To determine $\alpha_1, \ldots, \alpha_3$, we need to compute the solution at an additional point, e.g.

$$u_{i+\frac{1}{2}} = u_h\left(\frac{x_i + x_{i+1}}{2}\right).$$

At the support nodes we get:

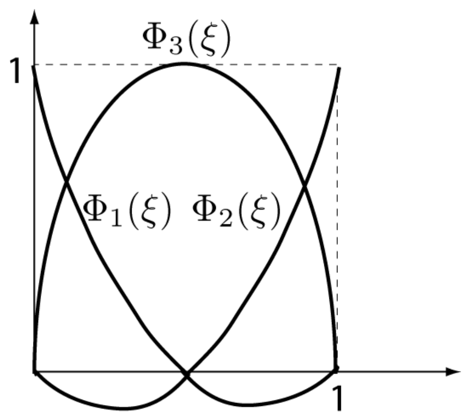
$$\begin{aligned}
u_i &= u_h(0) = \alpha_1 \\
u_{i+1} &= u_h(1) = \alpha_1 + \alpha_2 + \alpha_3 \\
u_{i+\frac{1}{2}} &= u_h\left(\frac{1}{2}\right) = \alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{4}\alpha_3
\end{aligned}$$

Solving for α_i, $i = 1, \ldots, 3$ leads to

$$u_h(\xi) = u_i\Phi_1(\xi) + u_{i+1}\Phi_2(\xi) + u_{i+\frac{1}{2}}\Phi_3(\xi)$$

with

$$\begin{aligned}
\Phi_i(\xi) &= 2\left(\xi - \frac{1}{2}\right)(\xi - 1) \\
\Phi_2(\xi) &= 2\xi\left(\xi - \frac{1}{2}\right) \\
\Phi_3(\xi) &= 4\xi\,(1 - \xi\ )
\end{aligned}$$

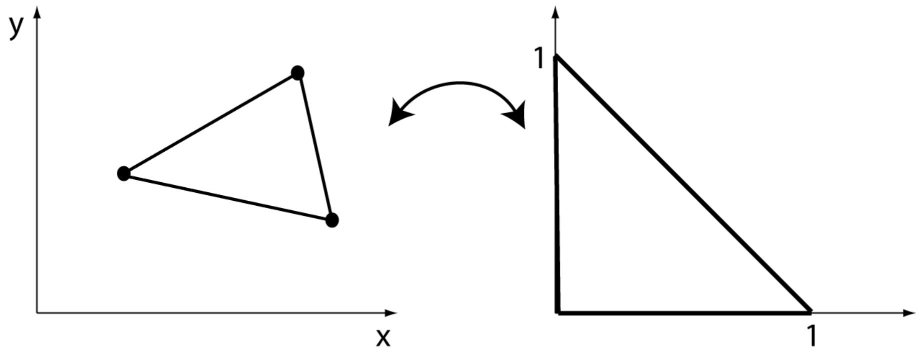


The sub-matrices A_{I_k} are now 3×3 matrices and can be computed like in the linear case.

Note 9. *The gradients $\nabla\Phi_i(\xi)$ are no longer constant over the reference element. Thus, one needs to employ a numerical integral solver.*

5.11.4 Finite Elements in $\mathbb{R}^2$

Let us translate the previous concept to the two-dimensional case. Grid elements are now triangles.



The affine transformation is a bi-linear, bijective, mapping with the following properties:

$$
\begin{aligned}
x &= x_1 + (x_2 - x_1)\xi + (x_3 - x_1)\eta \\
y &= y_1 + (y_2 - y_1)\xi + (y_3 - y_1)\eta \\
\Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
\begin{pmatrix} \xi \\ \eta \end{pmatrix} &= \underbrace{\begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}^{-1}}_{A^{-1}} \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix} \\
&= \frac{1}{\det A} \begin{pmatrix} y_3 - y_1 & x_1 - x_3 \\ y_1 - y_2 & x_2 - x_1 \end{pmatrix} \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix}
\end{aligned}
$$

with $\det A = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$. For the partial derivatives we get:

$$
\begin{aligned}
u_x &= u_\xi \xi_x + u_\eta \eta_x \\
u_y &= u_\xi \eta_y + u_\eta \eta_y
\end{aligned}
$$

and

$$
\begin{aligned}
\xi_x &= \frac{y_3 - y_1}{\det A} \\
\eta_x &= \frac{y_2 - y_1}{\det A} \\
\xi_y &= \frac{x_3 - x_1}{\det A} \\
\eta_y &= \frac{x_2 - x_1}{\det A}
\end{aligned}
$$

With $\nabla \Phi_i$, $i = 1, 2, 3$ and $\xi_x, \xi_y, \eta_x, \eta_y$ the matrix entries of A_{I_k} are fully determined.

Choosing the Φ_i: linear elements

We can write our solution as

$$\begin{aligned} u_h(\xi,\eta) &= \alpha_1 + \alpha_2\xi + \alpha_3\eta \\ u_j &:= u_h(\overline{P_j}), \quad j = 1,2,3 \end{aligned}$$

$\overline{P_j}$ are the corner points of the reference triangle. It follows:

$$\begin{aligned} u_1 &= u_h(0,0) = \alpha_1 \\ u_2 &= u_h(1,0) = \alpha_1 + \alpha_2 \\ u_3 &= u_h(0,1) = \alpha_1 + \alpha_3 \end{aligned}$$

and

$$u_h(\xi,\eta) = u_1 + (u_2 - u_1)\xi + (u_3 - u_1)\eta = (1 - \xi - \eta)u_1 + \xi u_2 + \eta u_3.$$

With $\Phi_1 = 1 - \xi - \eta$, $\Phi_2 = \xi$, $\Phi_3 = \eta$ we get

$$u_h(\xi,\eta) = u_1\Phi_1 + u_2\Phi_2 + u_3\Phi_3.$$

with

$$\begin{aligned} \Phi_i(\overline{P_j}) &= \delta_{ij} \quad i,j = 1,2,3 \\ \sum_{i=1}^{3} \Phi_i(\xi,\eta) &= 1 \quad \xi,\eta \in \bar{T} \end{aligned}$$