

## 3 Modeling Neurons I

In this first chapter on modeling neurons, we will derive simple models that only consider the temporal evolution of the membrane potential. Neglecting the spatial extension of neurons can be motivated by the fact that neurons may be assumed electrotonically compact, thus the detailed spatio-temporal dynamics can be neglected. This assumption is, however, only valid to a certain extent.

In a first step we will derive the connection between the membrane potential and charge exchange across the membrane. Given the equation

$$C_m V = Q \quad (3.1)$$

the temporal change of the membrane potential  $V_m$  can be written as

$$C_m \frac{dV}{dt} = \frac{dQ}{dt} \quad (3.2)$$

Here,  $\frac{dQ}{dt}$  is comprised of membrane channel activity, synaptic currents, and/or electrode currents.

$$\begin{aligned} \Rightarrow Q &:= \sum \text{contributions} = -I_m + I_e + I_s \\ \Rightarrow C_m \frac{dV}{dt} &= -I_m + I_e \\ \Leftrightarrow c_m \frac{dV}{dt} &= -i_m + \frac{I_e}{A} \quad (\text{RECAP: } i_m = \sum_i g_i (V - E_i)) \end{aligned}$$

### 3.1 Integrate-and-Fire models

Integrate and Fire (I&F) models neglect the biophysics of action potentials. Therefore, we can only model the changes in  $V_m$  up to the action potential threshold ( $\approx -55 - (-50)$  mV). We will need to add the additional condition of resetting  $V_m$  to the resting (or reset) potential  $V_{reset}$ .

#### 3.1.1 Leaky I&F model

The *leaky Integrate and Fire* model only considers passive properties of the membrane. Equation (3.2) can be written as

$$\begin{aligned} c_m \frac{dV}{dt} &= -g_L (V - E_L) + \frac{I_e}{A} \\ \underbrace{\frac{\tau_m}{c_m r_m}}_{\substack{\leftarrow \frac{1}{g_L} \\ r_m}} \frac{dV}{dt} &= E_L - V + R_m I_e \end{aligned} \quad (3.3)$$

with the extra condition

$$V \equiv V_{th} \Rightarrow AP \rightsquigarrow V = V_{reset} \quad (3.4)$$

### Solving the differential equation (3.3)

1. assuming that  $I_e$  is constant and given the initial condition  $V(t_0)$ , we can derive the following analytic solution:

$$V(t) = E_L + R_m I_e + (V(t_0) - (E_L + R_m I_e)) e^{-\frac{t-t_0}{\tau_m}} \quad (3.5)$$

2. If  $I_e$  is not constant over time, we can approximate  $I_e$  with a piecewise linear function. Solve  $V(t + \Delta t)$  with  $I_e|_{\Delta t}$  constant:

$$\begin{aligned} \Rightarrow V(t + \Delta t) &= E_L + R_m I_e + (V(t) - (E_L + R_m I_e)) e^{-\frac{\Delta t}{\tau_m}} \\ \Rightarrow \text{solve at discrete timepoints } t_0 + k\Delta t. \\ \Delta t \rightarrow 0 &\Rightarrow \text{Convergence to continuous solution} \end{aligned}$$

### 3.1.2 Measurable quantities with leaky I&F model

#### 1. Firing rate

Experimentally the firing rate  $r_{fire}$  can be recorded by injecting a constant current  $I_{inj}$  and then measuring the time between action potentials  $t_{fire}$ .

**Definition 6.** *The firing rate is defined as*

$$r_{fire} := \frac{1}{t_{fire}} \quad (3.6)$$

The rate  $r_{fire}$  can also be computed by using the leaky I&F model. We need to solve

$$\begin{aligned} V(t_{fire}) &= V_{th} = E_L + R_m I_{inj} + (V_{reset} - E_L - R_m I_{inj}) \exp\left(-\frac{t_{fire}}{\tau_m}\right) \\ \Rightarrow \frac{V_{th} - (E_L + R_m I_{inj})}{V_{reset} - E_L - R_m I_{inj}} &= \exp\left(-\frac{t_{fire}}{\tau_m}\right) \\ \Leftrightarrow t_{fire} &= \tau_m \ln\left(\frac{V_{reset} - E_L - R_m I_{inj}}{V_{th} - E_L + R_m I_{inj}}\right) \\ \Rightarrow r_{fire} &= \frac{1}{\tau_m} \ln\left(\frac{V_{reset} - E_L - R_m I_{inj}}{V_{th} - E_L + R_m I_{inj}}\right)^{-1} \end{aligned} \quad (3.7)$$

