma - \frac{2D}{\varepsilon}\int\rho|\nabla\rho|^2.
\end{aligned} \tag{3.32}$$

Using Hölder's inequality, the advection terms in (3.32) can be estimated as

$$\int u\cdot\nabla\rho\Delta\rho + \int u\cdot\nabla\sigma\Delta\sigma \leq \|u\|_{L^6}\left(\|\nabla\rho\|_{L^3}\|\Delta\rho\|_{L^2} + \|\nabla\sigma\|_{L^3}\|\Delta\sigma\|_{L^2}\right). \tag{3.33}$$

We use the Gagliardo-Nirenberg inequality

$$\|\nabla f\|_{L^3} \leq C\|\Delta f\|_{L^2}^{\frac{d}{6}}\|\nabla f\|_{L^2}^{1-\frac{d}{6}} + C\|\nabla f\|_{L^2}$$

the estimate (3.2), and Young's inequality with exponents $\frac{12}{6+d}$ and $\frac{12}{6-d}$ in (3.33) to obtain

$$\begin{aligned}
&\int u\cdot\nabla\rho\Delta\rho + \int u\cdot\nabla\sigma\Delta\sigma \\
&\leq C\|u\|_{L^6}\left(\|\nabla\rho\|_{L^2}^{1-\frac{d}{6}}\|\Delta\rho\|_{L^2}^{1+\frac{d}{6}} + \|\nabla\sigma\|_{L^2}^{1-\frac{d}{6}}\|\Delta\sigma\|_{L^2}^{1+\frac{d}{6}}\right) + C\|u\|_{L^6}\left(\|\nabla\rho\|_{L^2}\|\Delta\rho\|_{L^2} + \|\nabla\sigma\|_{L^2}\|\Delta\sigma\|_{L^2}\right) \\
&\leq \frac{D}{4}\left(\|\Delta\rho\|_{L^2}^2 + \|\Delta\sigma\|_{L^2}^2\right) + C\left(\|\rho\|_{L^6}^2\|\nabla\Phi\|_{L^\infty}^2 + \|\rho\|_{L^6}^{\frac{12}{6-d}}\|\nabla\Phi\|_{L^\infty}^{\frac{12}{6-d}}\right)\left(\|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2\right).
\end{aligned}$$

The other terms in (3.32) can be estimated using Hölder's inequality, Young's inequality, and Ladyzhenskaya's inequalities in two or three dimensions

$$\|f\|_{L^4} \leq C\|f\|_{L^2}^{1-\frac{d}{4}}\|\nabla f\|_{L^2}^{\frac{d}{4}}.$$

The resulting estimates for (3.32) is

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\rho\|_{L^2}^2+\|\nabla\sigma\|_{L^2}^2\right)+2D\left(\|\Delta\rho\|_{L^2}^2+\|\Delta\sigma\|_{L^2}^2\right)+\frac{2D}{\varepsilon}\int\sigma|\nabla\rho|^2\\
&\quad\le\frac{D}{2}\left(\|\Delta\rho\|_{L^2}^2+\|\Delta\sigma\|_{L^2}^2\right)+C\left(\|\rho\|_{L^6}^2\|\nabla\Phi\|_{L^\infty}^2+\|\rho\|_{L^6}^{\frac{12}{6-d}}\|\nabla\Phi\|_{L^\infty}^{\frac{12}{6-d}}\right)\left(\|\nabla\rho\|_{L^2}^2+\|\nabla\sigma\|_{L^2}^2\right)\\
&\qquad+C\|\nabla\Phi\|_{L^\infty}^2\left(\|\nabla\rho\|_{L^2}^2+\|\nabla\sigma\|_{L^2}^2\right)+C\|\rho\|_{L^2}\left(\|\nabla\rho\|_{L^4}^2+\|\nabla\sigma\|_{L^4}^2\right)\\
&\quad\le D\left(\|\Delta\rho\|_{L^2}^2+\|\Delta\sigma\|_{L^2}^2\right)\\
&\qquad+C\left(\|\nabla\Phi\|_{L^\infty}^2+\|\rho\|_{L^6}^2\|\nabla\Phi\|_{L^\infty}^2+\|\rho\|_{L^6}^{\frac{12}{6-d}}\|\nabla\Phi\|_{L^\infty}^{\frac{12}{6-d}}+\|\rho\|_{L^2}+\|\rho\|_{L^2}^{\frac{4}{4-d}}\right)\left(\|\nabla\rho\|_{L^2}^2+\|\nabla\sigma\|_{L^2}^2\right),
\end{aligned}
$$

where in the last line we used the Young's inequality with exponents $\frac{4}{d}$ and $\frac{4}{4-d}$ and elliptic estimates. Integrating this inequality in time, we obtain

$$
\begin{aligned}
&\|\nabla\rho(t)\|_{L^2}^2+\|\nabla\sigma(t)\|_{L^2}^2+D\int_0^t\|\Delta\rho(\tau)\|_{L^2}^2+\|\Delta\sigma(\tau)\|_{L^2}^2\,\mathrm{d}\tau\\
&\quad\le\|\nabla\rho(0)\|_{L^2}^2+\|\nabla\sigma(0)\|_{L^2}^2\\
&\qquad+C\sup_{\tau\in[0,t]}\left(\|\nabla\Phi(\tau)\|_{L^\infty}^2+\|\rho(\tau)\|_{L^6}^2\|\nabla\Phi(\tau)\|_{L^\infty}^2+\|\rho(\tau)\|_{L^6}^{\frac{12}{6-d}}\|\nabla\Phi(\tau)\|_{L^\infty}^{\frac{12}{6-d}}+\|\rho(\tau)\|_{L^2}+\|\rho(\tau)\|_{L^2}^{\frac{4}{4-d}}\right)\\
&\qquad\cdot\int_0^t\|\nabla\rho(\tau)\|_{L^2}^2+\|\nabla\sigma(\tau)\|_{L^2}^2\,\mathrm{d}\tau\\
&\quad\le C,
\end{aligned}
$$

where the last line follows from (3.6)–(3.7) and (3.13).

