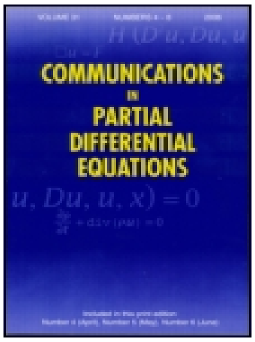

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The Harnack inequality for second-order parabolic equations with divergence-free drifts of low regularity

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ABSTRACT
We establish the Harnack inequality for advection-diffusion equations with divergence-free drifts by adapting the classical Moser technique to parabolic equations with drifts with regularity lower than the scale invariant spaces.

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1. Introduction

In this paper, we address the qualitative properties of solutions to the parabolic equation

$$u_t - \Delta u + b \cdot \nabla u = 0 \quad \text{in } \Omega, \tag{1.1}$$

where b is a given divergence free vector field of low regularity, and Ω is a space-time domain. The study of such equations with non-smooth drifts $b(x,t)$ is motivated by the need to understand the qualitative and quantitative properties of nonlinear partial differential equations, where the drift depends on the solution u and its first derivatives and for which we often do not have a priori bounds available except in some very low regularity spaces. Advection-diffusion equations of the form (1.1) often arise in applications with the additional divergence-free condition $\operatorname{div} b = 0$, in particular, in problems involving incompressible fluids. Several important recent papers have addressed regularity of the solutions of the linear advection-diffusion equations with very little smoothness assumptions on the divergence free drift [2, 3, 6, 11, 16, 17, 20] (cf. also [1, 4, 10, 12]). Here we study this problem for the parabolic equation (1.1) with a divergence-free "supercritical" drift b. Criticality here refers to the following property: the usual parabolic rescaling $x \to \lambda x$, $t \to \lambda^2 t$ leaves the equation invariant if the drift term in the equation satisfies $b \in L_t^q L_x^p$ with $2/q + n/p = 1$. Accordingly, we say that the drift is critical if this relation holds, is subcritical if $2/q + n/p < 1$ and is supercritical if $2/q + n/p > 1$.

The subcritical drifts were addressed in the classical paper [15] (cf. also [5, 14, 19]). Our main result is the Harnack-type inequality for parabolic advection-diffusion equations with a supercritical drift. We use the notation

$$Q_R^*(x_0, t_0) = \{(x,t) \in \mathbb{R}^{n+1} : |x - x_0| < R,\ t_0 < t < t_0 + R^2\} \tag{1.2}$$

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for the parabolic cylinder centered at the bottom and

$$Q_R(x_0,t_0) = \{(x,t) \in \mathbb{R}^{n+1}\colon |x-x_0| < R,\ t_0 - R^2 < t < t_0\} \tag{1.3}$$

for the parabolic cylinder centered at the top. For simplicity, we write $Q_R = Q_R(0,0)$.

Theorem 1.1. *Let u be a nonnegative Lipschitz solution to the parabolic equation*

$$u_t - \Delta u + b \cdot \nabla u = 0 \quad \text{in } \Omega, \tag{1.4}$$

that is,

$$\int_\Omega \partial_t u\varphi + \int_\Omega \partial_j u \partial_j \varphi + \int_\Omega b_j \partial_j u \varphi = 0 \tag{1.5}$$

for any Lipschitz function $\varphi \geq 0$ in Ω and $\varphi = 0$ in Ω^c. Assume that $b \in L^{\bar{q}}(\Omega) \cap L^\infty L^2(\Omega)$ with $n/2+1 < \bar{q} \leq n+2$ and div $b = 0$ *in the sense of distributions. Then for any $Q_{2R} \subset \Omega$,*

$$\sup_{Q_{R/2}(0,-3R^2)} u \leq \left(C + C(R^{1-(n+2)/\bar{q}}\|b\|_{L^{\bar{q}}})^{1/(2-(n+2)/\bar{q})}\right)^{C(n)/p_0} \inf_{Q_R} u, \tag{1.6}$$

where $p_0 = 1/(CM_R^C)$ and $M_R = 1 + (R^{1-n/2}\|b\|_{L^\infty L^2})^2 + R^{1-(n+2)/\bar{q}}\|b\|_{L^{\bar{q}}}$.

Here, and elsewhere in this paper, the symbol C denotes a large constant which depends on the parameters $\bar{q}$ and n, and on the domain $\Omega \subset \mathbb{R}^{n+1}$. Also, we denote the anisotropic Lebesgue spaces by $L^pL^q(\Omega) = L_t^pL_x^q(\Omega)$, and in the case when $p = q$ by $L^q(\Omega) = L_{x,t}^q(\Omega)$.

Remark 1.2. The statement of Theorem 1.1 remains valid for divergence-free drifts b in the more general anisotropic supercritical Lebesgue space $L^lL^{\bar{q}}(\Omega)$ with $1 \leq 2/l + n/\bar{q} < 2$ and satisfying the additional condition $b \in L^\infty L^2(\Omega)$. In Lemma 2.1, we use these anisotropic spaces for the drift, while in Lemma 3.1 we set $l = \bar{q}$ for convenience of the presentation. We explain modifications needed to treat the more general case after the proof of Lemma 3.3.

Remark 1.3. The requirement on u to be a Lipschitz generalized solution to (1.1) is sufficient in order to guarantee that the drift term

$$\int_\Omega b_j \partial_j u \varphi \tag{1.7}$$

is bounded when b belongs to a low regularity (supercritical) space. Note that the Lipschitz assumption can be relaxed to obtain the Harnack inequality (1.6) for weak solutions satisfying $u \in L^\infty L^2(\Omega)$ and $\nabla u \in L^2L^2(\Omega)$ provided some additional regularity assumptions are imposed on the drift (cf. [16], page 19).

The qualitative properties of solutions to the equation (1.1) have been extensively studied in the past. In particular, Harnack's inequality for the second order parabolic equation

$$u_t - \partial_i(a_{ij}(x,t)\partial_j u) = 0$$

in the self-adjoint form, with measurable strongly elliptic coefficients a_{ij} was obtained in the seminal work of Moser [14] for subcritical drifts and no lower order terms. In [13], Lieberman established the Harnack inequality in the case of non-zero lower-order coefficients, when the drift is in a subcritical Morrey space.

Recently, Nazarov and Ural'tseva proved in [16] that the assumptions on the divergence free drift b may be significantly relaxed to allow it to lie in the scale invariant (critical) Morrey spaces $M_{l,q}^{n/q+2/l-1}$ for all q and l satisfying $1 \leq n/q + 2/l < 2$. Seregin et al. (cf. [17]) established the Harnack inequality when b belongs to $L^\infty(BMO^{-1})$, which is also a critical (scale-invariant) condition. In our previous paper [9], we obtained a Harnack inequality for elliptic equations with supercritical divergence-free drifts. The purpose of the present paper is to address the more challenging parabolic case. Note that the approach from [9] does not apply here.

It is well-known that Harnack-type inequality implies Hölder regularity of solutions to (1.1) when the drift lies in a *critical* (scale-invariant) space. However, due to *supercritical* assumptions on the drift in Theorem 1.1, one cannot deduce from (1.6) the Hölder continuity of the solutions. Note that in general the solutions to (1.1) with *supercritical* drifts may not satisfy even weaker continuity properties. For instance, in [18] it was proven that in two dimension the solutions of (1.1) may become discontinuous in finite time provided the drift b is divergence free and $b \in L^\infty L^p(\mathbb{R}^2)$ with $p \in [1, 2)$. On the other hand, using the Harnack inequality (1.6), we were able to obtain in [8] the uniform continuity of solutions to (1.1) with a divergence-free drift that belongs to a slightly supercritical logarithmic Morrey space.

The paper is organized as follows. In Section 2, we establish the local boundedness of nonnegative Lipschitz subsolutions to (1.1) by using Moser's iteration. This result of independent interest was previously obtained in [16]. However, the bound (2.2) with an explicit dependence on the parameters is needed for establishing the validity of Theorem 1.1, and thus we provide our proof here for completeness. The rest of the paper, Section 3, is devoted to the lower bound of the infimum of Lipschitz supersolutions to (1.1), stated in Lemma 3.1. We proceed by deriving consecutive estimates on the nonnegative supersolution $w = \log_+(u/K)$, where the constant K is determined in the initial step (cf. Lemma 3.2) and depends on the values of the supersolution u to (1.1). Here we follow the approach of Lieberman [13]. We emphasize that this initial step requires an additional assumption on the drift $b \in L^\infty L^2(\Omega)$ which was not needed in the elliptic case (cf. [9]). In Lemma 3.3, we establish an estimate which allows us to bootstrap the initial bounds on w from Lemma 3.2 to higher L^σ-norms for any $\sigma \in [1, (n+2)/n)$. Using Lemma 3.3, we also obtain a bound on $\|\nabla w\|_{L^2}$ in Subsection 3.3, which is essential for estimating higher norms. Then, the aforementioned estimates on all the higher norms are deduced by using Moser's iteration technique (see Subsection 3.4). The lower bound on the infimum then follows from the auxiliary assertion in Lemma 3.5. Our main result is a consequence of Lemmas 2.1 and 3.1.

