if

$$a(u,v) = (f,v)_0 \quad \forall v \in H_0^1(\Omega) \tag{5.34}$$

$$a(u,v) := \int_\Omega \sum_{i,k} a_{ik}\partial_i u \partial_k v + a_0 uv dx \tag{5.35}$$

Theorem 7. *Let L be a second order differential operator. Then the boundary value problem*

$$\begin{aligned} Lu &= f \text{ in } \Omega \\ u &= 0 \text{ on } \Gamma \end{aligned}$$

has a weak solution in $H_0^1(\Omega)$ and is minimum of the problem

$$\frac{1}{2}a(v,v) - (f,v)_0 \longrightarrow \min \text{ in } H_0^1(\Omega)$$

Example 2. *For the model problem*

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \Gamma \end{aligned}$$

the corresponding bi-linear form is

$$a(u,v) = \int_\Omega \nabla u \nabla v dx.$$

The variational problem then is:
Find $u \in H_0^1(\Omega)$, such that

$$(\nabla u, \nabla v)_0 = (f,v)_0 \quad \forall v \in H_0^1(\Omega)$$

This u is found as solution of the minimization problem

$$\frac{1}{2}\int_\Omega \nabla u \nabla v dx - (f,v)_0 \longrightarrow \min .$$

5.10 Galerkin method

We now need to decide how to approximate the function space associated with the boundary value problem. Consider the variational problem
Find $u \in H^1(\Omega)$ with

$$a(u,v) = (f,v).$$

Example 3. *Let the boundary value problem be defined by*

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= u_0 \text{ on } \Gamma. \end{aligned}$$

The weak formulation then reads:
Find $u \in H_0^1(\Omega)$ with

$$\int_\Omega \nabla u \nabla v dx = \int_\Omega f v dx \quad \forall v \in H_0^1(\Omega)$$

with

$$\begin{aligned} a(u,v) &:= \int_\Omega \nabla u \nabla v dx \\ (f,v) &:= \int_\Omega f v dx \end{aligned}$$

Problem. We have to solve a minimization problem over an infinite-dimensional space $H^m(\Omega)$ or $H_0^m(\Omega)$. Thus, we need to find an approximating finite dimensional space for numerically solving the minimization problem.

Idea 2. *Replace the solution space V with V_h, where V_h is finite dimensional, i.e. has a finite basis.*

Transform above minimization problem to

$$J(v) := \frac{1}{2} a(v,v) - \langle l, v \rangle \longrightarrow \min_{V_h} \tag{5.36}$$

$u_h \in V_h$ is solution of the minimization problem, if

$$a(u_h, v) = \langle l, v \rangle \quad \forall v \in V_h.$$

The finite problem

Let $\{\psi_1, \psi_2, \dots, \psi_n\}$ be a basis of V_h. Then

$$a(u_h, v) = \langle l, v \rangle \quad \forall v \in V_h$$

is equivalent to

$$a(u_h, \psi_i) = \langle l, \psi_i \rangle \quad i = 1, 2, \dots, N.$$

$u_h \in V_h$ can be written as a linear combination of ψ_i:

$$u_h = \sum_{k=1}^N z_k \psi_k, \tag{5.37}$$

where z_k are the unknowns that need to be computed. This leads to an equation system

$$\sum_{k=1}^N a(\psi_k, \psi_i) z_k = \langle l, \psi_i \rangle \quad i = 1, 2, \dots N$$

by inserting $u_h = \sum_{k=1}^N z_k \psi_k$ in $a(u_h, \psi_i) = \langle l, \psi_i \rangle \, \forall i = 1, \dots, N$ and using the linearity of $a(\cdot, \cdot)$. With $A_{ik} := a(\psi_k, \psi_i)$ and $b_i := \langle, \psi_i \rangle$ the equation system can be written as

$$Az = b$$

Note 5. *If $a(.,.)$ is V-elliptic, then matrix A is positive definite:*

$$\begin{aligned} z^t A z &= \sum_{i,k} z_i A_{ik} z_k = a\left(\sum_k z_k \psi_k, \sum_i z_i \psi_i\right) \\ &= a(u_h, u_h) \geq C_E \|u_h\|_V^2 \end{aligned}$$

So how well does $u_h \in V_h$ approximate $u \in V$?

Lemma 3. *(Céa lemma)*
Let the bilinear form a be V-elliptic and bounded and $H_0^m(\Omega) \subset V \subset H^m(\Omega)$. Then

$$\|u - u_h\|_m \leq \frac{C_S}{C_E} \inf_{v_h \in V_h} \|u - v_h\|_m. \tag{5.38}$$

Proof. We know that

$$\begin{aligned} a(u, v) &= \langle l, v \rangle \quad \forall v \in V, \\ a(u_h, v) &= \langle l, v \rangle \quad \forall v \in V_h. \end{aligned}$$

Since $V_h \subset V$ we can write

$$a(u, v) - a(u_h, v) = a(u - u_h, v) = 0 \text{ for } v \in V_h.$$

With $v_h \in V_h$ we can express $v \in V_h$ by

$$v = v_h - u_h \in V_h.$$

$\Rightarrow a(u - u_h, v_h - u_h) = 0$. Since $a(\cdot, \cdot)$ is elliptic and bounded, we get

$$\begin{aligned} C_E \|u - u_h\|_m^2 &\leq a(u - u_h, u - u_h) = a(u - u_h, u - u_h) + a(u - u_h, v_h - u_h) \\ &\leq C_S \|u - u_h\|_m \|u - v_h\|_m \end{aligned}$$

It follows, that

$$\begin{aligned} \|u - u_h\|_m &\leq \frac{C_S}{C_E} \|u - v_h\|_m \quad \forall v_h \in V_h \\ \Rightarrow \|u - u_h\|_m &\leq \frac{C_S}{C_E} \inf_{v_h \in V_h} \|u - v_h\|_m \end{aligned}$$

□

Note 6. *Since $a(u - u_h, v) = 0$ the approximation error $u - u_h$ is orthogonal to V.*

Conclusion from the Céa lemma

The better V_h approximates V, the smaller the distance $\|u - u_h\|_m$ between u and u_h becomes.