For $r>2$, we differentiate the two equations in (1.8)–(1.9), and then multiply by $\nabla\rho|\nabla\rho|^{r-2}$ and $\nabla\sigma|\nabla\sigma|^{r-2}$ respectively, integrate over $\mathbb{T}^d$, and integrate by parts to obtain

$$
\begin{aligned}
\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^r}^r=&-\int|\nabla\rho|^{r-2}\nabla\rho\cdot(\nabla u)^*\nabla\rho-D\int|\nabla\rho|^{r-2}|\nabla\nabla\rho|^2-\frac{4D(r-2)}{r^2}\int\left|\nabla|\nabla\rho|^{\frac{r}{2}}\right|^2\\
&-D\int|\nabla\rho|^{r-2}\Delta\rho\nabla\sigma\cdot\nabla\Phi-D\int|\nabla\rho|^{r-2}\Delta\rho\sigma\Delta\Phi-D\int\nabla\rho\cdot\nabla|\nabla\rho|^{r-2}\sigma\Delta\Phi\\
&-D(r-2)\int\nabla\rho\cdot(\nabla\nabla\rho)\cdot\nabla\rho|\nabla\rho|^{r-4}\nabla\sigma\cdot\nabla\Phi
\end{aligned}
\tag{3.34}
$$

and

$$
\begin{aligned}
\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\sigma\|_{L^r}^r=&-\int|\nabla\sigma|^{r-2}\nabla\sigma\cdot(\nabla u)^*\nabla\sigma-D\int|\nabla\sigma|^{r-2}|\nabla\nabla\sigma|^2-\frac{4D(r-2)}{r^2}\int\left|\nabla|\nabla\sigma|^{\frac{r}{2}}\right|^2\\
&-D\int|\nabla\sigma|^{r-2}\Delta\sigma\nabla\rho\cdot\nabla\Phi-D\int|\nabla\sigma|^{r-2}\Delta\sigma\rho\Delta\Phi-D\int\nabla\sigma\cdot\nabla|\nabla\sigma|^{r-2}\rho\Delta\Phi\\
&-D(r-2)\int\nabla\sigma\cdot(\nabla\nabla\sigma)\cdot\nabla\sigma|\nabla\sigma|^{r-4}\nabla\rho\cdot\nabla\Phi,
\end{aligned}
\tag{3.35}
$$

where $(\nabla u)^*$ represents the transpose matrix of ∇u.

For simplicity, we denote

$$
Y=\|\nabla\rho\|_{L^r}^r+\|\nabla\sigma\|_{L^r}^r=\|R\|_{L^2}^2+\|S\|_{L^2}^2,\quad R=|\nabla\rho|^{\frac{r}{2}},\quad S=|\nabla\sigma|^{\frac{r}{2}}.
$$

Adding (3.34) to (3.35) and using (1.10), Hölder's inequality, and Young's inequality, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Y + \mathcal{D}_1 \le r\int |\nabla u|(R^2+S^2) + \frac{Dr}{2}\int |\nabla\rho|^{r-2}|\nabla\nabla\rho|^2 + \frac{Dr}{2}\int |\nabla\sigma|^{r-2}|\nabla\nabla\sigma|^2 \\ + 2Dr((r-2)^2+1)\|\nabla\Phi\|_{L^\infty}^2\left(\|\nabla\rho\|_{L^r}^{r-2}\|\nabla\sigma\|_{L^r}^2 + \|\nabla\sigma\|_{L^r}^{r-2}\|\nabla\rho\|_{L^r}^2\right) \\ + \frac{2Dr}{\varepsilon^2}((r-2)^2+1)\int \rho^2\left(|\nabla\rho|^{r-2}\sigma^2 + |\nabla\sigma|^{r-2}\rho^2\right),\end{aligned}$$

where $\mathcal{D}_1$ is the dissipation term

$$\mathcal{D}_1 = Dr\int |\nabla\rho|^{r-2}|\nabla\nabla\rho|^2 + Dr\int|\nabla\sigma|^{r-2}|\nabla\nabla\sigma|^2 + \frac{4D(r-2)}{r}\left(\|\nabla R\|_{L^2}^2 + \|\nabla S\|_{L^2}^2\right).$$

Therefore, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Y + \mathcal{D}_2 \le r\left(\int |\nabla u|(R^2+S^2)\right) + 2Dr((r-2)^2+1)\|\nabla\Phi\|_{L^\infty}^2 Y \\ + \frac{2Dr}{\varepsilon^2}((r-2)^2+1)\int \rho^2(\rho^2+\sigma^2)\left(R^{\frac{2r-4}{r}} + S^{\frac{2r-4}{r}}\right),\end{aligned} \tag{3.36}$$

where

$$\begin{aligned}\mathcal{D}_2 &= \frac{Dr}{2}\int |\nabla\rho|^{r-2}|\nabla\nabla\rho|^2 + \frac{Dr}{2}\int|\nabla\sigma|^{r-2}|\nabla\nabla\sigma|^2 + \frac{4D(r-2)}{r}\left(\|\nabla R\|_{L^2}^2 + \|\nabla S\|_{L^2}^2\right) \\ &\ge \frac{4D(r-2)}{r}\left(\|R\|_{H^1}^2 - \|R\|_{L^2}^2 + \|S\|_{H^1}^2 - \|S\|_{L^2}^2\right).\end{aligned}$$

We first note that from Ladyzhenskaya's inequality, Young's inequality, and (3.3)

$$\begin{aligned}\int |\nabla u|(R^2+S^2) &\le \|\nabla u\|_{L^2}\left(\|R\|_{L^4}^2 + \|S\|_{L^4}^2\right) \\ &\le \|\nabla u\|_{L^2}\left(\|R\|_{L^2}^{2-\frac{d}{2}}\|\nabla R\|_{L^2}^{\frac{d}{2}} + \|R\|_{L^2}^2 + \|S\|_{L^2}^{2-\frac{d}{2}}\|\nabla S\|_{L^2}^{\frac{d}{2}} + \|S\|_{L^2}^2\right) \\ &\le \frac{D(r-2)}{r^2}\left(\|R\|_{H^1}^2 + \|S\|_{H^1}^2\right) + C\|\nabla\Phi\|_{L^\infty}^{\frac{4}{4-d}}\|\nabla\rho\|_{L^2}^{\frac{4}{4-d}}\left(\|R\|_{L^2}^2 + \|S\|_{L^2}^2\right).\end{aligned} \tag{3.37}$$

By Hölder's inequality with exponents $\frac{r}{2}$ and $\frac{2}{2-\frac{4}{r}}$, we have

$$\begin{aligned}&\frac{2Dr}{\varepsilon^2}((r-2)^2+1)\int \rho^2(\rho^2+\sigma^2)\left(R^{\frac{2r-4}{r}} + S^{\frac{2r-4}{r}}\right) \\ &\quad\le C\left(\|R\|_{L^2}^2 + \|S\|_{L^2}^2\right)^{\frac{r-2}{r}}\left(\|\rho\|_{L^{2r}} + \|\sigma\|_{L^{2r}}\right)^4 \\ &\quad\le \|R\|_{L^2}^2 + \|S\|_{L^2}^2 + C\left(\|\rho\|_{L^{2r}} + \|\sigma\|_{L^{2r}}\right)^{2r}.\end{aligned} \tag{3.38}$$