2. Local boundedness

In this section, we show that any nonnegative Lipschitz subsolution of (1.1) is locally bounded when the divergence free drift belongs to the anisotropic Lebesgue spaces $L^l L^{\bar{q}}(\Omega)$ for all l and $\bar{q}$ satisfying $1 \leq 2/l + n/\bar{q} < 2$.

Lemma 2.1. *Assume that u is a nonnegative Lipschitz subsolution to the equation*

$$u_t - \Delta u + b \cdot \nabla u = 0 \tag{2.1}$$

with $b \in L^l L^{\bar{q}}(\Omega)$ *for* $1 \le 2/l + n/\bar{q} < 2$ *and* $\operatorname{div} b \le 0$ *in the sense of distributions. Then for any* $Q_R \subset \Omega$, $p > 0$, *and* $0 < \theta < \tau < 1$

$$\sup_{Q_{\theta R}} u \le C\left(1 + \left(R^{1-2/l-n/\bar{q}}\|b\|_{L^l(L^{\bar{q}}(\Omega))}\right)^{1/(2-2/l-n/\bar{q})}\right)^{(n+2)/p} R^{-(n+2)/p}\|u\|_{L^p(Q_{\tau R})}, \quad (2.2)$$

where $C = C(n, p, l, \bar{q}, \theta, \tau)$ *is a positive constant.*

Proof of Lemma 2.1. Let u be a nonnegative Lipschitz subsolution of (2.1) in Ω, that is,

$$\int_\Omega \partial_t u\varphi + \int_\Omega \partial_j u \partial_j \varphi + \int_\Omega b_j \partial_j u\varphi \le 0 \quad (2.3)$$

for any Lipschitz function $\varphi \ge 0$ in Ω and $\varphi = 0$ in Ω^c.

Without loss of generality, we may assume that that $R = 1$. We use in (2.3) test functions of the form

$$\varphi = \left(\frac{\beta}{2} + 1\right) u^{\beta+1}\eta^{2\gamma}\chi_{\{t\le T\}}$$

with a Lipschitz cut-off function η in Q_τ, such that $0 \le \eta \le 1$, and the constants $\beta > 0$ and $\gamma > 0$ to be set later—below, we let $\beta \to +\infty$ with γ remaining fixed. This gives, for $T \in (-\tau^2, 0)$

$$\begin{aligned}&\left(\frac{\beta}{2}+1\right)\int_{Q_\tau}(\partial_t u)(u^{\beta+1})\eta^{2\gamma}\chi_{\{t\le T\}} + \left(\frac{\beta}{2}+1\right)\int_{Q_\tau}(\partial_j u)\left(\partial_j(u^{\beta+1})\right)\eta^{2\gamma}\chi_{\{t\le T\}}\\ &\quad+\left(\frac{\beta}{2}+1\right)\int_{Q_\tau}u^{\beta+1}\partial_j u\left(\partial_j(\eta^{2\gamma})\right)\chi_{\{t\le T\}} + \left(\frac{\beta}{2}+1\right)\\ &\quad\times\int_{Q_\tau}b_j u^{\beta+1}(\partial_j u)\eta^{2\gamma}\chi_{\{t\le T\}} \le 0. \end{aligned} \quad (2.4)$$

Set $w = u^{\beta/2+1}$, so that

$$\partial_j w = \left(\frac{\beta}{2}+1\right)u^{\beta/2}\partial_j u.$$

Using (2.4), we get, integrating the first term by parts in time:

$$\begin{aligned}&\frac{1}{2}\int_{B_\tau} w^2\eta^{2\gamma}\Big|_{t=T} + \frac{\beta+1}{\beta/2+1}\int_{Q_\tau}|\nabla w|^2\eta^{2\gamma}\chi_{\{t\le T\}}\\ &\quad\le -2\gamma\int_{Q_\tau}(\partial_j w)w\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\le T\}} - \int_{Q_\tau}b_j(\partial_j w)w\eta^{2\gamma}\chi_{\{t\le T\}}\\ &\quad+\gamma\int_{Q_\tau}w^2\eta^{2\gamma-1}(\partial_t\eta)\chi_{\{t\le T\}}. \end{aligned} \quad (2.5)$$

Here we have utilized the fact that $\eta(x, -\tau^2) = 0$. For the first term in the right side of (2.5) we have

$$-2\gamma\int_{Q_\tau}(\partial_j w)w\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\le T\}} = \gamma\int_{Q_\tau}w^2\left(\eta^{2\gamma-1}\Delta\eta + (2\gamma-1)\eta^{2\gamma-2}|\nabla\eta|^2\right)\chi_{\{t\le T\}}, \quad (2.6)$$

while for the second term

$$-\int_{Q_\tau} b_j(\partial_j w) w \eta^{2\gamma} \chi_{\{t\le T\}} = \frac{1}{2}\int_{Q_\tau} (\partial_j b_j) w^2 \eta^{2\gamma} \chi_{\{t\le T\}} + \gamma \int_{Q_\tau} b_j w^2 \eta^{2\gamma-1}(\partial_j \eta)\chi_{\{t\le T\}}$$
$$\le \gamma \int_{Q_\tau} b_j w^2 \eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\le T\}} \tag{2.7}$$

since $\operatorname{div} b \le 0$.

Next, let $\gamma_0 = 2/l + n/\bar{q}$. Then, by assumption, we have $\gamma_0 \in [1,2)$. We also choose $\gamma = 1/(2-\gamma_0)$, so that $\gamma\gamma_0 = 2\gamma - 1$. By Hölder's inequality we have the following estimate for the right side in (2.7):

$$\int_{Q_\tau} b_j w^2 \eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\le T\}} \le \int_{Q_\tau} |b_j| |w\eta^\gamma|^{\gamma_0} |w|^{2-\gamma_0} |\partial_j \eta| \chi_{\{t\le T\}}$$
$$\le \|b\|_{L_t^l L_x^{\bar{q}}} \|w\eta^\gamma \chi_{t\le T}\|_{L_t^s L_x^r}^{\gamma_0} \|w|\nabla\eta|^{1/(2-\gamma_0)}\chi_{t\le T}\|_{L^2_{t,x}}^{2-\gamma_0}. \tag{2.8}$$

Here s and r are determined by

$$\frac{1}{\bar{q}} + \frac{\gamma_0}{r} + \frac{2-\gamma_0}{2} = 1$$

and

$$\frac{1}{l} + \frac{\gamma_0}{s} + \frac{2-\gamma_0}{2} = 1.$$

It is easy to verify that $2/s + n/r = n/2$—this is how γ_0 was chosen. Now, Young's and the interpolation inequality

$$\|f\|_{L_t^s L_x^r} \le C\|f\|_{L_t^\infty L_x^2}^{1-\alpha} \|\nabla f\|_{L^2_{t,x}}^{\alpha} \tag{2.9}$$

with $2/s + n/r = n/2$ and $\alpha = n/2 - n/r$, applied to the right side of (2.8), imply

$$\int_{Q_\tau} b_j w^2 \eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\le T\}} \le \epsilon \|w\eta^\gamma \chi_{t\le T}\|_{L_t^s L_x^r}^2 + C\|b\|_{L_t^l L_x^{\bar{q}}}^{2/(2-\gamma_0)} \|w|\nabla\eta|^{1/(2-\gamma_0)}\chi_{t\le T}\|_{L^2_{t,x}}^2$$
$$\le \frac{1}{2}\left(\|w\eta^\gamma\chi_{t\le T}\|_{L_t^\infty L_x^2}^2 + \|\nabla(w\eta^\gamma)\chi_{t\le T}\|_{L^2_{t,x}}^2\right) + C\|b\|_{L_t^l L_x^{\bar{q}}}^{2/(2-\gamma_0)} \|w|\nabla\eta|^{1/(2-\gamma_0)}\chi_{t\le T}\|_{L^2_{t,x}}^2. \tag{2.10}$$

