Proof. Let $u, v \in V$, $t \in \mathbb{R}$. Consider

$$\begin{aligned} J(u+tv) &= \frac{1}{2}a(u+tv, u+tv) - f(u+tv) = \\ &= \frac{1}{2}\big(a(u,u) + 2ta(u,v) + t^2 a(v,v)\big) - f(u) - tf(v) \\ &= J(u) + t(a(u,v) - f(v)) + \frac{1}{2}t^2 a(v,v) \end{aligned}$$

1. Assume $a(u, v) = f(v)$ for $u \in V$:

$$\begin{aligned} \overset{t=1}{\Rightarrow} J(u+v) &= J(u) + (f(v) - f(v)) + \frac{1}{2}a(v,v) \\ &= J(u) + \frac{1}{2}a(v,v) > J(u) \end{aligned}$$

   $\Rightarrow u$ is minimum.

2. Let $u \in V$ be a minimum.

$$\begin{aligned} J(u+tv) &= J(u) + t(a(u,v) - f(v)) + \frac{1}{2}t^2 a(v,v) \\ \Rightarrow 0 = \frac{dJ(u+tv)}{dt} &= a(u,v) - f(v) + ta(v,v) \\ \Rightarrow J'(u+tv)|_{t=0} &= a(u,v) - f(v) \\ \Leftrightarrow a(u,v) &= f(v) \end{aligned}$$

$\Rightarrow a(u, v) = f(v) \iff u$ is minimum.

□

5.9.1 Second order elliptic differential operators

Consider the differential operator

$$Lu := -\sum_{i,k=1}^{n} \partial_i(a_{ik}\partial_k u) + a_0 u$$

with $a_0(x) \geq 0 \, (x \in \Omega)$. The associated problem

$$\begin{aligned} Lu &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$

can be assumed to be homogeneous:

$$\begin{aligned} w &:= u - g \\ \Rightarrow Lw &= f_1 \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

The following theorem sets up the connection between boundary value problem and variational problem.

Theorem 5. *(Minimal property)*
Every classical solution of the boundary value problem

$$-\sum_{i,k} \partial_i(a_{ik}\partial_k u) + a_0 u = f \text{ in } \Omega \tag{5.27}$$

$$u = 0 \text{ on } \partial\Omega \tag{5.28}$$

is solution of the minimization problem

$$J(v) := \int_\Omega \left(\frac{1}{2} \sum_{i,k} a_{ik}\partial_i v \partial_k v + \frac{1}{2} a_0 v^2 - f v \right) dx \to \min \tag{5.29}$$

for functions in $C^2(\Omega) \cap C^0(\bar{\Omega})$ *with zero boundary values.*

Proof. Green's formula states

$$\int_\Omega v\partial_i w + \partial_i v w dx = \int_{\partial\Omega} v \frac{\partial w}{\partial \vec{n}} ds$$

With $v|_{\partial\Omega} = 0$ follows

$$\int_\Omega v\partial_i w = -\int_\Omega \partial_i v w$$

and with $w = a_{ik}\partial_k u$

$$\int_\Omega v\partial_i(a_{ik}\partial_k u)dx = -\int_\Omega a_{ik}\partial_i v \partial_k u dx.$$

Set

$$a(u,v) := \int_\Omega \sum_{i,k} a_{ik}\partial_i u \partial_k v + a_0 uv dx$$

$$f(v) := \int_\Omega f v dx$$

and sum over i and j:

$$\Rightarrow \int_\Omega \sum_{i,j} v\partial_i(a_i k\partial_k u)dx = \underbrace{-\int_\Omega \sum_{i,j} a_{ik}\partial_i v \partial_k u dx}_{=a(u,v)-a_0 uv}.$$

Extending by $\int_\Omega f v dx$ delivers

$$a(u,v) - f(v) = \int_\Omega v \left(-\sum \partial_i(a_{ik}\partial_k u) + a_0 u - f \right) dx$$

$$= \int_\Omega v(Lu - f)dx \underset{\text{if } Lu=f}{=} 0$$

□

Thus, the characterization theorem states

$$\begin{gathered} u \text{ is solution of the variational problem and} \\ u \in C^2(\Omega) \cap C^0(\bar{\Omega}) \\ \Downarrow \\ u \text{ is classical solution} \end{gathered}$$

We now need to check whether a solution exists in $C^2(\Omega) \cap C^0(\bar{\Omega})$.

5.9.2 Existence and uniqueness of the variational problem

Consider the Dirichlet-Integral (energy functional)

$$J(u) := \int_\Omega |\nabla u|^2 dx = \int_\Omega \sum_{i=1}^n u_{x_i}^2(x) dx \tag{5.30}$$

From the minimal property we get

$$\begin{aligned} J(u) \to \min \Longleftrightarrow \Delta u &= 0 \text{ in } \Omega \\ u &= \varphi \text{ on } \Gamma \end{aligned}$$

Dirichlet principle

Dirichlet argued that since $J(u) \geq 0$ there must exist a minimum at some u.
$\Rightarrow$ There exists a solution to

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega \\ u &= \varphi \text{ on } \Gamma \end{aligned}$$

A counter example was formulated by Weierstraß (1870):
He considered the following problem:

$$J(u) = \int_0^1 u^2(x) dx \longrightarrow \min \text{ for } u \in C^0([0,1])$$

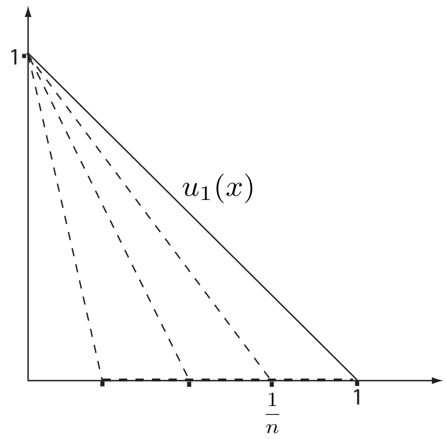
and $u(0) = 1$, $u(1) = 0$.
Now one can define the function sequence

$$u_n(x) = \begin{cases} 1 - nx & 0 \leq x \leq \frac{1}{n} \\ 0 & x > \frac{1}{n} \end{cases} \tag{5.31}$$

Thus, $\lim_{n \to 0} u_n = 0$, i.e. the infimum of $J(u)$ is

$$\inf J(u) = 0,$$

but is not reached for continuous functions and contradicts the Dirichlet principle. This example shows, that in order to guarantee a unique solution of the variational problem, the function space needs to be chosen wisely. This is summarized in the following theorem.