Using the inequalities (3.37)–(3.38) in (3.36), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}Y \le C\left(1 + \|\nabla\Phi\|_{L^\infty}^2 + \|\nabla\Phi\|_{L^\infty}^{\frac{4}{4-d}}\|\nabla\rho\|_{L^2}^{\frac{4}{4-d}}\right)Y + \left(\|\rho\|_{L^{2r}} + \|\sigma\|_{L^{2r}}\right)^{2r}.$$

By the bounds (3.6)–(3.7),(3.20), and Grönwall's inequality, we then deduce that $Y(t)$ has at most exponential growth in time $t > 0$,

$$\begin{aligned}
Y(t) &= \|\nabla\rho(t)\|_{L^r}^r + \|\nabla\sigma(t)\|_{L^r}^r \\
&\le \exp\left(C\int_0^t 1 + \|\nabla\Phi(\tau)\|_{L^\infty}^2 + \|\nabla\Phi(\tau)\|_{L^\infty}^{\frac{4}{4-d}}\|\nabla\rho(\tau)\|_{L^2}^{\frac{4}{4-d}}\,\mathrm{d}\tau\right) \\
&\quad \cdot \left[\|\nabla\rho(0)\|_{L^r}^r + \|\nabla\sigma(0)\|_{L^r}^r + \int_0^t \left(\|\rho(\tau)\|_{L^{2r}} + \|\sigma(\tau)\|_{L^{2r}}\right)^{2r}\mathrm{d}\tau\right] \\
&\le Ce^{C't},
\end{aligned}$$

where the constants $C, C' > 0$ depend on r, the parameters of the problem, and the initial data.

We finally use the bounds (3.3), (3.7), and (3.30) to obtain that

$$\|\nabla u(t)\|_{L^r} \le \|\nabla\rho(t)\|_{L^r}\|\nabla\Phi(t)\|_{L^\infty} \le Ce^{C''t}$$

for some constant $C'' \in \mathbb{R}$ depending on r, the parameters of the problem, and the initial data. □

4. Higher derivative a priori bounds

In this section, we present the a priori estimates of Theorem 1.3. We first note that the embedding $H^3(\mathbb{T}^d) \hookrightarrow W^{1,p}(\mathbb{T}^d)$ (for $p \ge 1$) and Theorem 1.1 imply the global existence of unique strong solutions together with the bounds (1.14)–(1.15) on the interval $[0,T]$ for any $T > 0$. We only need to show the propagation of H^3-regularity.

To prove the estimates in (1.16), we first establish the following lemma.

Lemma 4.1. *Let $c_1(0), c_2(0) \in H^3(\mathbb{T}^d)$ be nonnegative functions satisfying* (1.7)*. Suppose (ρ, σ, Φ, u) solves* (1.8)–(1.12) *with initial data* (1.13) *on the interval $[0,T]$. Then for any $t \in [0,T]$, we have*

$$\|\Delta\rho(t)\|_{L^2} + \|\Delta\sigma(t)\|_{L^2} + \int_0^t \|\nabla\Delta\rho(\tau)\|_{L^2}^2 + \|\nabla\Delta\sigma(\tau)\|_{L^2}^2 \le C, \tag{4.1}$$

for some constants $C > 0$.

Proof. We multiply (1.8) and (1.9) by $\Delta^2\rho$ and $\Delta^2\sigma$, respectively, and integrate over $\mathbb{T}^d$. Integrating by parts and (1.10) give

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Delta\rho\|_{L^2}^2 + \|\Delta\sigma\|_{L^2}^2\right) + D\left(\|\nabla\Delta\rho\|_{L^2}^2 + \|\nabla\Delta\sigma\|_{L^2}^2\right) + \frac{D}{\varepsilon}\int \sigma|\Delta\rho|^2 \\
&= \int \nabla\rho\cdot(\nabla u\nabla\Delta\rho) + \int \nabla\sigma\cdot(\nabla u\nabla\Delta\sigma) + \int u\cdot(\nabla\nabla\rho\nabla\Delta\rho) + \int u\cdot(\nabla\nabla\sigma\nabla\Delta\sigma) \\
&\quad + D\int \nabla\Delta\sigma\cdot\nabla\Phi\Delta\rho + 2D\int \nabla\nabla\sigma : \nabla\nabla\Phi\Delta\rho - \frac{3D}{\varepsilon}\int \nabla\sigma\cdot\nabla\rho\Delta\rho \\
&\quad + D\int \nabla\Delta\rho\cdot\nabla\Phi\Delta\sigma + 2D\int \nabla\nabla\rho : \nabla\nabla\Phi\Delta\sigma - \frac{3D}{\varepsilon}\int |\nabla\rho|^2\Delta\sigma - \frac{3D}{\varepsilon}\int \rho\Delta\sigma\Delta\rho \\
&= I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4} + I_{1,5},
\end{aligned} \tag{4.2}$$

where

$$
\begin{aligned}
I_{1,1} &= \int \nabla\rho\cdot(\nabla u\nabla\Delta\rho) + \int \nabla\sigma\cdot(\nabla u\nabla\Delta\sigma),\\
I_{1,2} &= \int u\cdot(\nabla\nabla\rho\nabla\Delta\rho) + \int u\cdot(\nabla\nabla\sigma\nabla\Delta\sigma),\\
I_{1,3} &= -\frac{3D}{\varepsilon}\int \nabla\sigma\cdot\nabla\rho\Delta\rho - \frac{3D}{\varepsilon}\int|\nabla\rho|^2\Delta\sigma,\\
I_{1,4} &= D\int \nabla\Delta\sigma\cdot\nabla\Phi\Delta\rho + D\int\nabla\Delta\rho\cdot\nabla\Phi\Delta\sigma,\\
I_{1,5} &= 2D\int\nabla\nabla\sigma:\nabla\nabla\Phi\Delta\rho + 2D\int\nabla\nabla\rho:\nabla\nabla\Phi\Delta\sigma - \frac{3D}{\varepsilon}\int\rho\Delta\sigma\Delta\rho.
\end{aligned}
$$