By (2.5), (2.6), and (2.10), we obtain, for any $-\tau^2 < T < 0$:

$$\int_{B_\tau} u^{\beta+2}(T)\eta^{2\gamma}(T) + \int_{Q_\tau} |\nabla(u^{\beta/2+1}\eta^\gamma)|^2 \chi_{\{t\le T\}}$$
$$\le C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\Delta\eta|\chi_{\{t\le T\}} + C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2\chi_{\{t\le T\}}$$
$$+ C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\partial_t\eta|\chi_{\{t\le T\}} + C\|b\|_{L_t^l L_x^{\bar{q}}}^{2/(2-\gamma_0)} \|u^{\beta/2+1}|\nabla\eta|^{1/(2-\gamma_0)}\chi_{t\le T}\|_{L^2_{t,x}}^2$$
$$+ \frac{1}{2}\|u^{\beta/2+1}\eta^\gamma\chi_{t\le T}\|_{L_t^\infty L_x^2}^2. \tag{2.11}$$

As this inequality holds for all $-\tau^2 < T < 0$, we may take the supremum over T to eliminate the $L_t^\infty L_x^2$-norm in the right side. Namely, from (2.11), we have

$$\sup_{T\in[-\tau^2,0]} \int_{B_\tau} u^{\beta+2}(T)\eta^{2\gamma}(T)$$

$$\le C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\Delta\eta| + C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2 + C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\partial_t\eta|$$

$$+ C\|b\|^{2/(2-\gamma_0)}_{L_t^l L_x^{\bar q}} \|u^{\beta/2+1}|\nabla\eta|^{1/(2-\gamma_0)}\|^2_{L^2_{t,x}} + \frac{1}{2}\|u^{\beta/2+1}\eta^\gamma\|^2_{L_t^\infty L_x^2} \tag{2.12}$$

and

$$\int_{Q_\tau} |\nabla(u^{\beta/2+1}\eta^\gamma)|^2 \le C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\Delta\eta| + C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2$$

$$+ C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\partial_t\eta| + C\|b\|^{2/(2-\gamma_0)}_{L_t^l L_x^{\bar q}} \|u^{\beta/2+1}|\nabla\eta|^{1/(2-\gamma_0)}\|^2_{L^2_{t,x}}$$

$$+ \frac{1}{2}\|u^{\beta/2+1}\eta^\gamma\|^2_{L_t^\infty L_x^2}. \tag{2.13}$$

Adding the last two estimates and absorbing the $L_t^\infty L_x^2$-norm, we obtain

$$\sup_{-\tau^2\le T\le 0} \int_{B_\tau} u^{\beta+2}(T)\eta^{2\gamma}(T) + \int_{Q_\tau} |\nabla(u^{\beta/2+1}\eta^\gamma)|^2$$

$$\le C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\Delta\eta| + C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2 + C\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\partial_t\eta|$$

$$+ C\|b\|^{2/(2-\gamma_0)}_{L_t^l L_x^{\bar q}} \|u^{\beta/2+1}|\nabla\eta|^{1/(2-\gamma_0)}\|^2_{L^2_{t,x}}, \tag{2.14}$$

with an increased constant $C > 0$. By the interpolation inequality (2.9), used on the left side of (2.14) with $r = s = 2(n+2)/n$, and $\alpha = n/(n+2)$, used together with Young's inequality, we get the following estimate:

$$\|u^{\beta/2+1}\eta^\gamma\|_{L^{2\chi}(Q_\tau)} \le C\left(\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\Delta\eta|\right)^{1/2} + C\left(\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2\right)^{1/2}$$

$$+ C\left(\int_{Q_\tau} u^{\beta+2}\eta^{2\gamma-1}|\partial_t\eta|\right)^{1/2} + C\|b\|^{1/(2-\gamma_0)}_{L^l L^{\bar q}} \|u^{\beta/2+1}|\nabla\eta|^{1/(2-\gamma_0)}\|_{L^2} \tag{2.15}$$

with $\chi = (n+2)/n$.

We shall now use (2.15) iteratively. We take a decreasing sequence $r_i > 0$, and at each step choose the cut-off function $\eta \in C_0^\infty(\Omega)$ such that

$$\eta \equiv 1 \ \text{ in } Q_{r_{i+1}}$$

$$\eta \equiv 0 \ \text{ in } Q^c_{r_i}$$

and

$$|\nabla\eta| \leq \frac{C}{r_i - r_{i+1}}$$
$$|\Delta\eta| \leq \frac{C}{(r_i - r_{i+1})^2}$$
$$|\partial_t\eta| \leq \frac{C}{(r_i - r_{i+1})^2}.$$

Then (2.15) gives

$$\|u^{\beta/2+1}\|_{L^{2\chi}(Q_{r_{i+1}})} \leq \frac{C}{r_i - r_{i+1}}\|u^{\beta/2+1}\|_{L^2(Q_{r_i})} + \frac{C\|b\|^{1/(2-\gamma_0)}_{L^lL^{\bar{q}}(Q_{r_i})}}{(r_i - r_{i+1})^{1/(2-\gamma_0)}}\|u^{\beta/2+1}\|_{L^2(Q_{r_i})}. \quad (2.16)$$

Let us choose β_i in (2.16) so that $\chi^i = \beta_i/2 + 1$. In addition, we set

$$r_i = \theta + \frac{(\tau - \theta)}{2^i}, \qquad i = 0, 1, 2, \ldots$$

so that $r_i - r_{i+1} = (\tau - \theta)/2^{i+1}$. Thus we obtain

$$\begin{aligned}\|u\|_{L^{2\chi^{i+1}}(Q_{r_{i+1}})} &\leq \left(\frac{C2^{i+1}}{\tau - \theta} + \frac{C2^{(i+1)/(2-\gamma_0)}}{(\tau - \theta)^{1/(2-\gamma_0)}}\|b\|^{1/(2-\gamma_0)}_{L^lL^{\bar{q}}(Q_{r_i})}\right)^{1/\chi^i}\|u\|_{L^{2\chi^i}(Q_{r_i})}\\ &\leq C^{1/\chi^i}2^{(i+1)/(\gamma_1\chi^i)}\left((\tau - \theta)^{-1} + \left((\tau - \theta)^{-1}\|b\|_{L^lL^{\bar{q}}(Q_{r_i})}\right)^{1/(2-\gamma_0)}\right)^{1/\chi^i}\\ &\quad \times \|u\|_{L^{2\chi^i}(Q_{r_i})}, \end{aligned} \quad (2.17)$$

where $\gamma_1 = \min\{2 - \gamma_0, 1\}$. By iteration, starting from $i = 0$, we conclude that the estimate (2.2) holds for $p \geq 2$.

Now, let $p \in (0, 2)$. The previous argument has shown that

$$\begin{aligned}\sup_{Q_\theta} u &\leq C\left((\tau - \theta)^{-1} + \left((\tau - \theta)^{-1}\|b\|_{L^lL^{\bar{q}}(Q_\tau)}\right)^{1/(2-\gamma_0)}\right)^{(n+2)/2}\|u\|_{L^2(B_\tau)}\\ &\leq C\left((\tau - \theta)^{-1} + \left((\tau - \theta)^{-1}\|b\|_{L^lL^{\bar{q}}(Q_\tau)}\right)^{1/(2-\gamma_0)}\right)^{(n+2)/2}\|u\|^{1-p/2}_{L^\infty(Q_\tau)}\|u\|^{p/2}_{L^p(Q_\tau)}, \end{aligned} \quad (2.18)$$

which implies

$$\sup_{Q_\theta} u \leq \frac{1}{2}\|u\|_{L^\infty(Q_\tau)} + C\left((\tau - \theta)^{-1} + \left((\tau - \theta)^{-1}\|b\|_{L^lL^{\bar{q}}(Q_\tau)}\right)^{1/(2-\gamma_0)}\right)^{(n+2)/p}\|u\|_{L^p(Q_\tau)}. \quad (2.19)$$

Now, the iteration argument of [7, Lemma 4.3] may be applied to complete the proof of Lemma 2.1 for $0 < p < 2$. □

3. The lower bound

The goal of this section is to prove Lemma 3.1, which establishes a lower bound of the infimum of a Lipschitz supersolution to (2.1).

Recall (cf. (1.2) and (1.3)) that we use the notation $Q_R^*(x_0, t_0)$ for the cylinder centered at the bottom and $Q_R(x_0, t_0)$ for the cylinder centered at the top, and $Q_R = Q_R(0, 0)$.