We need to guarantee that

1.  $\ln\left(\frac{V_{reset} - E_L - R_m I_{inj}}{V_{th} - E_L + R_m I_{inj}}\right) \neq 0$ . This is true for  $V_{reset} \neq V_{th}$ .
2.  $\frac{V_{reset} - E_L - R_m I_{inj}}{V_{th} - E_L + R_m I_{inj}} > 0$

Since the driving force  $V_{reset} - E_L$  needs to be overcome for action potential initiation, we know  $V_{reset} - E_L < R_m I_{inj}$ . Thus, we need to guarantee

$$\begin{aligned} & V_{tk} - E_L + R_m I_{inj} \stackrel{!}{<} 0 \\ \Leftrightarrow & R_m I_{inj} > V_{tk} - E_L \\ \Rightarrow & r_{fire} = \begin{cases} \frac{1}{\tau_m} \cdot \ln \left( \frac{V_{reset} - E_L - R_m I_{inj}}{V_{th} - E_L - R_m I_{inj}} \right)^{-1} & R_m I_{inj} > V_{th} - E \\ 0 & \text{else} \end{cases} \end{aligned}$$

For  $I_{inj}$  large enough:  $R_m I_{inj} + E_L > V_{th}$

$$\Rightarrow r_{fire} = \left( \tau_m \ln \left( \frac{V_{reset} - V_{th}}{V_{th} - E_L - R_m I_{inj}} \right) \right)^{-1}$$

Using Taylor expansion we can approximate  $\ln(1+x) \approx x$  and therefore:

$$r_{fire} \approx \left( \tau_m \left( \frac{V_{reset} - V_{th}}{V_{th} - E_L - R_m I_{inj}} \right) \right)^{-1} = \frac{V_{th} - E_L - R_m I_{inj}}{\tau_m (V_{reset} - V_{th})}$$

So we see that the firing rate develops linearly with  $I_{inj}$ .

### Modeling variable firing rates

Experimental data suggests that firing rates are typically not constant. Why do firing rates change? Because active channels depolarize and hyperpolarize the cell. Hyperpolarization through  $K^+$  channels reduces the firing rate. We can consider mimicking this  $K^+$ -channel influence by extending the leaky I&F model:

$$\tau_m \frac{dV}{dt} = E_L - V - \underbrace{r_m g_{sra} (V - E_K)}_{\text{new!}} + R_m I_e \quad (3.8)$$

Introducing a spike rate adaption conductance  $g_{sra}$ , which varies over time lets us control spike rate adaption over time. Let us assume the following for  $g_{sra}$ :

$$\tau_{sra} \frac{dg_{sra}}{dt} = -g_{sra}$$

By updating the conductance in each time step  $g_{sra} \rightarrow g_{sra} + \Delta g_{sra}$  we can choose  $\tau_{sra}$  to fit experimental data.

## 2. Binning of spikes

Choosing a time window  $t_i$  lets us define the number of spikes per time bin:

$$S_i := \frac{1}{t_i} \sum_{j=0}^{t_i} s_j \quad (3.9)$$

Comparing  $S_i (i = 1 \dots n)$  for individual neurons can inform us about how, e.g. visual input is processed. Evaluating  $S_i$  with respect to changes in the bin size (i.e. time window) allows us to investigate whether this quantity is invariant.

**Example:** Visual input (e.g. video sequence). Evaluate

1. firing rate of single neurons and their distribution
2.  $S_i$  of single neurons and their distribution
3.  $S_i^k$  for invariance

⇒ Assign scenes (faces, scenery) to above quantities.

### I&F + refractory phase

Introducing a time window in which no spikes are permitted, we can mimic the refractory phase without adding the biophysics of channel types. Alternatively we can add additional terms of the type  $g_X(V_m - E_X)$ , with  $\frac{dg}{dt} \neq 0$ .

### Summary

1. Integrate and Fire models are easy to handle
2. we can compute statistical information, such as firing rates, inter-spike intervals, spike count distributions etc. and match this data to specific information coding.
3. Integrate and Fire models have no spatial resolution (which is what makes them easy to handle)
4. No modeling of biophysical properties

Points 3 + 4 are the motivation for us to study membrane dynamics in more detail and add spatial resolution to the model.

### 3.1.3 Voltage-dependent conductances

Changes in the membrane potential lead to stochastic opening and closing of ion channels. In what way does this behavior depend on membrane potential and other factors (like synaptic neurotransmitters and other ion types)?

**Remember:** Hodgkin and Huxley set out to describe the biophysical behavior of channels in a deterministic approach. This is only valid if channels are present in large numbers. For postsynaptic receptors, for instance, this is typically not the case.

Suppose now that the opening probability of a channel is proportional to the fraction of open channels, relative to the total number.