For the term $I_{1,1}$, we apply Hölder's inequality, the bound (3.3), the Gagliardo-Nirenberg inequality

$$
\|\nabla f\|_{L^4} \le C\|\nabla\Delta f\|_{L^2}^{\frac{d}{8}}\|\nabla f\|_{L^2}^{1-\frac{d}{8}} + C\|\nabla f\|_{L^2},
$$

and Young's inequality to obtain

$$
\begin{aligned}
I_{1,1} &\le \|\nabla u\|_{L^4}\left(\|\nabla\rho\|_{L^4}\|\nabla\Delta\rho\|_{L^2} + \|\nabla\sigma\|_{L^4}\|\nabla\Delta\sigma\|_{L^2}\right)\\
&\le C\|\nabla\Phi\|_{L^\infty}\|\nabla\rho\|_{L^4}\left(\|\nabla\rho\|_{L^4}\|\nabla\Delta\rho\|_{L^2} + \|\nabla\sigma\|_{L^4}\|\nabla\Delta\sigma\|_{L^2}\right)\\
&\le C\|\nabla\Phi\|_{L^\infty}\left(\|\nabla\rho\|_{L^2}^{2-\frac{d}{4}} + \|\nabla\sigma\|_{L^2}^{2-\frac{d}{4}}\right)\left(\|\nabla\Delta\rho\|_{L^2}^{1+\frac{d}{4}} + \|\nabla\Delta\sigma\|_{L^2}^{1+\frac{d}{4}}\right)\\
&\quad + C\|\nabla\Phi\|_{L^\infty}\left(\|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2\right)\left(\|\nabla\Delta\rho\|_{L^2} + \|\nabla\Delta\sigma\|_{L^2}\right)\\
&\le \frac{D}{10}\left(\|\nabla\Delta\rho\|_{L^2}^2 + \|\nabla\Delta\sigma\|_{L^2}^2\right) + C\|\nabla\Phi\|_{L^\infty}^{\frac{8}{4-d}}\left(\|\nabla\rho\|_{L^2}^{\frac{16-2d}{4-d}} + \|\nabla\sigma\|_{L^2}^{\frac{16-2d}{4-d}}\right)\\
&\quad + C\|\nabla\Phi\|_{L^\infty}^2\left(\|\nabla\rho\|_{L^2}^4 + \|\nabla\sigma\|_{L^2}^4\right).
\end{aligned}
\tag{4.3}
$$

To estimate the term $I_{1,2}$, we use Hölder's inequality, Young's inequality, Ladyzhenskaya's inequality, and the bound (3.2)

$$
\begin{aligned}
I_{1,2} &\le \|u\|_{L^4}\left(\|\nabla\nabla\rho\|_{L^4}\|\nabla\Delta\rho\|_{L^2} + \|\nabla\nabla\sigma\|_{L^4}\|\nabla\Delta\sigma\|_{L^2}\right)\\
&\le \frac{5}{2D}\|u\|_{L^4}^2\left(\|\nabla\nabla\rho\|_{L^4}^2 + \|\nabla\nabla\sigma\|_{L^4}^2\right) + \frac{D}{10}\left(\|\nabla\Delta\rho\|_{L^2}^2 + \|\nabla\Delta\sigma\|_{L^2}^2\right)\\
&\le C\|u\|_{L^4}^2\left(\|\nabla\nabla\rho\|_{L^2}^{2-\frac{d}{2}} + \|\nabla\nabla\sigma\|_{L^2}^{2-\frac{d}{2}}\right)\left(\|\nabla\Delta\rho\|_{L^2}^{\frac{d}{2}} + \|\nabla\Delta\sigma\|_{L^2}^{\frac{d}{2}}\right) + C\|u\|_{L^4}^2\left(\|\nabla\nabla\rho\|_{L^2}^2 + \|\nabla\nabla\sigma\|_{L^2}^2\right)\\
&\quad + \frac{D}{10}\left(\|\nabla\Delta\rho\|_{L^2}^2 + \|\nabla\Delta\sigma\|_{L^2}^2\right)\\
&\le C\left(\|\nabla\Phi\|_{L^\infty}^2\|\rho\|_{L^4}^2 + \|\nabla\Phi\|_{L^\infty}^{\frac{8}{4-d}}\|\rho\|_{L^4}^{\frac{8}{4-d}}\right)\left(\|\nabla\nabla\rho\|_{L^2}^2 + \|\nabla\nabla\sigma\|_{L^2}^2\right)\\
&\quad + \frac{D}{5}\left(\|\nabla\Delta\rho\|_{L^2}^2 + \|\nabla\Delta\sigma\|_{L^2}^2\right).
\end{aligned}
\tag{4.4}
$$

To estimate the term $I_{1,3}$, we use Hölder's inequality, Ladyzhenskaya's inequality, the Gagliardo-Nirenberg interpolation inequality

$$
\|\Delta f\|_{L^2} \le C\|\nabla\Delta f\|_{L^2}^{\frac{1}{2}}\|\nabla f\|_{L^2}^{\frac{1}{2}} + C\|\nabla f\|_{L^2},
$$

and Young's inequality

$$
\begin{aligned}
I_{1,3} &\le \frac{3D}{\varepsilon}\|\nabla\rho\|_{L^4}\|\nabla\sigma\|_{L^4}\|\Delta\rho\|_{L^2} + \frac{3D}{\varepsilon}\|\nabla\rho\|_{L^4}^2\|\Delta\sigma\|_{L^2} \\
&\le C\left(\|\nabla\rho\|_{L^2} + \|\nabla\sigma\|_{L^2}\right)\left(\|\Delta\rho\|_{L^2}^{\frac{d}{2}} + \|\Delta\sigma\|_{L^2}^{\frac{d}{2}}\right)\left(\|\nabla\Delta\rho\|_{L^2}^{2-\frac{d}{2}} + \|\nabla\Delta\sigma\|_{L^2}^{2-\frac{d}{2}}\right) \\
&\quad + C\|\nabla\rho\|_{L^2}^2\|\nabla\sigma\|_{L^2} \\
&\le \frac{D}{10}\left(\|\nabla\Delta\rho\|_{L^2}^2 + \|\nabla\Delta\sigma\|_{L^2}^2\right) + C\left(\|\nabla\rho\|_{L^2}^{\frac{4}{d}} + \|\nabla\sigma\|_{L^2}^{\frac{4}{d}}\right)\left(\|\Delta\rho\|_{L^2}^2 + \|\Delta\sigma\|_{L^2}^2\right) \\
&\quad + C\|\nabla\rho\|_{L^2}^2\|\nabla\sigma\|_{L^2}.
\end{aligned}
\tag{4.5}
$$