Lemma 3.1. *Assume that u is a nonnegative Lipschitz supersolution to (2.1), and $b \in L^{\bar{q}}(\Omega) \cap L^\infty L^2(\Omega)$ with $n/2 + 1 < \bar{q} \leq n + 2$ and $\operatorname{div} b = 0$ in the sense of distributions. Then there exists a (small) positive number $p_0 = p_0(n, \bar{q}, R, M_R)$ such that*

$$\left(CR^{-n-2} \int_{Q_R^*(0,-4R^2)} u^{p_0} \right)^{1/p_0} \leq \exp\left(1 + (R^{1-(n+2)/\bar{q}} \|b\|_{L^{\bar{q}}})^{1/(2-(n+2)/\bar{q})}\right)^{C(n)} \inf_{Q_R} u \quad (3.1)$$

with

$$M_R = 1 + (R^{1-n/2} \|b\|_{L^\infty L^2})^2 + R^{1-(n+2)/\bar{q}} \|b\|_{L^{\bar{q}}}. \quad (3.2)$$

We establish the proof of Lemma 3.1 in several steps, successively improving the estimate. We primarily work with the function

$$v = \log(u/K)$$

with a constant K to be determined. If u is a supersolution to (2.1), then v is also a supersolution to (2.1). More precisely, v satisfies the inequality

$$|\nabla v|^2 \leq v_t - \Delta v + b \cdot \nabla v \quad \text{in } \Omega. \quad (3.3)$$

Next, we obtain various bounds on $w = v_+$.

3.1. A bound on $\int w^\alpha$ for $\alpha \in (0, 1)$

We begin with the following initial estimate on $w = v_+ = \log_+(u/K)$. Note that the constant K we choose in (3.4) below does depend on the solution $u(x, t)$.

Lemma 3.2. *Let $\eta(x) = C(1 - |x|^2/(9R^2))_+$ be normalized so that $\int_{\mathbb{R}^n} \eta^2(x)\, dx = 1$, and set*

$$K = \exp\left(\int_{B_{3R}} \eta^2(x) \log u(x, 4R^2)\, dx \right). \quad (3.4)$$

Then for $\alpha \in (0, 1)$ we have

$$\int_{Q_{2R}^*} w^\alpha \, dx\, dt \leq CM_0 R^{n+2} \quad (3.5)$$

with $M_0 = 1 + (R^{1-n/2} \|b\|_{L^\infty L^2})^2$.

Proof of Lemma 3.2. Again, without loss of generality, we assume that $R = 1$. We multiply (3.3) by the cut-off $\eta^2(x)$ and integrate over $B_3 \times (t_1, t_2)$ with $0 \leq t_1 < t_2 \leq 4$ in order to obtain

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{B_3}|\nabla v(x,t)|^2\eta^2(x)\,dx\,dt\\&\le\int_{B_3}v(x,t_2)\eta^2(x)\,dx-\int_{B_3}v(x,t_1)\eta^2(x)\,dx+2\int_{t_1}^{t_2}\int_{B_3}(\partial_j v(x,t))\eta(x)(\partial_j\eta(x))\,dx\,dt\\&\quad+\int_{t_1}^{t_2}\int_{B_3}b_j(x,t)(\partial_j v(x,t))\eta^2(x)\,dx\,dt.\end{aligned}\tag{3.6}$$

After rearranging the terms and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\int_{B_3}v(x,t_1)\eta^2(x)\,dx-\int_{B_3}v(x,t_2)\eta^2(x)\,dx+\int_{t_1}^{t_2}\int_{B_3}|\nabla v(x,t)|^2\eta^2(x)\,dx\,dt\\&\le 2\int_{t_1}^{t_2}\int_{B_3}(\partial_j v(x,t))\eta(x)(\partial_j\eta(x))\,dx\,dt+\int_{t_1}^{t_2}\int_{B_3}b_j(x,t)(\partial_j v(x,t))\eta^2(x)\,dx\,dt\\&\le\frac{1}{2}\int_{t_1}^{t_2}\int_{B_3}|\nabla v(x,t)|^2\eta^2(x)\,dx\,dt+C\|\nabla\eta(x)\|^2_{L^2(B_3\times(t_1,t_2)))}\\&\quad+C\|b\|^2_{L_t^\infty L_x^2(B_3\times(t_1,t_2))}\|\eta\|^2_{L_t^2L_x^\infty(B_3\times(t_1,t_2))}.\end{aligned}\tag{3.7}$$

Absorbing the first term on the far right side of (3.7) leads to

$$\begin{aligned}&\int_{B_3}v(x,t_1)\eta^2(x)\,dx-\int_{B_3}v(x,t_2)\eta^2(x)\,dx+\frac{1}{2}\int_{t_1}^{t_2}\int_{B_3}|\nabla v(x,t)|^2\eta^2(x)\,dx\,dt\\&\le C\left(1+\|b\|^2_{L^\infty L^2(\Omega)}\right)(t_2-t_1)\end{aligned}\tag{3.8}$$

since $0\le\eta\le 1$. Now, we set $M_0=1+\|b\|^2_{L^\infty L^2(\Omega)}$. Using weighted Poincaré's inequality (cf. [13, Lemma 6.12]) on the left side of (3.8), we get

$$\begin{aligned}&\int_{B_3}v(x,t_1)\eta^2(x)\,dx-\int_{B_3}v(x,t_2)\eta^2(x)\,dx\\&+\frac{1}{C}\int_{t_1}^{t_2}\int_{B_3}\left|v(x,t)-\int_{B_3}v(x,t)\eta^2(x)\,dx\right|^2\eta^2(x)\,dx\,dt\le C_0M_0(t_2-t_1)\end{aligned}\tag{3.9}$$

where $C_0>0$ is a fixed constant.

For the rest of the proof we may proceed as in the proof of [13, Lemma 6.21]. Consider the function

$$p(x,t)=v(x,t)-C_0M_0(4-t),\tag{3.10}$$

defined as a translation of v in time by the term coming from the right side of (3.9). Note that the constant K in (3.4) was chosen so that

$$\int_{B_3}v(x,4)\eta^2(x)\,dx=0,\tag{3.11}$$

and (3.9) and (3.11) imply that

$$\int_{B_3}v(x,t)\eta^2(x)\,dx\le C_0M_0(4-t),\qquad 0\le t\le 4.\tag{3.12}$$

Based on $\eta(x)$ being uniformly positive for $|x| \leq 2$, we claim the upper bound

$$\left|\{(x,t) \in Q_2^* : p(x,t) > \mu\}\right| \leq \frac{C|Q_2^*|}{\mu}, \qquad \mu \geq 1 \tag{3.13}$$

on the size of the level sets of p. In order to show (3.13), fix $\mu \geq 1$ and denote

$$Q_\mu(t) = \{x \in B_2 : p(x,t) > \mu\}, \qquad 0 \leq t \leq 4. \tag{3.14}$$

Also, let

$$P(t) = \int_{B_3} v(x,t)\eta^2(x)\,dx - C_0 M_0(4-t) \tag{3.15}$$

for $0 \leq t \leq 4$ and observe that $P(4) = 0$. Now, we may rewrite (3.9) in the form

$$P(t_1) - P(t_2) + \frac{1}{C}\int_{t_1}^{t_2}\int_{B_3} |p(x,t) - P(t)|^2\,dx\,dt \leq 0. \tag{3.16}$$

By (3.12), we have $P(t) \leq 0$ for $0 \leq t \leq 4$. Thus $p(x,t) - P(t) > \mu - P(t) > 0$ on $Q_\mu(t)$, which together with (3.16) gives

$$P(t_1) - P(t_2) + \frac{1}{C|Q_2^*|}\int_{t_1}^{t_2} |Q_\mu(t)|(\mu - P(t))^2\,dt \leq 0 \tag{3.17}$$

for $0 \leq t_1 \leq t_2 \leq 4$. Dividing by $t_2 - t_1$ and taking the limit $t_2 \to t_1$, we get

$$-P'(t) + \frac{|Q_\mu(t)}{C|Q_2^*|}(\mu - P(t))^2 \leq 0 \tag{3.18}$$

or equivalently

$$\frac{|Q_\mu(t)|}{C|Q_2^*|} \leq \frac{P'(t)}{(\mu - P(t))^2} \tag{3.19}$$

for $0 \leq t \leq 4$. Integrating (3.19) in time, we finally obtain

$$\frac{1}{C|Q_2^*|}\left|\{(x,t) \in Q_2^* : p(x,t) > \mu\}\right| \leq \frac{1}{\mu - P(4)} - \frac{1}{\mu - P(0)} \leq \frac{1}{\mu}, \tag{3.20}$$

where we utilized $P(4) = 0$ in the last inequality. Thus, the validity of (3.13) is established.

