Continuity of the solution yields $B_1 = B_2$ for $x = 0$. Requiring that the solution is bounded for $x \to \infty$ yields

$$v(x) = B \cdot \exp(-|x|/\lambda).$$

**Determining B:**
In an environment $[-\varepsilon, \varepsilon]$ around zero we get

$$\lambda^2 \frac{d^2v}{dx^2} = v - r_m i_e. \tag{4.15}$$

Let $I_e$ be the injected current. Then

$$i_e = \frac{I_e}{2\pi a \Delta x}; \quad \Delta x = 2\varepsilon$$

and we can compute $\frac{d^2v}{dx^2}$:

$$\frac{dv}{dx} = \begin{cases} -\frac{B}{\lambda}\exp(-x/\lambda) & x > 0 \\ \frac{B}{\lambda}\exp(-x/\lambda) & x < 0 \end{cases}$$

Therefore there is a discontinuous jump $\frac{2B}{\lambda}$ in $\frac{d^2v}{dx^2}$.
$\Rightarrow$ The difference quotient for $\frac{d^2v}{dx^2}$ around zero yields

$$\frac{d^2v}{dx^2} \approx -\frac{2B}{\lambda \Delta x} \tag{4.16}$$

$$\Rightarrow \lambda^2 \frac{d^2v}{dx^2} = \underbrace{v}_{v - v_{rest} \text{ small}} - r_m i_e = -r_m i_e \tag{4.17}$$

$$\Leftrightarrow \lambda^2 \left(-\frac{2B}{\lambda \Delta x}\right) = -r_m \frac{I_e}{2\pi a \Delta x} \Rightarrow B = \frac{r_m \cdot I_e}{4\pi a \lambda} \tag{4.18}$$

$$\text{With} \quad R_\lambda := \frac{r_m}{2\pi a \lambda} \quad \text{we get}$$

$$v(x) = \frac{I_e R_\lambda}{2} \exp\left(-\frac{|x|}{\lambda}\right) \tag{4.19}$$

## 4.2 Multi-Compartment models

Complex neuronal structures are difficult to integrate into an analytic approach. We can, however, approximate neuronal morphology by identifying a finite number of points on the cellular domain, and register diameter and connectivity data (discretization of the computational domain). The underlying model equations then transform into a system of coupled ordinary differential equations, that can be solved numerically.

- Compartmentalize neuron morphology into finite number of cylinders
- Each compartment is considered to be isopotential, i.e. associate one degree of freedom with each cylinder
- Solve for the membrane potential in each compartment $\mu$

**Flux balance in compartment $\mu$ yields:**

$$c_m \frac{dV_\mu}{dt} = -i_m^\mu + \frac{I_e^\mu}{A_\mu} + g_{\mu,\mu+1}(V_{\mu+1} - V_\mu) + g_{\mu,\mu-1}(V_{\mu-1} - V_\mu) \tag{4.20}$$

in an unbranched cable.
Here, $g_{\mu,\mu+1}$ describes the axial flow of charges between coupled compartments $\mu$ and $\mu + 1$.
From Ohm's law we get (assuming $\mu$ and $\mu + 1$ have the same size):

$$\begin{aligned}
&\Rightarrow \quad R_L = \frac{r_L L}{\pi a^2} \qquad (4.21)\\
&\Rightarrow \quad I^\mu = \frac{\pi a^2}{r_L L} \cdot (V_{\mu+1} - V_\mu)\\
&\Rightarrow \quad i^\mu = \underbrace{\frac{1}{2\pi a L}}_{\text{surface area of } \mu'} \cdot \frac{\pi a^2}{r_L L} (V_{\mu+1} - V_\mu) = \underbrace{\frac{a}{2 r_L L^2}}_{g_{\mu,\mu+1}} (V_{\mu+1} - V_\mu)\\
&\Rightarrow \quad g_{\mu,\mu+1} = \frac{a}{2 r_L L^2}
\end{aligned}$$

**Differently sized compartments**

Now, let $L_\mu$ be the length and $a_\mu$ the radius of $\mu$ ($L_{\mu+1}, a_{\mu+1}$, respectively)
The axial resistance between the center points of $\mu$ and $\mu + 1$ then is

$$g_{\mu,\mu+1} = \frac{1}{\frac{r_L L_\mu}{2\pi a_\mu^2} + \frac{r_L L_{\mu+1}}{2\pi a_{\mu+1}^2}} \cdot \underbrace{\frac{1}{2\pi a_\mu L_\mu}}_{\text{Oberfläche } \mu} \tag{4.22}$$

$$g_{\mu,\mu+1} = \frac{a_\mu a_{\mu+1}^2}{r_L L_\mu (a_{\mu+1}^2 + a_\mu^2 L_{\mu+1})} \tag{4.23}$$

These values can be precomputed. Solving the cable equation on a y-shaped domain consisting of two compartments on the main branch and one compartment each on the branches and a current injection $I_e$ in the first compartment yields the following set of coupled equations:

$$
\begin{aligned}
c_m \frac{dV_1}{dt} &= -i_1 + \frac{I_e}{A_1} + g_{1,2}(V_2 - V_1) \\
c_m \frac{dV_2}{dt} &= -i_2 + g_{2,3}(V_3 - V_2) + g_{2,1}(V_1 - V_2) + g_{2,4}(V_4 - V_2) \\
c_m \frac{dV_3}{dt} &= -i_3 + g_{3,2}(V_2 - V_3) \\
c_m \frac{dV_4}{dt} &= -i_4 + g_{4,2}(V_2 - V_4)
\end{aligned}
$$

This can be written in matrix/vector format and solved numerically:

$$
c_m \begin{pmatrix} \frac{dV_1}{dt} \\ \frac{dV_2}{dt} \\ \frac{dV_3}{dt} \\ \frac{dV_4}{dt} \end{pmatrix} = - \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{pmatrix} + \begin{pmatrix} \frac{I_e}{A_1} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -g_{1,2} & g_{1,2} & 0 & 0 \\ g_{2,1} & -(g_{2,3} + g_{2,1} + g_{2,4}) & g_{2,3} & 0 \\ 0 & g_{3,2} & -g_{3,2} & 0 \\ 0 & g_{4,2} & 0 & -g_{4,2} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}
$$

General purpose simulators, such as NEURON, solve such coupled ODE equations numerically. There is a large body of numerical methods dedicated to solving such problems, which will be subject to a later chapter.