The estimates for the other terms in (4.2) are similar. By Hölder's inequality, Young's inequality, the elliptic estimates, and Ladyzhenskaya's inequality, we obtain

$$
\begin{aligned}
I_{1,4} &\le D\|\nabla\Phi\|_{L^\infty}\left(\|\nabla\Delta\sigma\|_{L^2}\|\Delta\rho\|_{L^2} + \|\nabla\Delta\rho\|_{L^2}\|\Delta\sigma\|_{L^2}\right) \\
&\le \frac{5}{4}\|\nabla\Phi\|_{L^\infty}^2\left(\|\Delta\rho\|_{L^2}^2 + \|\Delta\sigma\|_{L^2}^2\right) + \frac{D}{5}\left(\|\nabla\Delta\rho\|_{L^2}^2 + \|\nabla\Delta\sigma\|_{L^2}^2\right),
\end{aligned}
\tag{4.6}
$$

and

$$
\begin{aligned}
I_{1,5} &\le C\|\nabla\nabla\Phi\|_{L^2}\|\Delta\rho\|_{L^4}\|\Delta\sigma\|_{L^4} + C\|\rho\|_{L^2}\|\Delta\rho\|_{L^4}\|\Delta\sigma\|_{L^4} \\
&\le C\|\rho\|_{L^2}\left(\|\Delta\rho\|_{L^2}^{2-\frac{d}{2}} + \|\Delta\sigma\|_{L^2}^{2-\frac{d}{2}}\right)\left(\|\nabla\Delta\rho\|_{L^2}^{\frac{d}{2}} + \|\nabla\Delta\sigma\|_{L^2}^{\frac{d}{2}}\right) \\
&\le C\|\rho\|_{L^2}^{\frac{4}{4-d}}\left(\|\Delta\rho\|_{L^2}^2 + \|\Delta\sigma\|_{L^2}^2\right) + \frac{D}{5}\left(\|\nabla\Delta\rho\|_{L^2}^2 + \|\nabla\Delta\sigma\|_{L^2}^2\right).
\end{aligned}
\tag{4.7}
$$

Using the estimates (4.3)–(4.7) in (4.2), we conclude

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\Delta\rho\|_{L^2}^2 + \|\Delta\sigma\|_{L^2}^2\right) + \frac{D}{10}\left(\|\nabla\Delta\rho\|_{L^2}^2 + \|\nabla\Delta\sigma\|_{L^2}^2\right) + \frac{D}{\varepsilon}\int\sigma|\Delta\rho|^2 \\
&\le C\left(\|\nabla\Phi\|_{L^\infty}^2\|\rho\|_{L^4}^2 + \|\nabla\Phi\|_{L^\infty}^{\frac{8}{4-d}}\|\rho\|_{L^4}^{\frac{8}{4-d}} + \|\rho\|_{L^2}^{\frac{4}{4-d}} + \|\nabla\Phi\|_{L^\infty}^2 + \|\nabla\rho\|_{L^2}^{\frac{4}{d}} + \|\nabla\sigma\|_{L^2}^{\frac{4}{d}}\right)\left(\|\Delta\rho\|_{L^2}^2 + \|\Delta\sigma\|_{L^2}^2\right) \\
&\quad + C\|\nabla\rho\|_{L^2}^2\|\nabla\sigma\|_{L^2} + C\|\nabla\Phi\|_{L^\infty}^2\left(\|\nabla\rho\|_{L^2}^4 + \|\nabla\sigma\|_{L^2}^4\right) + C\|\nabla\Phi\|_{L^\infty}^{\frac{8}{4-d}}\left(\|\nabla\rho\|_{L^2}^{\frac{16-2d}{4-d}} + \|\nabla\sigma\|_{L^2}^{\frac{16-2d}{4-d}}\right).
\end{aligned}
\tag{4.8}
$$

For simplicity, we denote

$$
\begin{aligned}
Z &= \|\Delta\rho\|_{L^2}^2 + \|\Delta\sigma\|_{L^2}^2, \\
W_1 &= \|\nabla\Phi\|_{L^\infty}^2\|\rho\|_{L^4}^2 + \|\nabla\Phi\|_{L^\infty}^{\frac{8}{4-d}}\|\rho\|_{L^4}^{\frac{8}{4-d}} + \|\rho\|_{L^2}^{\frac{4}{4-d}} + \|\nabla\Phi\|_{L^\infty}^2 + \|\nabla\rho\|_{L^2}^{\frac{4}{d}} + \|\nabla\sigma\|_{L^2}^{\frac{4}{d}}, \\
W_2 &= \|\nabla\rho\|_{L^2}^2\|\nabla\sigma\|_{L^2} + \|\nabla\Phi\|_{L^\infty}^2\left(\|\nabla\rho\|_{L^2}^4 + \|\nabla\sigma\|_{L^2}^4\right) + \|\nabla\Phi\|_{L^\infty}^{\frac{8}{4-d}}\left(\|\nabla\rho\|_{L^2}^{\frac{16-2d}{4-d}} + \|\nabla\sigma\|_{L^2}^{\frac{16-2d}{4-d}}\right).
\end{aligned}
$$

From (1.14)(iii), it follows that

$$
\int_0^t Z(\tau)\,\mathrm{d}\tau \le C. \tag{4.9}
$$

By (1.14)(i)–(iii), we have that

$$
\sup_{\tau\in[0,t]} W_1(\tau) \le C, \tag{4.10}
$$

and that

$$
\begin{aligned}
\int_0^t W_2(\tau)\,\mathrm{d}\tau \le& \sup_{\tau\in[0,t]} \|\nabla\sigma(\tau)\|_{L^2} \cdot \int_0^t \|\nabla\rho(\tau)\|_{L^2}^2\,\mathrm{d}\tau \\
&+ \sup_{\tau\in[0,t]} \left(\|\nabla\rho(\tau)\|_{L^2}^4 + \|\nabla\sigma(\tau)\|_{L^2}^4\right) \cdot \int_0^t \|\nabla\Phi(\tau)\|_{L^\infty}^2\,\mathrm{d}\tau \\
&+ \sup_{\tau\in[0,t]} \left(\|\nabla\rho(\tau)\|_{L^2}^{\frac{16-2d}{4-d}} + \|\nabla\sigma(\tau)\|_{L^2}^{\frac{16-2d}{4-d}}\right) \cdot \int_0^t \|\nabla\Phi(\tau)\|_{L^\infty}^{\frac{8}{4-d}}\,\mathrm{d}\tau \\
\le& C.
\end{aligned}
\tag{4.11}
$$

We integrate (4.8) in time and then use the fact that $\sigma \ge 0$ and the bounds (4.9)–(4.11) to conclude (4.1). □

Next, we propagate the H^3 regularity of the solutions.

Lemma 4.2. *Let $c_1(0), c_2(0) \in H^3(\mathbb{T}^d)$ be nonnegative functions satisfying* (1.7)*. Suppose (ρ,σ,Φ,u) solves* (1.8)–(1.12) *with initial data* (1.13) *on the interval $[0,T]$. Then for any $t\in[0,T]$, we have*

$$
\|\nabla\Delta\rho(t)\|_{L^2} + \|\nabla\Delta\sigma(t)\|_{L^2} + \int_0^t \|\Delta^2\rho(\tau)\|_{L^2}^2 + \|\Delta^2\sigma(\tau)\|_{L^2}^2 \le C, \tag{4.12}
$$

$$
\|u\|_{H^3} \le C, \tag{4.13}
$$

for some constants $C>0$.