Using (3.13), we obtain the bound

$$\begin{aligned}\int_{\{(x,t)\in Q_2^* : p(x,t)>1\}} p^\alpha\,dx\,dt &= \alpha\int_1^\infty \mu^{\alpha-1}\left|\{(x,t) \in Q_2^* : p(x,t) > \mu\}\right|\,d\mu \\ &\leq C\alpha|Q_2^*|\int_1^\infty \mu^{\alpha-2}\,d\mu \leq C|Q_2^*| \end{aligned}\tag{3.21}$$

since $\alpha \in (0,1)$. We conclude the proof of (3.5) by noting that the function w satisfies $w^\alpha \leq Cp^\alpha + CM_0^\alpha$ if $p \geq 1$ and $w^\alpha \leq C + CM_0^\alpha$ if $p < 1$. □

3.2. A bound on $\int w^\sigma$ for $\sigma \in [1,(n+2)/n)$

From now on, without loss of generality, we assume that $R = 1$. As before, we work with $w = v_+ = \log_+(u/K)$ with a constant K defined in (3.4). The function w is a supersolution to the equation for v, that is,

$$|\nabla w|^2 \leq w_t - \Delta w + b \cdot \nabla w \tag{3.22}$$

since it is a maximum of two supersolutions, $v_1 = v(x,t)$ and $v_2 \equiv 0$. We need the following inequality that bootstraps bounds for the L^α-norms with $\alpha \in (0,1)$ we have obtained in Lemma 3.2 to higher norms.

Lemma 3.3. *For any $\sigma \in [1,(n+2)/n)$ and any $\alpha \in (0,1)$, we have*

$$\|w\|_{L^\sigma(Q_1^*)} \leq C(1 + \|b\|_{L^{\bar{q}}})^C \|w\|_{L^\alpha(Q_2^*)}, \tag{3.23}$$

where $C = C(\alpha,\sigma,n,\bar{q})$.

Proof of Lemma 3.3. Let η be a Lipschitz cut-off in Q_2^* with $0 \leq \eta \leq 1$; note that unlike in the proof of Lemma 3.2 the cut-off here also depends on time. We multiply (3.22) by the function

$$(w+1)^{2\beta}\eta^{2\gamma}\chi_{\{t\geq T\}},$$

with $\beta \in (-1/2,0)$, $\gamma > 1$ to be determined, and $T \in (0,4)$, and integrate over Q_2^* to obtain

$$\begin{aligned}
&\frac{1}{2\beta+1}\int_{B_2}(w+1)^{2\beta+1}\eta^{2\gamma}\Big|_{t=T} + \int_{Q_2^*}|\nabla w|^2(w+1)^{2\beta}\eta^{2\gamma}\chi_{\{t\geq T\}}\\
&\leq 2\beta\int_{Q_2^*}|\nabla w|^2(w+1)^{2\beta-1}\eta^{2\gamma}\chi_{\{t\geq T\}} + 2\gamma\int_{Q_2^*}(\partial_j w)(w+1)^{2\beta}\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\geq T\}}\\
&\quad - \frac{2\gamma}{2\beta+1}\int_{Q_2^*} b_j(w+1)^{2\beta+1}\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\geq T\}}\\
&\quad - \frac{2\gamma}{2\beta+1}\int_{Q_2^*}(w+1)^{2\beta+1}\eta^{2\gamma-1}(\partial_t\eta)\chi_{\{t\geq T\}}.
\end{aligned} \tag{3.24}$$

Here we have used the condition div $b=0$. The first term on the right side is negative since $\beta \in (-1/2,0)$, while integration by parts in the second term on the right gives

$$\begin{aligned}
&2\gamma\int(\partial_j w)(w+1)^{2\beta}\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\geq T\}} = \frac{2\gamma}{2\beta+1}\int\partial_j\left((w+1)^{2\beta+1}\right)\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\geq T\}}\\
&= -\frac{2\gamma(2\gamma-1)}{2\beta+1}\int(w+1)^{2\beta+1}\eta^{2\gamma-2}|\nabla\eta|^2\chi_{\{t\geq T\}}\\
&\quad - \frac{2\gamma}{2\beta+1}\int(w+1)^{2\beta+1}\eta^{2\gamma-1}(\Delta\eta)\chi_{\{t\geq T\}}.
\end{aligned} \tag{3.25}$$

This, together with (3.24) leads to

$$\begin{aligned}
&\frac{1}{2\beta+1}\int_{B_2}(w+1)^{2\beta+1}\eta^{2\gamma}\Big|_{t=T} + \int_{Q_2^*}(w+1)^{2\beta}|\nabla w|^2\eta^{2\gamma}\chi_{\{t\geq T\}}\\
&\leq -\frac{2\gamma}{2\beta+1}\int_{Q_2^*} b_j(w+1)^{2\beta+1}\eta^{2\gamma-1}(\partial_j\eta)\chi_{\{t\geq T\}}\\
&\quad - \frac{2\gamma}{2\beta+1}\int_{Q_2^*}(w+1)^{2\beta+1}\left(\eta^{2\gamma-1}\partial_t\eta + (2\gamma-1)\eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}\Delta\eta\right)\chi_{\{t\geq T\}}.
\end{aligned} \tag{3.26}$$

We may use the inequality

$$|\nabla((w+1)^{\beta+1/2}\eta^\gamma)|^2 \le 2(\beta+1/2)^2(w+1)^{2\beta-1}|\nabla w|^2\eta^{2\gamma} + 2\gamma^2(w+1)^{2\beta+1}\eta^{2\gamma-2}|\nabla\eta|^2 \tag{3.27}$$

on the left side of (3.26). In addition, as $w > 0$, we have $(w+1)^{2\beta-1} \le (w+1)^{2\beta}$, which altogether gives

$$\begin{aligned}
&\int_{B_2}(w+1)^{2\beta+1}\eta^{2\gamma}|_{t=T} + \int_{Q_2^*}|\nabla((w+1)^{\beta+1/2}\eta^\gamma)|^2\chi_{\{t\ge T\}}\\
&\le C\gamma\int_{Q_2^*}|b_j|(w+1)^{2\beta+1}\eta^{2\gamma-1}|\partial_j\eta|\chi_{\{t\ge T\}}\\
&\quad + C\gamma^2\int_{Q_2^*}(w+1)^{2\beta+1}\left(\eta^{2\gamma-1}|\partial_t\eta| + \eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}|\Delta\eta|\right)\chi_{\{t\ge T\}}.
\end{aligned}\tag{3.28}$$

An application of the interpolation inequality (2.9) with $r = s = 2(n+2)/n$ leads to

$$\begin{aligned}
&\|(w+1)^{\beta+1/2}\eta^\gamma\|^2_{L^{2(n+2)/n}} \le C\|(w+1)^{\beta+1/2}\eta^\gamma\|^2_{L^\infty L^2} + C\|\nabla((w+1)^{\beta+1/2}\eta^\gamma)\|^2_{L^2}\\
&\le C\gamma\int_{Q_2^*}|b_j|(w+1)^{2\beta+1}\eta^{2\gamma-1}|\partial_j\eta|\chi_{\{t\ge T\}}\\
&\quad + C\gamma^2\int_{Q_2^*}(w+1)^{2\beta+1}\left(\eta^{2\gamma-1}|\partial_t\eta| + \eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}|\Delta\eta|\right)\chi_{\{t\ge T\}}.
\end{aligned}\tag{3.29}$$

Next, we may estimate the drift term in (3.29) with the help of Hölder's inequality as

$$\begin{aligned}
&C\gamma\int_{Q_2^*}|b|(w+1)^{2\beta+1}\eta^{2\gamma-1}|\nabla\eta|\\
&= C\gamma\int_{Q_2^*}|b|(w+1)^{(2\beta+1)(1-\lambda)}(w+1)^{(2\beta+1)\lambda}\eta^{2\gamma-1}|\nabla\eta|\\
&\le C\gamma\|b\|_{L^{\bar q}}\|(w+1)^{(2\beta+1)(1-\lambda)}\|_{L^{1/(1-\lambda)}}\|(w+1)^{(2\beta+1)\lambda}\eta^{2\gamma-1}\|_{L^{(n+2)/(n\lambda)}}\|\nabla\eta\|_{L^\infty},
\end{aligned}\tag{3.30}$$

where

$$\frac{1}{\bar q} + 1 - \lambda + \frac{n\lambda}{n+2} = 1.$$

Therefore, λ is given by $\lambda = (n+2)/(2\bar q)$ and $\lambda \in [1/2, 1)$, as $1 \le (n+2)/\bar q < 2$ by assumption. Using Young's inequality, this leads to

$$\begin{aligned}
&C\gamma\int_{Q_2^*}|b|(w+1)^{2\beta+1}\eta^{2\gamma-1}|\nabla\eta|\\
&\le C\gamma\|b\|_{L^{\bar q}}\|(w+1)^{2\beta+1}\|^{1-\lambda}_{L^1}\|(w+1)^{2\beta+1}\eta^{(2\gamma-1)/\lambda}\|^{\lambda}_{L^{\frac{n+2}{n}}}\|\nabla\eta\|_{L^\infty}\\
&\le \frac{1}{2}\|(w+1)^{\beta+1/2}\eta^{(2\gamma-1)/(2\lambda)}\|^2_{L^{2(n+2)/n}}\\
&\quad + \left(C\gamma\|b\|_{L^{\bar q}}\|\nabla\eta\|_{L^\infty}\right)^{1/(1-\lambda)}\|(w+1)^{\beta+1/2}\|^2_{L^2}.
\end{aligned}\tag{3.31}$$

