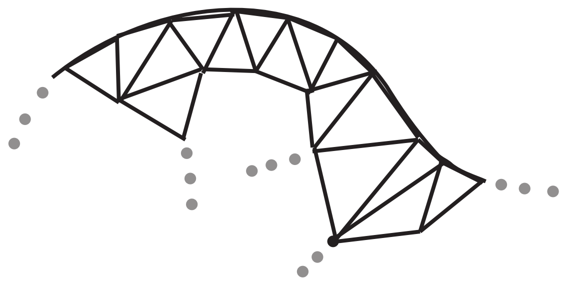


**Definition 15.** *(Finite Element space)*
*Given discretization $\Omega_h$ of $\Omega \subset \mathbb{R}^d$, the space*

$$V_h^p(\mathcal{T}) = \{u \in H^1 : \text{ für alle } T \in \mathcal{T} : u|_T \in \mathbb{P}_p\}$$

*is called the conforming Finite Element space of order p.*

From the example of Courant a general method for $\mathbb{R}^d$, $d = 1, 2, 3$, can be derived.

### 5.11.3 Finite Elements in $\mathbb{R}^1$

Let us consider the Helmholtz equation

$$-\Delta u + u = f \text{ with } a(u, v) = \int_\Omega (\nabla u \nabla v + uv) dx \tag{5.39}$$

Let $\mathcal{N} = \{a = x_0, x_1, x_2, \ldots, x_{N+1} = b\}$ be the discretization of domain $[a, b]$ with element width $h_i = x_{i+1} - x_i$ and local basis functions $\varphi_i$ with the following property:

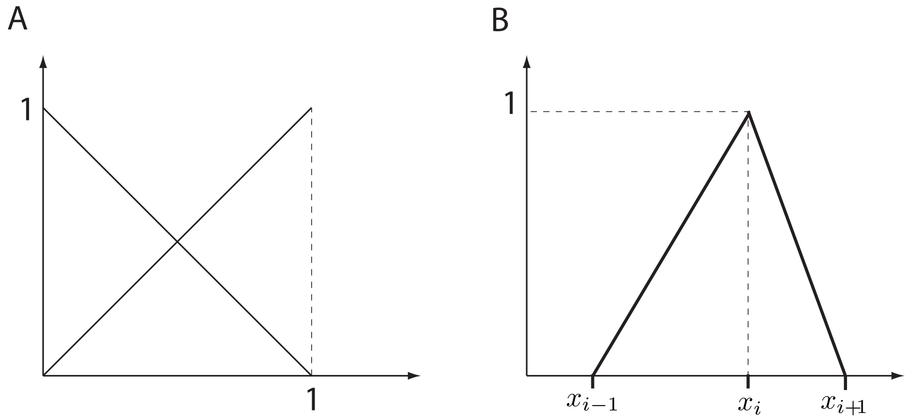
$$\varphi_i(x_j) = \delta_{ij}$$

Then the solution $u_h$ has the form

$$u_h = \sum_{i=1}^{N} a_i \varphi_i.$$

#### Computing $u_h$ on a reference interval

The basis functions $\varphi_i$ can be expressed by form functions $\Phi_i$. Using a bijective, affine, transformational mapping, the interval $[x_i, x_{i+1}]$ can be mapped to the reference interval $[0, 1]$. The solution $u_i$ can then be computed on the interval $[0, 1]$, expressed by the form functions and later transferred back to the original element.