Proof. We multiply (1.8) and (1.9) by $-\Delta^3\rho$ and $-\Delta^3\sigma$ respectively, integrate over $\mathbb{T}^d$. We integrate by parts and use (1.10) to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\Delta\rho\|_{L^2}^2 + \|\nabla\Delta\sigma\|_{L^2}^2\right) + D\left(\|\Delta^2\rho\|_{L^2}^2 + \|\Delta^2\sigma\|_{L^2}^2\right) + \frac{D}{\varepsilon}\int \sigma|\nabla\Delta\rho|^2 \\
&\quad = I_{2,1} + I_{2,2} + I_{2,3} + I_{2,4} + I_{2,5} + I_{2,6},
\end{aligned}
\tag{4.14}
$$

where

$$
\begin{aligned}
I_{2,1} &= \int \Delta u\cdot(\nabla\rho\Delta^2\rho + \nabla\sigma\Delta^2\sigma) + \int u\cdot(\nabla\Delta\rho\Delta^2\rho + \nabla\Delta\sigma\Delta^2\sigma) + 2\int \nabla u : (\nabla\nabla\rho\Delta^2\rho + \nabla\nabla\sigma\Delta^2\sigma),\\
I_{2,2} &= D\int \nabla\Delta\rho\cdot\nabla\nabla\Delta\sigma\nabla\Phi + D\int \nabla\Delta\sigma\cdot\nabla\nabla\Delta\rho\nabla\Phi,\\
I_{2,3} &= -\frac{2D}{\varepsilon}\int \nabla\Delta\rho\cdot(\nabla\nabla\rho\nabla\sigma) - \frac{5D}{\varepsilon}\int \nabla\Delta\sigma\cdot(\nabla\nabla\rho\nabla\rho) - \frac{3D}{\varepsilon}\int \nabla\Delta\rho\cdot(\nabla\nabla\sigma\nabla\rho)\\
&\quad - \frac{D}{\varepsilon}\int \nabla\Delta\rho\cdot\nabla\rho\Delta\sigma - \frac{3D}{\varepsilon}\int \nabla\Delta\rho\cdot\nabla\sigma\Delta\rho - \frac{4D}{\varepsilon}\int \nabla\Delta\sigma\cdot\nabla\rho\Delta\rho,\\
I_{2,4} &= 2D\int \nabla\rho\cdot(\nabla\nabla\nabla\sigma : \nabla\nabla\Phi) + 2D\int \nabla\sigma\cdot(\nabla\nabla\nabla\rho : \nabla\nabla\Phi),\\
I_{2,5} &= -\frac{3D}{\varepsilon}\int \nabla\Delta\rho\cdot\nabla\Delta\sigma\rho,\\
I_{2,6} &= 2D\int \nabla\rho\cdot(\nabla\nabla\nabla\Phi : \nabla\nabla\sigma) + 2D\int \nabla\sigma\cdot(\nabla\nabla\nabla\Phi : \nabla\nabla\rho).
\end{aligned}
$$

First, from (1.11) and the fact that the Leray projector commutes with the Laplacian, we find that

$$
\|\Delta u\|_{L^2} \le C\|\Delta(\rho\nabla\Phi)\|_{L^2} \le C\|\Delta\rho\|_{L^2}\|\nabla\Phi\|_{L^\infty} + C\|\nabla\rho\|_{L^6}\|\rho\|_{L^3}, \tag{4.15}
$$

where we also used Hölder's inequality and the equation (1.10) in the second inequality.

For the terms involving velocity u, we use Hölder's inequalities for L^2-L^2-L^∞ or L^2-L^3-L^6, the estimate (3.3), the Gagliardo-Nirenberg interpolation inequality

$$\|\nabla\nabla f\|_{L^3} \le C\|\Delta^2 f\|_{L^2}^{\frac{d}{12}}\|\Delta f\|_{L^2}^{1-\frac{d}{12}} + C\|\Delta f\|_{L^2},$$

the embeddings $H^2(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ and $H^1(\mathbb{T}^d) \hookrightarrow L^6(\mathbb{T}^d)$, $d = 2, 3$, the bound (4.15), and Young's inequality,

$$\begin{aligned}
I_{2,1} &\le \|\Delta u\|_{L^2}\left(\|\nabla\rho\|_{L^\infty}\|\Delta^2\rho\|_{L^2} + \|\nabla\sigma\|_{L^\infty}\|\Delta^2\sigma\|_{L^2}\right) + \|u\|_{L^\infty}\left(\|\Delta^2\rho\|_{L^2}\|\nabla\Delta\rho\|_{L^2} + \|\Delta^2\sigma\|_{L^2}\|\nabla\Delta\sigma\|_{L^2}\right)\\
&\quad + 2\|\nabla u\|_{L^6}\left(\|\nabla\nabla\rho\|_{L^3}\|\Delta^2\rho\|_{L^2} + \|\nabla\nabla\sigma\|_{L^3}\|\Delta^2\sigma\|_{L^2}\right)\\
&\le \|\Delta u\|_{L^2}\left(\|\nabla\rho\|_{L^\infty}\|\Delta^2\rho\|_{L^2} + \|\nabla\sigma\|_{L^\infty}\|\Delta^2\sigma\|_{L^2}\right) + \|u\|_{L^\infty}\left(\|\Delta^2\rho\|_{L^2}\|\nabla\Delta\rho\|_{L^2} + \|\Delta^2\sigma\|_{L^2}\|\nabla\Delta\sigma\|_{L^2}\right)\\
&\quad + C\|\nabla\rho\|_{L^6}\|\nabla\Phi\|_{L^\infty}\left(\|\Delta\rho\|_{L^2}^{1-\frac{d}{12}} + \|\Delta\sigma\|_{L^2}^{1-\frac{d}{12}}\right)\left(\|\Delta^2\rho\|_{L^2}^{1+\frac{d}{12}} + \|\Delta^2\sigma\|_{L^2}^{1+\frac{d}{12}}\right)\\
&\quad + C\|\nabla\rho\|_{L^6}\|\nabla\Phi\|_{L^\infty}\left(\|\Delta\rho\|_{L^2} + \|\Delta\sigma\|_{L^2}\right)\left(\|\Delta^2\rho\|_{L^2} + \|\Delta^2\sigma\|_{L^2}\right)\\
&\le \frac{D}{5}\left(\|\Delta^2\rho\|_{L^2}^2 + \|\Delta^2\sigma\|_{L^2}^2\right) + C\|\Delta u\|_{L^2}^2\left(\|\nabla\rho\|_{L^\infty}^2 + \|\nabla\sigma\|_{L^\infty}^2\right) + C\|u\|_{H^2}^2\left(\|\nabla\Delta\rho\|_{L^2}^2 + \|\nabla\Delta\sigma\|_{L^2}^2\right)\\
&\quad + C\left(\|\nabla\rho\|_{L^6}^{\frac{24}{12-d}}\|\nabla\Phi\|_{L^\infty}^{\frac{24}{12-d}} + \|\nabla\rho\|_{L^6}^2\|\nabla\Phi\|_{L^\infty}^2\right)\left(\|\Delta\rho\|_{L^2}^2 + \|\Delta\sigma\|_{L^2}^2\right)\\
&\le \frac{D}{5}\left(\|\Delta^2\rho\|_{L^2}^2 + \|\Delta^2\sigma\|_{L^2}^2\right) + C\left(\|\Delta\rho\|_{L^2}^2\|\nabla\Phi\|_{L^\infty}^2 + \|\rho\|_{H^2}^2\|\rho\|_{L^3}^2\right)\left(\|\nabla\Delta\rho\|_{L^2}^2 + \|\nabla\Delta\sigma\|_{L^2}^2\right)\\
&\quad + C\left(\|\rho\|_{H^2}^{\frac{24}{12-d}}\|\nabla\Phi\|_{L^\infty}^{\frac{24}{12-d}} + \|\rho\|_{H^2}^2\|\nabla\Phi\|_{L^\infty}^2\right)\left(\|\Delta\rho\|_{L^2}^2 + \|\Delta\sigma\|_{L^2}^2\right).
\end{aligned} \tag{4.16}$$