Setting $\gamma = 1/(2(1-\lambda)) \geq 1$ so that $(2\gamma - 1)/2\lambda = \gamma$ and using (3.29) and (3.31), we obtain

$$\begin{aligned} &\|(w+1)^{\beta+\frac{1}{2}}\eta^\gamma\|^2_{L^{\frac{2(n+2)}{n}}} \\ &\leq \frac{1}{2}\|(w+1)^{\beta+\frac{1}{2}}\eta^\gamma\|^2_{L^{\frac{2(n+2)}{n}}} + \left(C\|b\|_{L^{\bar{q}}}\|\nabla\eta\|_{L^\infty}\right)^{2\gamma} \|(w+1)^{\beta+\frac{1}{2}}\|^2_{L^2} \\ &\quad + C\int_{Q_2^*}(w+1)^{2\beta+1}\left(\eta^{2\gamma-1}|\partial_t\eta| + \eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}|\Delta\eta|\right). \end{aligned} \tag{3.32}$$

The first term on the right may be absorbed into the left side:

$$\begin{aligned} &\|(w+1)^{\beta+\frac{1}{2}}\eta^\gamma\|^2_{L^{\frac{2(n+2)}{n}}(Q_2^*)} \\ &\leq \left(C\|b\|_{L^{\bar{q}}}\|\nabla\eta\|_{L^\infty}\right)^{2\gamma} \|(w+1)^{\beta+\frac{1}{2}}\|^2_{L^2} \\ &\quad + C\int_{Q_2^*}(w+1)^{2\beta+1}\left(\eta^{2\gamma-1}|\partial_t\eta| + \eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}|\Delta\eta|\right). \end{aligned} \tag{3.33}$$

We now once again use an iteration procedure, applied to a decreasing sequence of parabolic cylinders Q_{r_i} with $r_{i+1} < r_i$. Choosing the cut-off η such that $\eta \equiv 1$ in $Q^*_{r_{i+1}}$ and $\eta \equiv 0$ in $(Q^*_{r_i} \cup Q_{r_i})^c$, we have, from (3.33):

$$\begin{aligned} \|(w+1)^{\beta+\frac{1}{2}}\|^2_{L^{\frac{2(n+2)}{n}}(Q^*_{r_{i+1}})} &\leq \left(\frac{C\gamma\|b\|_{L^{\bar{q}}}}{r_i - r_{i+1}}\right)^{2\gamma} \|(w+1)^{\beta+\frac{1}{2}}\|^2_{L^2(Q^*_{r_i})} \\ &\quad + \frac{C}{(r_i - r_{i+1})^2}\|(w+1)^{\beta+\frac{1}{2}}\|^2_{L^2(Q^*_{r_i})}, \end{aligned}$$

or equivalently

$$\|(w+1)^{2\beta+1}\|_{L^{(n+2)/n}(Q^*_{r_{i+1}})} \leq C(r_i - r_{i+1})^{-2\gamma}(\|b\|^{2\gamma}_{L^{\bar{q}}} + 1)\|(w+1)^{2\beta+1}\|_{L^1(Q^*_{r_i})}, \tag{3.34}$$

since $\gamma \geq 1$. Set $\chi = (n+2)/n$, pick $\alpha \in (0,1)$, and consider $\sigma \in [1, (n+2)/n)$. Possibly increasing σ and decreasing α we may assume that $\sigma = \chi^j\alpha$ with $j \in \mathbb{N}$. We shall use (3.34) with

$$\beta_i = \frac{\chi^i\alpha - 1}{2}$$

for $i = 0, \ldots, j$ so that $2\beta_0 + 1 = \alpha$ and $2\beta_j + 1 = \sigma$, and $r_i = 1 + 2^{-i}$. Then (3.34) implies the recursive relation

$$\|w+1\|_{L^{\chi^{i+1}\alpha}(Q^*_{r_{i+1}})} \leq C2^{2\gamma(i+1)/\chi^i}(\|b\|^{2\gamma}_{L^{\bar{q}}} + 1)^{1/\chi^i}\|w+1\|_{L^{\chi^i\alpha}(Q^*_{r_i})}, \tag{3.35}$$

and a finite number of iterations gives (3.23). □

Remark 3.4. Observe that if $b \in L^l L^{\bar{q}}(\Omega)$ with $1 \leq 2/l + n/\bar{q} < 2$, we may bound the drift term (3.30) using the same idea as in (2.8) and (2.10) but with w^2 replaced by $w^{2\beta+1}$, and then apply the interpolation inequality (2.9).

3.3. An estimate for $\int |\nabla w|^2$

The next step is to obtain bounds on $\|\nabla w\|_{L^2}$. Recall that w satisfies

$$|\nabla w|^2 \le w_t - \Delta w + b \cdot \nabla w. \tag{3.36}$$

Multiplying (3.36) by $\eta^2 \chi_{\{t\ge T\}}$ and integrating over Q_2^* gives

$$\int_{B_2} w\eta^2\Big|_{t=T} + \int_{Q_2^*} |\nabla w|^2\eta^2 \le 2\int_{Q_2^*} (\partial_j w)\eta(\partial_j \eta) - 2\int_{Q_2^*} b_j w\eta\partial_j\eta - 2\int_{Q_2^*} w\eta\partial_t\eta, \tag{3.37}$$

where we used div $b = 0$. After estimating the right side, we get

$$\begin{aligned}\int_{B_2} w\eta^2\Big|_{t=T} + \int_{Q_2^*} |\nabla w|^2\eta^2 &\le C\|\nabla\eta\|_{L^2}^2 + C\|b\|_{L^{\bar q}}\|w\eta\|_{L^{\bar q^*}}\|\nabla\eta\|_{L^\infty} + C\|\eta_t\|_{L^\infty}\|w\eta\|_{L^1}\\ &\le C(1 + \|b\|_{L^\infty L^2}^2 + \|b\|_{L^{\bar q}})^C = CM^C,\end{aligned} \tag{3.38}$$

where $M = 1 + \|b\|_{L^\infty L^2}^2 + \|b\|_{L^{\bar q}}$. In the last inequality, we used Lemmas 3.2 and 3.3 with $\sigma = \bar q^*$ and $\sigma = 1$, respectively, where $\bar q^* < (n+2)/n$, as $1/\bar q + 1/\bar q^* = 1$ and $(n+2)/2 < \bar q \le n+2$.

Note that with the bound (3.38) in hand we may extend the argument in the proof of Lemma 3.3 to include $\beta \in [0, 1/2]$. Namely, in that proof we have considered $\beta \in (-1/2, 0)$ and dropped the first term in the right side of (3.24) simply because β was negative. Now, we may rely on (3.38) to bound this term in (3.24). The rest of the argument in the proof of Lemma 3.3 did not rely on the negativity of β. As the aforementioned term in (3.24) involves the product $|\nabla w|^2(w+1)^{2\beta-1}$ while (3.38) estimates $|\nabla w|^2$, we would still need the restriction $\beta \le 1/2$.