Existence and uniqueness

Theorem 6. *(Lax-Milgram)*
Let V be a closed and convex set in Hilbert space H and $a : H \times H \longrightarrow \mathbb{R}$ an elliptic bi-linear form. For every $l \in H'$ the variational problem

$$J(v) := \frac{1}{2}a(v,v) - \langle l, v\rangle \longrightarrow \min \tag{5.32}$$

has exactly one solution in V.

Proof. J is bounded from below:

$$\begin{aligned} J(v) &\geq \frac{1}{2}C_E\|v\|^2 - \|l\|\|v\| \\ &= \frac{1}{2}C_E\|v\|^2 - \|l\|\|v\| + \|l\|^2 - \|l\|^2 \\ &= \frac{1}{2C_E}(C_E^2\|v\|^2 - 2C_E\|l\|\|v\| + \|l\|^2) - \frac{1}{2C_E}\|l\|^2 \\ &= \frac{1}{2C_E}(C_E\|v\| - \|l\|)^2 - \frac{1}{2C_E}\|l\|^2 \geq -\frac{\|l\|^2}{2C_E} \end{aligned}$$

Now set $c_1 := \inf\{J(v) : v \in V\}$. Let (v_n) be a sequence with

$$\lim_{n\to\infty} v_n = c_1.$$

Then $C_E\|v_n - v_m\|^2 \leq a(v_n - v_m, v_n - v_m)$, since $a(.,.)$ is elliptic. With the help of the parallelogram equation

$$\|v_n + v_m\|^2 + \|v_n - v_m\|^2 = 2(\|v_n\|^2 + \|v_m\|^2) \tag{5.33}$$

we get

$$\begin{aligned} a(v_n + v_m, v_n + v_m) + a(v_n - v_m, v_n - v_m) &= 2(a(v_n, v_n) + a(v_m, v_m)) \\ \Leftrightarrow a(v_n - v_m, v_n - v_m) &= 2a(v_n, v_n) + 2a(v_m, v_m) - a(v_n + v_m, v_n + v_m) \end{aligned}$$

Applying this yields

$$\begin{aligned} C_E\|v_n - v_m\|^2 &\leq a(v_n - v_m, v_n - v_m) \\ &= 2a(v_n, v_n) + 2a(v_m, v_m) - a(v_n + v_m, v_n + v_m) \end{aligned}$$

With $J(v) = \frac{1}{2}a(v,v) - \langle l, v\rangle$ we get

$$\begin{aligned} 4J(v_n) &= 2a(v_n, v_n) - 4\langle l, v_n\rangle \\ &= 4J(v_n) - 4\langle l, v_n\rangle + 4J(v_m) - 4\langle l, v_m\rangle - \left(8 \cdot J\left(\frac{v_n + v_m}{2}\right) - 4\langle l, v_n + v_m\rangle\right) \\ &= 4J(v_n) + 4J(v_m) - 8J\left(\frac{v_n + v_m}{2}\right) \leq 4J(v_n) + 4J(v_m) - 8c_1 \end{aligned}$$

Demanding V to be convex yields

$$\frac{v_n + v_m}{2} \in V.$$

Since $\lim_{n\to\infty} v_n = c_1$, it follows $\lim_{n\to\infty} 4J(v_{n,m}) = 4c_1$.

$$\Rightarrow C_E\|v_n - v_m\|^2 \longrightarrow 0 \text{ for } n, m \to \infty,$$

so $\|v_n - v_m\|^2 \longrightarrow 0$.
$\Rightarrow (v_n)$ is a Cauchy sequence in H. H is a Hilbert space, therefore there exists a $u \in H$ and with V closed also $u \in V$ with $\lim_{n\to\infty} v_n = u$ and $J(u) = \lim_{n\to\infty} J(v_n) = \inf_{v\in V} J(v)$.
$\Rightarrow J(v) = \frac{1}{2}a(v,v) - \langle l, v\rangle \longrightarrow \min$ has a solution $u \in V$.

Uniqueness: Let u_1 and u_2 be solutions, and also limits of Cauchy sequences. Then $u_1, u_2, u_1, u_2, \ldots$ also converges.
$\Rightarrow u_1, u_2, u_1, u_2, \ldots$ is a Cauchy sequence. This, however only can hold if $u_1 = u_2 \Rightarrow$ Uniqueness.

□

5.9.3 Weak solution of the boundary value problem

In finite difference methods the convergence proofs called for strong regularity of the solution. Differentiability was considered in the strong sense. To use the function analytical properties of Hilbert spaces with integrability conditions (instead of classical differentiability), we make use of weak differentiability, which is consistent with strong differentiability almost everywhere.

Definition 13. *A function $u \in H_0^1(\Omega)$ is called* weak solution *of the second order boundary value problem*

$$\begin{aligned} Lu &= f \text{ in } \Omega \\ u &= 0 \text{ on } \Gamma \end{aligned}$$