For the term $I_{2,2}$, we use Hölder's inequality and Young's inequality to get

$$\begin{aligned}
I_{2,2} &\le D\|\nabla\Phi\|_{L^\infty}\left(\|\nabla\Delta\rho\|_{L^2}\|\nabla\nabla\Delta\sigma\|_{L^2} + \|\nabla\Delta\sigma\|_{L^2}\|\nabla\nabla\Delta\rho\|_{L^2}\right)\\
&\le C\|\nabla\Phi\|_{L^\infty}^2\left(\|\nabla\Delta\rho\|_{L^2}^2 + \|\nabla\Delta\sigma\|_{L^2}^2\right) + \frac{D}{5}\left(\|\Delta^2\rho\|_{L^2}^2 + \|\Delta^2\sigma\|_{L^2}^2\right).
\end{aligned} \tag{4.17}$$

By Hölder's inequality for L^2-L^4-L^4 and the Gagliardo-Nirenberg inequalities

$$\begin{aligned}
\|\nabla\Delta f\|_{L^2} &\le C\|\Delta^2 f\|_{L^2}^{\frac{1}{2}}\|\Delta f\|_{L^2}^{\frac{1}{2}} + C\|\Delta f\|_{L^2},\\
\|\Delta f\|_{L^4} &\le C\|\Delta^2 f\|_{L^2}^{\frac{d+4}{12}}\|\nabla f\|_{L^2}^{\frac{8-d}{12}} + C\|\nabla f\|_{L^2},\\
\|\nabla f\|_{L^4} &\le C\|\Delta^2 f\|_{L^2}^{\frac{d}{12}}\|\nabla f\|_{L^2}^{1-\frac{d}{12}} + C\|\nabla f\|_{L^2},
\end{aligned}$$

we obtain

$$
\begin{aligned}
I_{2,3} &\le C\|\nabla\Delta\rho\|_{L^2}\|\Delta\rho\|_{L^4}\|\nabla\sigma\|_{L^4} + C\|\nabla\Delta\sigma\|_{L^2}\|\Delta\rho\|_{L^4}\|\nabla\rho\|_{L^4} + C\|\nabla\Delta\rho\|_{L^2}\|\Delta\sigma\|_{L^4}\|\nabla\rho\|_{L^4} \\
&\le C\left(\|\Delta^2\rho\|_{L^2}^{\frac12}\|\Delta\rho\|_{L^2}^{\frac12} + \|\Delta\rho\|_{L^2}\right)\left(\|\Delta^2\rho\|_{L^2}^{\frac{d+4}{12}}\|\nabla\rho\|_{L^2}^{\frac{8-d}{12}} + \|\nabla\rho\|_{L^2}\right)\left(\|\Delta^2\sigma\|_{L^2}^{\frac{d}{12}}\|\nabla\sigma\|_{L^2}^{1-\frac{d}{12}} + \|\nabla\sigma\|_{L^2}\right) \\
&\quad + C\left(\|\Delta^2\sigma\|_{L^2}^{\frac12}\|\Delta\sigma\|_{L^2}^{\frac12} + \|\Delta\sigma\|_{L^2}\right)\left(\|\Delta^2\rho\|_{L^2}^{\frac{d+4}{12}}\|\nabla\rho\|_{L^2}^{\frac{8-d}{12}} + \|\nabla\rho\|_{L^2}\right)\left(\|\Delta^2\rho\|_{L^2}^{\frac{d}{12}}\|\nabla\rho\|_{L^2}^{1-\frac{d}{12}} + \|\nabla\rho\|_{L^2}\right) \\
&\quad + C\left(\|\Delta^2\rho\|_{L^2}^{\frac12}\|\Delta\rho\|_{L^2}^{\frac12} + \|\Delta\rho\|_{L^2}\right)\left(\|\Delta^2\sigma\|_{L^2}^{\frac{d+4}{12}}\|\nabla\sigma\|_{L^2}^{\frac{8-d}{12}} + \|\nabla\sigma\|_{L^2}\right)\left(\|\Delta^2\rho\|_{L^2}^{\frac{d}{12}}\|\nabla\rho\|_{L^2}^{1-\frac{d}{12}} + \|\nabla\rho\|_{L^2}\right) \\
&\le C\left(\|\Delta^2\rho\|_{L^2}^{\frac{5+d}{6}} + \|\Delta^2\sigma\|_{L^2}^{\frac{5+d}{6}}\right)\left(\|\Delta\rho\|_{L^2}^{\frac12} + \|\Delta\sigma\|_{L^2}^{\frac12}\right)\left(\|\nabla\rho\|_{L^2}^{\frac{10-d}{6}} + \|\nabla\sigma\|_{L^2}^{\frac{10-d}{6}}\right) \\
&\quad + C\left(\|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2\right)\left(\|\Delta\rho\|_{L^2} + \|\Delta\sigma\|_{L^2}\right),
\end{aligned}
$$

which, by Young's inequality, implies

$$
\begin{aligned}
I_{2,3} \le \frac{D}{5}\left(\|\Delta^2\rho\|_{L^2}^2 + \|\Delta^2\sigma\|_{L^2}^2\right) + C\left(\|\nabla\rho\|_{L^2}^{\frac{20-2d}{7-d}} + \|\nabla\sigma\|_{L^2}^{\frac{20-2d}{7-d}}\right)\left(\|\Delta\rho\|_{L^2}^{\frac{6}{7-d}} + \|\Delta\sigma\|_{L^2}^{\frac{6}{7-d}}\right) \\
+ C\left(\|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2\right)\left(\|\Delta\rho\|_{L^2} + \|\Delta\sigma\|_{L^2}\right).
\end{aligned} \tag{4.18}
$$