3.4. Bound on $\int w^{2\beta+1}$ for $\beta \ge 1/2$

We now extend the bound for

$$\int w^{2\beta+1}$$

to $\beta \ge 1/2$. As in the proof of Lemma 3.3, we let η be a Lipschitz cut-off in Q_2^* with $0 \le \eta \le 1$. This time, we multiply (3.22) by the function

$$w^{2\beta}\eta^{2\gamma}\chi_{\{t\ge T\}}$$

with $\beta \ge 1/2$ and $T \in (0, 4)$, and integrate over Q_2^*, using the divergence-free condition on b:

$$\begin{aligned}&\frac{1}{2\beta+1}\int_{B_2} w^{2\beta+1}\eta^{2\gamma}\Big|_{t=T} + \int_{Q_2^*} |w|^{2\beta}|\nabla w|^2\eta^{2\gamma}\\ &\le 2\beta\int_{Q_2^*} |\nabla w|^2 w^{2\beta-1}\eta^{2\gamma} + 2\gamma\int_{Q_2^*} (\partial_j w)w^{2\beta}\eta^{2\gamma-1}\partial_j\eta\\ &\quad - \frac{2\gamma}{2\beta+1}\int_{Q_2^*} b_j w^{2\beta+1}\eta^{2\gamma-1}\partial_j\eta - \frac{2\gamma}{2\beta+1}\int_{Q_2^*} w^{2\beta+1}\eta^{2\gamma-1}\partial_t\eta.\end{aligned} \tag{3.39}$$

For the first term in the right side of (3.39), we use the inequality

$$2\beta|w|^{2\beta-1} \le \frac{1}{2}|w|^{2\beta} + (4\beta)^{2\beta-1}, \tag{3.40}$$

and for the second:

$$\begin{aligned} 2\gamma \int (\partial_j w) w^{2\beta} \eta^{2\gamma-1} \partial_j \eta &= \frac{2\gamma}{2\beta+1} \int \partial_j (w^{2\beta+1}) \eta^{2\gamma-1} \partial_j \eta \\ &= -\frac{2\gamma(2\gamma-1)}{2\beta+1} \int w^{2\beta+1} \eta^{2\gamma-2} |\nabla \eta|^2 - \frac{2\gamma}{2\beta+1} \int w^{2\beta+1} \eta^{2\gamma-1} \Delta \eta. \end{aligned} \tag{3.41}$$

Together with (3.39) this gives

$$\begin{aligned} &\frac{1}{2\beta+1} \int_{B_2} w^{2\beta+1} \eta^{2\gamma} \Big|_{t=T} + \frac{1}{2} \int_{Q_2^*} |w|^{2\beta} |\nabla w|^2 \eta^{2\gamma} \\ &\le (4\beta)^{2\beta-1} \int_{Q_2^*} |\nabla w|^2 \eta^{2\gamma} - \frac{2\gamma}{2\beta+1} \int_{Q_2^*} b_j w^{2\beta+1} \eta^{2\gamma-1} \partial_j \eta \\ &\quad - \frac{2\gamma}{2\beta+1} \int_{Q_2^*} w^{2\beta+1} \left(\eta^{2\gamma-1} \partial_t \eta + (2\gamma-1) \eta^{2\gamma-2} |\nabla \eta|^2 + \eta^{2\gamma-1} \Delta \eta \right). \end{aligned} \tag{3.42}$$

Applying the estimate (3.40) for the second term on the left side of (3.42), we obtain

$$\begin{aligned} &\frac{1}{2\beta+1} \int_{B_2} w^{2\beta+1} \eta^{2\gamma} |_{t=T} + 2\beta \int_{Q_2^*} |w|^{2\beta-1} |\nabla w|^2 \eta^{2\gamma} \\ &\le 2(4\beta)^{2\beta-1} \int_{Q_2^*} |\nabla w|^2 \eta^{2\gamma} + \frac{2\gamma}{2\beta+1} \int_{Q_2^*} |b_j| w^{2\beta+1} \eta^{2\gamma-1} |\partial_j \eta| \\ &\quad + \frac{2\gamma}{2\beta+1} \int_{Q_2^*} w^{2\beta+1} \left(\eta^{2\gamma-1} |\partial_t \eta| + (2\gamma-1) \eta^{2\gamma-2} |\nabla \eta|^2 + \eta^{2\gamma-1} |\Delta \eta| \right). \end{aligned} \tag{3.43}$$

Next, we multiply (3.43) by $(2\beta+1)$ and use the inequality

$$|\nabla(|w|^{\beta+1/2} \eta^\gamma)|^2 \le 2(\beta+1/2)^2 |w|^{2\beta-1} |\nabla w|^2 \eta^{2\gamma} + 2\gamma^2 |w|^{2\beta+1} \eta^{2\gamma-2} |\nabla \eta|^2 \tag{3.44}$$

on the left side to get

$$\begin{aligned} &\int_{B_2} w^{2\beta+1} \eta^{2\gamma} \Big|_{t=T} + \int_{Q_2^*} |\nabla(|w|^{\beta+1/2} \eta^\gamma)|^2 \\ &\le C(4\beta)^{2\beta} \int_{Q_2^*} |\nabla w|^2 \eta^{2\gamma} + C\gamma \int_{Q_2^*} |b_j| w^{2\beta+1} \eta^{2\gamma-1} |\partial_j \eta| \\ &\quad + C\gamma^2 \int_{Q_2^*} w^{2\beta+1} \left(\eta^{2\gamma-1} |\partial_t \eta| + \eta^{2\gamma-2} |\nabla \eta|^2 + \eta^{2\gamma-1} |\Delta \eta| \right). \end{aligned} \tag{3.45}$$

We use the interpolation inequality (2.9) with $r = s = 2(n+2)/n$ to write

$$\begin{aligned}\|w^{\beta+1/2}\eta^\gamma\|^2_{L^{2(n+2)/n}} &\le C\|w^{\beta+1/2}\eta^\gamma\|^2_{L^\infty L^2} + C\|\nabla(w^{\beta+1/2}\eta^\gamma)\|^2_{L^2}\\ &\le C(4\beta)^{2\beta}\int_{Q_2^*}|\nabla w|^2\eta^{2\gamma} + C\gamma\int_{Q_2^*}|b_j|w^{2\beta+1}\eta^{2\gamma-1}|\partial_j\eta|\\ &\quad + C\gamma^2\int_{Q_2^*}w^{2\beta+1}\left(\eta^{2\gamma-1}|\partial_t\eta| + \eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}|\Delta\eta|\right).\end{aligned} \tag{3.46}$$

First, we note that unlike in the proof of Lemma 3.3 we now have the uniform estimate (3.38) for the gradient:

$$\int_{Q_2^*}|\nabla w|^2 \le CM^C. \tag{3.47}$$

Next, we estimate the drift term in (3.46) similarly to what we did in the proof of Lemma 3.3. Namely, we may write

$$\begin{aligned}C\gamma\int_{Q_2^*}|b|w^{2\beta+1}\eta^{2\gamma-1}|\nabla\eta| &= C\gamma\int_{Q_2^*}|b|w^{(2\beta+1)(1-\lambda)}w^{(2\beta+1)\lambda}\eta^{2\gamma-1}|\nabla\eta|\\ &\le C\gamma\|b\|_{L^{\bar q}}\|w^{(2\beta+1)(1-\lambda)}\|_{L^{1/(1-\lambda)}}\|w^{(2\beta+1)\lambda}\eta^{2\gamma-1}\|_{L^{(n+2)/(n\lambda)}}\|\nabla\eta\|_{L^\infty}\end{aligned} \tag{3.48}$$

with $\lambda = (n+2)/(2\bar q) \in [1/2, 1)$. An application of Young's inequality gives

$$\begin{aligned}&C\gamma\int_{Q_2^*}|b|w^{2\beta+1}\eta^{2\gamma-1}|\nabla\eta|\\ &\le C\gamma\|b\|_{L^{\bar q}}\|w^{2\beta+1}\|^{1-\lambda}_{L^1}\|w^{2\beta+1}\eta^{(2\gamma-1)/\lambda}\|^\lambda_{L^{(n+2)/n}}\|\nabla\eta\|_{L^\infty}\\ &\le \frac{1}{2}\|w^{\beta+1/2}\eta^{(2\gamma-1)/(2\lambda)}\|^2_{L^{2(n+2)/n}} + \left(C\gamma\|b\|_{L^{\bar q}}\|\nabla\eta\|_{L^\infty}\right)^{1/(1-\lambda)}\|w^{\beta+1/2}\|^2_{L^2}.\end{aligned} \tag{3.49}$$

As before, choosing $\gamma = 1/(2(1-\lambda))$, we may absorb the first term in the right side of (3.49) into the left side of (3.46). Thus, we obtain

$$\begin{aligned}\|(w\eta)^{\beta+1/2}\|^2_{L^{2(n+2)/n}} &\le CM^C(4\beta)^{2\beta} + \left(C\gamma\|b\|_{L^{\bar q}}\|\nabla\eta\|_{L^\infty}\right)^{2\gamma}\|w^{\beta+1/2}\|^2_{L^2}\\ &\quad + C\gamma^2\int_{Q_2^*}w^{2\beta+1}\left(\eta^{2\gamma-1}|\partial_t\eta| + \eta^{2\gamma-2}|\nabla\eta|^2 + \eta^{2\gamma-1}|\Delta\eta|\right).\end{aligned} \tag{3.50}$$

