5.11 Finite element method

The finite element method informs us about how to choose the basis of V_h. The guiding principles are

1. use basis functions of low order,
2. use basis functions with compact support.

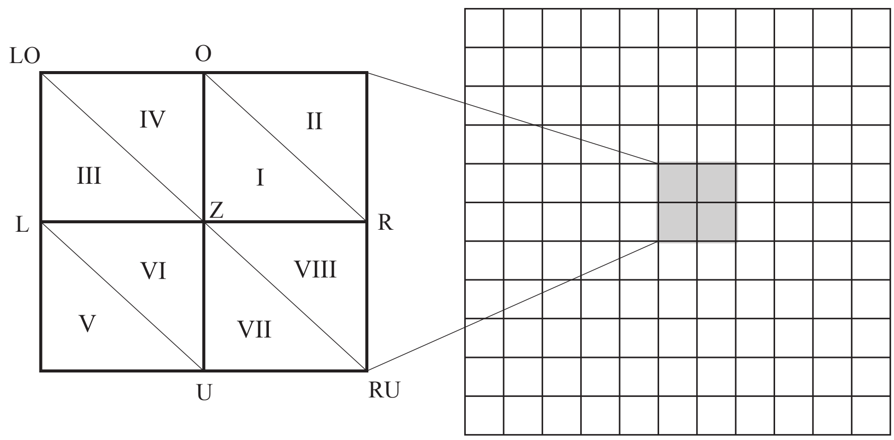
The approximation properties of low order functions are limited, but since they are easy to work with, they become a preferred choice. Better approximation can then be controlled by the grid resolution h. Compact support makes solving the matrix entries, for which integrals need to be computed numerically, easy.
These guiding principles will lead to larger systems, compared to the classical Galerkin idea of choosing few, but good basis functions. With ample computing resources the Finite element method outsources the computational tasks and leaves us with an easy to design algorithm. Courant demonstrated the key steps of the finite element method in the following example in 1943.

5.11.1 Courant example

Consider the Poisson problem

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega = (0,1) \times (0,1) \\ u &= 0 \text{ on } \Gamma \end{aligned}$$

Ω is subdivided into 8 triangles I-VIII, with nodes L, R, O, U, LO, RU and Z.



We choose V_h in the following way:

$$V_h = \{v \in C(\bar{\Omega}) : v \text{ linear and } v|_\Gamma = 0\}.$$

Thus, v can be expressed by

$$v(x,y) = a + bx + cy.$$

	I	II	III	IV	V	VI	VII	VIII
$\partial_1\psi_Z$	$\frac{-1}{h}$	0	$\frac{1}{h}$	0	0	$\frac{1}{h}$	0	$\frac{-1}{h}$
$\partial_2\psi_Z$	$\frac{-1}{h}$	0	0	$\frac{-1}{h}$	0	$\frac{1}{h}$	$\frac{1}{h}$	0

Since the values in the grid nodes are known, a, b and c can be determined. With N inner grid nodes we have $\dim V_h = N$, meaning we need N basis functions for V_h. Let the basis functions $\{\psi_i\}_{i=1}^N$ be defined by

$$\psi_i(K_j) = \delta_{ij} \text{ with } K_j = Node(j),\ j = 1, \ldots, N.$$

The basis function above the inner node Z therefore is

1. linear in each triangle, and
2. zero in the surrounding nodes.

They fulfill the finite element principles. We can now compute the derivatives of the basis function ψ_Z in the triangles I-VIII (see table) and solve the system

$$Au = b$$

with

$$A_{ij} = a(\psi_i, \psi_j).$$

In the Courant example we need to compute $a(\psi_Z, \psi_Z)$, $a(\psi_Z, \psi_O)$, $a(\psi_Z, \psi_U)$, $a(\psi_Z, \psi_L)$, $a(\psi_Z, \psi_R)$, $a(\psi_Z, \psi_{LO})$, and $a(\psi_Z, \psi_{RU})$.

For $a(\psi_Z, \psi_Z)$ we get:

$$\begin{aligned}
a(\psi_Z, \psi_Z) &= \int_\Omega (\nabla\psi_Z)^2 dxdy = \int_{I-VIII} (\nabla\psi_Z)^2 dxdy \\
&= 2 \cdot \int_{I+III+IV} \left((\partial_1\psi_Z)^2 + (\partial_2\psi_Z)^2\right) \\
&= 2 \cdot \int_{I+III} (\partial_1\psi_Z)^2 dxdy + 2 \cdot \int_{I+IV} (\partial_2\psi_Z)^2 dxdy \\
&= \frac{2}{h^2} \int_{I+III} dxdy + \frac{2}{h^2} \int_{I+IV} dxdy \\
\Rightarrow a(\psi_Z, \psi_Z) &= 4
\end{aligned}$$

For $a(\psi_Z, \psi_O)$ we get:

$$\begin{aligned}
a(\psi_Z, \psi_O) &= \int_{I-VIII} \nabla\psi_Z \nabla\psi_O dxdy \\
&= \int_{I+IV} \nabla\psi_Z \nabla\psi_O dxdy = \int_{I+IV} \partial_1\psi_Z\partial_1\psi_O + \partial_2\psi_Z\partial_2\psi_O dxdy \\
&= \int_{I+IV} \partial_2\psi_Z\partial_2\psi_O dxdy = \int_{I+IV} -\frac{1}{h}\cdot\frac{1}{h} dxdy \\
&= -\frac{1}{h}\cdot\int_{I+IV} dxdy = -1
\end{aligned}$$

From a symmetry argument we get:

$$a(\psi_Z, \psi_O) = a(\psi_Z, \psi_U) = a(\psi_Z, \psi_L) = a(\psi_Z, \psi_R) = -1$$

and we can verify that

$$a(\psi_Z, \psi_{RU}) = a(\psi_Z, \psi_{LO}) = 0$$

This leads to a matrix stencil of the form

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

Note 7. *The identity between the stencils of finite difference and finite element method does* not *hold in general.*

The Courant example leads us to the next steps:

1. We need a triangulation of the domain, with specific properties.
2. We need basis functions over the discrete computational domain.

5.11.2 Triangulations

A domain with curved, differentiable, boundaries can be approximated by piecewise linear functions.

Definition 14. *(valid triangulation)*
Subdivision $\mathcal{T} = \{T_1, T_2, \ldots, T_M\}$ of Ω into triangles is called valid, *if the following properties are fulfilled:*

1. $\bar{\Omega} = \bigcup_{i=1}^{M} T_i$.
2. $T_i \cap T_j$ *consists of exactly one point, then this point is corner point of T_i and T_j.*
3. $T_i \cap T_j$ *consists of multiple points, then $T_i \cap T_j$ is an edge of T_i and T_j.*

Points 2 and 3 define *conforming grids.*