**Transformational mapping**

Let $I_i = [x_i, x_{i+1}]$ and $\xi \in [0, 1]$. Then the following functions can be defined:

$$\begin{aligned} x_{I_i} : [0,1] &\longrightarrow I_i \\ \xi &\longmapsto x_i + h_i\xi \\ \xi_{I_i} : I_i &\longrightarrow [0,1] \\ x &\longmapsto \frac{(x - x_i)}{h_i} \end{aligned}$$

This defines a bijection between $[0, 1]$ and $[x_i, x_{i+1}]$. On the reference interval the solution can be expressed as:

$$u_h(\xi) = \alpha_1 + \alpha_2\xi.$$

Here, $u_i = u_h(0) = \alpha_1$ and $u_{i+1} = u_h(1) = \alpha_1 + \alpha_2$.

$$\begin{aligned} \Rightarrow u_h(\xi) &= \alpha_1 + \alpha_2\xi = u_i + (u_{i+1} - u_i)\xi \\ &= (1 - \xi)u_i + \xi u_{i+1} = u_i\Phi_1(\xi) + u_{i+1}\Phi_2(\xi) \end{aligned}$$

**Note 8.** *The following property applies to the form functions:*

$$\forall \xi \in [0, 1] : \Phi_1(\xi) + \Phi_2(\xi) = 1$$

For the basis functions we get:

$$\varphi_i(x) = \begin{cases} \Phi_2(\xi(x)) & x \in I_{i-1} \\ \Phi_1(\xi(x)) & x \in I_i \\ 0 & else \end{cases}$$

Now we can compute the matrix entries of the linear system in terms of the form functions.

### Computing the system matrix entries

We need to compute $a(\varphi_i, \varphi_j)$ of system matrix $A$ (in our case for the Helmholtz problem):

$$a(\varphi_i, \varphi_j) = \sum_{k=1}^{N} \int_{I_k} \nabla\varphi_i(x)\nabla\varphi_j(x) + \varphi_i(x)\varphi_j(x)\, dx.$$

The functions $\varphi_i, \varphi_j$ can be replaced by $\Phi_n$ and $\Phi_m$ ($n, m \in \{1, 2\}$).

$$\begin{aligned}
\Rightarrow (A_{I_k})_{nm} &= \int_{I_k} \nabla_x\Phi_n(\xi(x))\nabla_x\Phi_m(\xi(x)) + \Phi_n(\xi(x))\Phi_m(\xi(x))dx \\
&= (x_{k+1} - x_k)\int_0^1 \nabla_\xi\Phi_n(\xi)\xi'(x(\xi))\xi'(x(\xi))\nabla_\xi\Phi_m(\xi) + \Phi_n\Phi_m d\xi \\
&= h_k \int_0^1 \frac{1}{h_k^2}\nabla\Phi_n\nabla\Phi_m + \Phi_n\Phi_m d\xi
\end{aligned}$$

We see that the matrix is comprised of $2 \times 2$ sub-matrices $A_{I_k}$.
The matrix entries of $A_{I_k}$ are:

$$\begin{aligned}
(A_{I_k})_{11} &= \int_0^1 \frac{1}{h_k}\nabla\Phi_1\nabla\Phi_1 + \Phi_1\Phi_1 \cdot h_k d\xi \\
&= \int_0^1 \frac{1}{h_k} + h_k(1-\xi)^2 d\xi = \frac{1}{h_k} + \frac{1}{3}h_k \\
(A_{I_k})_{12} &= \int_0^1 \frac{1}{h_k}\nabla\Phi_1\nabla\Phi_2 + \Phi_1\Phi_2 \cdot h_k d\xi \\
&= \int_0^1 -\frac{1}{h_k} + \xi(1-\xi) \cdot h_k d\xi \\
&= -\frac{1}{h_k} + \frac{1}{6}h_k \\
(A_{I_k})_{21} &= -\frac{1}{h_k} + \frac{1}{6}h_k \\
(A_{I_k})_{22} &= \frac{1}{h_k} + \frac{1}{3}h_k
\end{aligned}$$

$$\Rightarrow A_{I_k} = \tfrac{1}{h_k}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + h_k \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}$$

### Quadratic elements

Instead of linear approximation, we can use quadratic approximation (still low order) which should increase the approximation quality. This leads to

$$u_h(\xi) = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2 \text{ on } I = [0, 1].$$