We are now ready to perform the iteration process. We set $r_i = 1 + 2^{-i}$ for $i = 0, 1, 2, \dots$ and choose the cut-off η such that $\eta \equiv 1$ in $Q^*_{r_{i+1}}$ and $\eta \equiv 0$ in $(Q^*_{r_i} \cup Q_{r_i})^c$. Then (3.50) gives at each iteration step

$$\begin{aligned}&\|w^{\beta+1/2}\|^2_{L^{2(n+2)/n}(Q^*_{r_{i+1}})}\\ &\le CM^C(4\beta)^{2\beta} + \left(\frac{C\gamma}{r_i - r_{i+1}}\|b\|_{L^{\bar q}}\right)^{2\gamma}\|w^{\beta+1/2}\|^2_{L^2(Q^*_{r_i})}\\ &\quad + \frac{C\gamma^2}{(r_i - r_{i+1})^2}\|w^{\beta+1/2}\|^2_{L^2(Q^*_{r_i})} \le CM^C(4\beta)^{2\beta}\\ &\quad + \left(\frac{C\gamma}{r_i - r_{i+1}}\right)^{2\gamma}(\|b\|^{2\gamma}_{L^{\bar q}} + 1)\|w^{\beta+1/2}\|^2_{L^2(Q^*_{r_i})}\end{aligned} \tag{3.51}$$

since $\gamma \geq 1$. Thus, we have the relation

$$\|w^{\beta+1/2}\|^2_{L^{2(n+2)/n}(Q^*_{r_{i+1}})} \leq CM^C \left(\frac{C\gamma}{r_i - r_{i+1}}\right)^{2\gamma} \left((4\beta)^{2\beta} + \|w^{\beta+1/2}\|^2_{L^2(Q^*_{r_i})}\right) \tag{3.52}$$

between consecutive scales. As in the proof of Lemma 3.3 we use it with $\beta_i = (\chi^i - 1)/2$ where $\chi = (n+2)/n$ but this time we may allow β (and thus i) to be arbitrarily large. We obtain

$$\begin{aligned} \|w\|^{\chi^i}_{L^{\chi^{i+1}}(Q^*_{r_{i+1}})} &\leq CM^C 2^{\gamma(i+1)} \left((2\chi^i)^{\chi^i} + \|w\|^{\chi^i}_{L^{\chi^i}(Q^*_{r_i})}\right) \\ &\leq CM^C 2^{\gamma(i+1)} \left(2\chi^i + \|w\|_{L^{\chi^i}(Q^*_{r_i})}\right)^{\chi^i} \end{aligned} \tag{3.53}$$

for $i = 0, 1, 2, \ldots$ Iterating the inequality

$$\|w\|_{L^{\chi^{i+1}}(Q^*_{r_{i+1}})} \leq (CM)^{C/\chi^i} 2^{2\gamma(i+1)/\chi^i} \left(2\chi^i + \|w\|_{L^{\chi^i}(Q^*_{r_i})}\right) \tag{3.54}$$

obtained from (3.52) by taking $1/\chi^i$ power on both sides, we get

$$\|w\|_{L^{\chi^{i+1}}(Q^*_{r_{i+1}})} \leq CM^C \left(\chi^{i+1} + \|w\|_{L^1(Q^*_1)}\right). \tag{3.55}$$

By Lemmas 3.2 and 3.3, we have

$$\|w\|_{L^1(Q^*_1)} \leq CM^C \|w\|_{L^\alpha(Q^*_2)} \leq CM^C \tag{3.56}$$

which together with (3.55) implies

$$\|w\|_{L^{\chi^{i+1}}(Q^*_{r_{i+1}})} \leq CM^C \chi^{i+1}. \tag{3.57}$$

Thus, we may conclude

$$\left(\int_{Q^*_1} w^{2\beta+1}\right)^{1/(2\beta+1)} \leq CM^C(2\beta+1) \tag{3.58}$$

for all $\beta > 0$, and

$$\int_{Q^*_1} \frac{(p_0 w)^{2\beta+1}}{(2\beta+1)!} \leq (Cp_0 M^C e)^{2\beta+1} \leq \frac{1}{2^{2\beta+1}} \tag{3.59}$$

provided $p_0 = (2CM^C e)^{-1}$. The last inequality leads to the estimate

$$\int_{Q^*_R} \left(\frac{u}{K}\right)^{p_0} \leq CR^{n+2}, \tag{3.60}$$

where the constant K is defined in (3.4) and

$$M_R = 1 + (R^{1-n/2}\|b\|_{L^\infty L^2})^2 + R^{1-(n+2)/\bar{q}}\|b\|_{L^{\bar{q}}}.$$

We apply Lemma 3.2 and (3.60) to the translated in time cylinder $Q^*_R(0, -4R^2)$ and obtain

$$\int_{Q^*_R(0,-4R^2)} \left(\frac{u}{K}\right)^{p_0} \leq CR^{n+2} \tag{3.61}$$

with $K = \exp(\int_{B_{3R}} \eta^2(x) \log u(x, 0)\, dx)$.

If u is a supersolution to (2.1), then $\log_+(K/u)$ is a subsolution to (2.1). The last ingredient in the proof of Lemma 3.1 is the following result.

Lemma 3.5. *We have*

$$\sup_{Q_R} \log_+ \left(\frac{K}{u}\right) \leq C\left(1 + (R^{1-(n+2)/\bar{q}}\|b\|_{L^{\bar{q}}})^{1/(2-(n+2)/\bar{q})}\right)^{C(n)}, \tag{3.62}$$

where

$$K = \exp\left(\int_{B_{3R}} \eta^2(x) \log u(x,0)\, dx\right). \tag{3.63}$$

Proof of Lemma 3.5. We apply Lemma 2.1 for the positive subsolution $\log_+(K/u)$ to (2.1) with $p \in (0,1)$ to obtain

$$\sup_{Q_R} \log_+ \frac{K}{u} \leq C\left(1 + (R^{1-(n+2)/\bar{q}}\|b\|_{L^{\bar{q}}})^{1/(2-(n+2)/\bar{q})}\right)^{(n+2)/p} R^{-(n+2)/p} \left\|\log_+ \frac{K}{u}\right\|_{L^p(Q_{2R})}. \tag{3.64}$$

Now, let $v = \log(u/K)$ with K given by (3.63). We have $v = -\log(K/u)$ and $\log_+(K/u) = \log_-(u/K)$. The choice of K implies that

$$\int_{B_{3R}} \eta^2(x) v(x,0)\, dx = 0.$$

We may proceed as in the proof of Lemma 3.2 to conclude

$$\left\|\log_+ \frac{K}{u}\right\|_{L^p(Q_{2R})} \leq CR^{(n+2)/p}, \tag{3.65}$$

which, combined with (3.64) proves (3.62). □

Proof of Lemma 3.1. Lemma 3.5 is, actually, an upper bound on K, or a lower bound on $\inf_{Q_R} u$:

$$K \leq C \exp\left(\left(1 + (R^{1-(n+2)/\bar{q}}\|b\|_{L^{\bar{q}}})^{1/(2-(n+2)/\bar{q})}\right)^{C(n)}\right) \inf_{Q_R} u, \tag{3.66}$$

which together with (3.60) gives

$$\left(CR^{-n-2} \int_{Q_R^*(0,-4R^2)} u^{p_0}\right)^{1/p_0} \leq K \leq C \exp\left(1 + (R^{1-(n+2)/\bar{q}}\|b\|_{L^{\bar{q}}})^{1/(2-(n+2)/\bar{q})}\right)^{C(n)} \inf_{Q_R} u. \tag{3.67}$$

Thus, the proof of Lemma 3.1 is complete. □

Proof of Theorem 1.1. The Harnack inequality (1.6) is obtained as a direct consequence of Lemmas 2.1 and 3.1. Indeed, combining the estimates (2.2) and (3.1), we conclude

$$\sup_{Q_{R/2}(0,-3R^2)} u \leq \left(C + C(R^{1-(n+2)/\bar{q}}\|b\|_{L^{\bar{q}}})^{1/(2-(n+2)/\bar{q})}\right)^{C(n)/p_0} \inf_{Q_R} u \tag{3.68}$$

for any Lipschitz solutions u to (2.1), where $p_0 = 1/(CM_R^C)$ and M_R given in (3.2). □

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