The estimates for $I_{2,4}$ and $I_{2,5}$ follow from (1.10), Hölder's inequality, the Sobolev embeddings $H^2(\mathbb{T}^d) \hookrightarrow L^\infty(\mathbb{T}^d)$ and $H^1(\mathbb{T}^d) \hookrightarrow L^6(\mathbb{T}^d)$, and Young's inequality,

$$
\begin{aligned}
I_{2,4} &\le C\|\rho\|_{L^3}\left(\|\nabla\rho\|_{L^6}\|\nabla\Delta\sigma\|_{L^2} + \|\nabla\sigma\|_{L^6}\|\nabla\Delta\rho\|_{L^2}\right) \\
&\le C\|\rho\|_{L^3}\left(\|\rho\|_{H^2} + \|\sigma - \bar\sigma\|_{H^2}\right)\left(\|\nabla\Delta\rho\|_{L^2} + \|\nabla\Delta\sigma\|_{L^2}\right) \\
&\le C\|\rho\|_{L^3}^2\left(\|\rho\|_{H^2}^2 + \|\sigma - \bar\sigma\|_{H^2}^2\right) + C\left(\|\nabla\Delta\rho\|_{L^2}^2 + \|\nabla\Delta\sigma\|_{L^2}^2\right),
\end{aligned} \tag{4.19}
$$

and

$$
I_{2,5} \le C\|\rho\|_{L^\infty}\|\nabla\Delta\rho\|_{L^2}\|\nabla\Delta\sigma\|_{L^2} \le C\|\rho\|_{H^2}\|\nabla\Delta\rho\|_{L^2}\|\nabla\Delta\sigma\|_{L^2}. \tag{4.20}
$$

Finally, we use (1.10), Hölder's inequality, and Ladyzhenskaya's inequality to obtain

$$
\begin{aligned}
I_{2,6} &\le C\|\nabla\rho\|_{L^4}^2\|\Delta\sigma\|_{L^2} + C\|\nabla\sigma\|_{L^4}\|\nabla\rho\|_{L^4}\|\Delta\rho\|_{L^2} \\
&\le C\left(\|\nabla\rho\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2\right)\left(\|\Delta\rho\|_{L^2} + \|\Delta\sigma\|_{L^2}\right) \\
&\quad + C\left(\|\nabla\rho\|_{L^2}^{2-\frac{d}{2}} + \|\nabla\sigma\|_{L^2}^{2-\frac{d}{2}}\right)\left(\|\Delta\rho\|_{L^2}^{1+\frac{d}{2}} + \|\Delta\sigma\|_{L^2}^{1+\frac{d}{2}}\right).
\end{aligned} \tag{4.21}
$$

Gathering the estimates (4.16)–(4.21) into (4.14), we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\Delta\rho\|_{L^2}^2+\|\nabla\Delta\sigma\|_{L^2}^2\right)+\frac{D}{5}\left(\|\Delta^2\rho\|_{L^2}^2+\|\Delta^2\sigma\|_{L^2}^2\right)+\frac{D}{\varepsilon}\int\sigma|\nabla\Delta\rho|^2\\
&\quad\le C\left(1+\|\rho\|_{H^2}+\|\nabla\Phi\|_{L^\infty}^2+\|\Delta\rho\|_{L^2}^2\|\nabla\Phi\|_{L^\infty}^2+\|\rho\|_{H^2}^2\|\rho\|_{L^3}^2\right)\left(\|\nabla\Delta\rho\|_{L^2}^2+\|\nabla\Delta\sigma\|_{L^2}^2\right)\\
&\qquad+C\|\rho\|_{L^3}^2\left(\|\rho\|_{H^2}^2+\|\sigma-\bar{\sigma}\|_{H^2}^2\right)+C\left(\|\nabla\rho\|_{L^2}^{2-\frac{d}{2}}+\|\nabla\sigma\|_{L^2}^{2-\frac{d}{2}}\right)\left(\|\Delta\rho\|_{L^2}^{1+\frac{d}{2}}+\|\Delta\sigma\|_{L^2}^{1+\frac{d}{2}}\right)\\
&\qquad+C\left(\|\nabla\rho\|_{L^2}^{\frac{20-2d}{7-d}}+\|\nabla\sigma\|_{L^2}^{\frac{20-2d}{7-d}}\right)\left(\|\Delta\rho\|_{L^2}^{\frac{6}{7-d}}+\|\Delta\sigma\|_{L^2}^{\frac{6}{7-d}}\right)\\
&\qquad+C\left(\|\rho\|_{H^2}^{\frac{24}{12-d}}\|\nabla\Phi\|_{L^\infty}^{\frac{24}{12-d}}+\|\rho\|_{H^2}^2\|\nabla\Phi\|_{L^\infty}^2\right)\left(\|\Delta\rho\|_{L^2}^2+\|\Delta\sigma\|_{L^2}^2\right)\\
&\qquad+C\left(\|\nabla\rho\|_{L^2}^2+\|\nabla\sigma\|_{L^2}^2\right)\left(\|\Delta\rho\|_{L^2}+\|\Delta\sigma\|_{L^2}\right).
\end{aligned}
\tag{4.22}
$$

We drop the dissipation terms, integrate in time, and use the bounds (1.14)(i)–(iii) and (4.1) to obtain

$$
\|\nabla\Delta\rho(t)\|_{L^2}^2+\|\nabla\Delta\sigma(t)\|_{L^2}^2\le C.
$$

Going back to (4.22), we conclude that

$$
\int_0^t\|\Delta^2\rho(\tau)\|_{L^2}+\|\Delta^2\sigma(\tau)\|_{L^2}\,\mathrm{d}\tau\le C.
$$

Finally, applying Leray's projection to (1.11), then using the Leibnitz rule, Hölder's inequality, and (4.12), we conclude (4.13). □

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