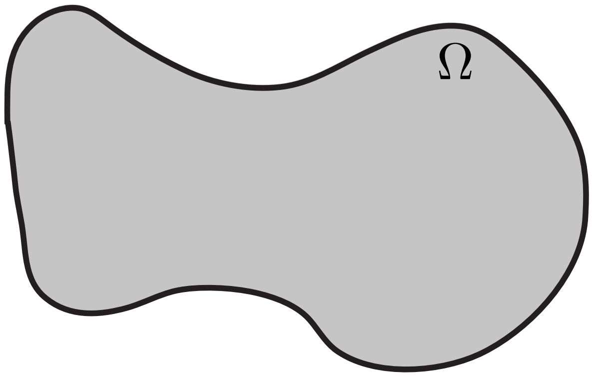
## 5.2 Solving the filter equation

In this section we will consider the filter equation on a structured image. Making use of the inherent geometric structure and the possibility of separating the time and space components of the diffusion equation in a numerical solve, we will discuss the *finite difference* method. Solving first in time (ODE solver), then in space (method of lines), allows us to consider the Poisson equation

$$-\Delta u = f \quad \text{on } \Omega \tag{5.16}$$

with a known right hand side $f$ on a domain $\Omega$.




The PDE is initially defined on a continuous domain, i.e. at infinitely many points.

**Idea 1.** *Choose a finite set of points in $\Omega$ in which $-\Delta u = f$ is satisfied. This leads to an approximation of the continuous domain.*

## 5.3 Domain discretization

Consider the unit square

$$\Omega = \{(x, y) : 0 < x < 1, 0 < y < 1\}. \tag{5.17}$$

We can then *discretize* the domain in the following way:

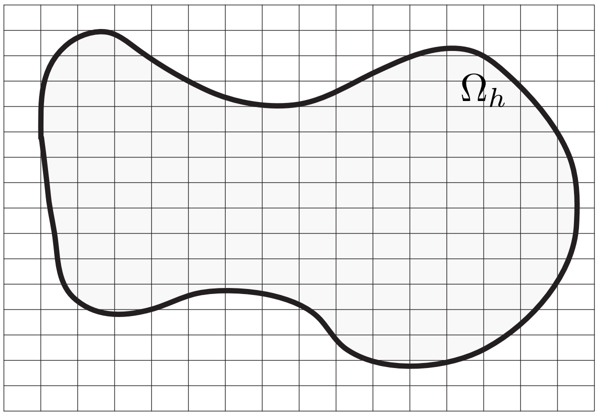
1. Define a structured grid $\Omega_h$. A step width $h$ gives us:

$$\Omega_h = \left\{(x, y) \in \Omega : \frac{x}{h}, \frac{y}{h} \in \mathbb{Z}\right\}.$$

2. Require the PDE to be fulfilled in $\Omega_h$.
    - replace $u(x)$ with $u_h(x)$. Here, $u(x)$ is the continuous and $u_h(x)$ the discrete solution.

- Approximate the differential operator:

$$\lim_{h\to 0} \frac{u(x+h)-u(x)}{h} \approx \frac{u(x+h)-u(x)}{h}$$




## 5.4 Approximation properties in $\mathbb{R}^1$

First, consider the one-dimensional problem

$$\begin{aligned} u''(x) &= f(x) \quad \text{in } \Omega = (0,1) \\ u(0) &= \varphi_0 \\ u(1) &= \varphi_1 \end{aligned}$$

Approximation of the first derivative can be chosen in multiple ways, e.g.:

1. $\delta^+ u(x) = \frac{u(x+h)-u(x)}{h}$
2. $\delta^- u(x) = \frac{u(x)-u(x-h)}{h}$
3. $\delta^0 u(x) = \frac{u(x+h)-u(x-h)}{2h}$

For the second derivative left and right differences $\delta^+$ und $\delta^-$ can be combined:

$$u''(x) = \delta^+\delta^- u(x) = \frac{\frac{u(x+h)-u(x)}{h} - \frac{u(x)-u(x-h)}{h}}{h} = \frac{u(x+h)-2u(x)+u(x-h)}{h^2} \tag{5.18}$$

**Lemma 1.** *Let* $[x-h, x+h] \subset \bar{\Omega}$. *Then*

1. $\delta^\pm u(x) = u'(x) + hR$ *with* $|R| \le \frac{1}{2}\|u''\|_\infty$
2. $\delta^0 u(x) = u'(x) + h^2 R$ *with* $|R| \le \frac{1}{6}\|u'''\|_\infty$

*3.* $\delta^+\delta^- u(x) = u''(x) + h^2 R$ *with* $|R| \leq \frac{1}{12}\|u^{(4)}\|_\infty$

**Proof.**
Point 1.:

$$\begin{aligned}
u(x \pm h) &= u(x) \pm hu'(x) + \frac{h^2}{2}u''(x) + \ldots \\
&= u(x) \pm hu'(x) + \frac{h^2}{2}u''(\xi), \text{ with } x \leq \xi \leq x+h \\
\Leftrightarrow \frac{u(x+h)-u(x)}{h} &= u'(x) + \frac{h}{2}u''(\xi) \\
&\leq u'(x) + \frac{h}{2}\|u''\|_\infty \\
&\Rightarrow 1.
\end{aligned}$$

Point 2.: We can apply Taylor expansion around $x \pm h$:

$$\begin{aligned}
u(x+h) &= u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(\xi) \\
u(x-h) &= u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u'''(\tilde{\xi})
\end{aligned}$$

Subtraction yields:

$$\begin{aligned}
\delta^0 u(x) &= \frac{2hu'(x) + \frac{h^3}{6}(u'''(\xi) + u'''(\tilde{\xi}))}{2h} \\
&= u'(x) + \frac{h^2}{12}(u'''(\xi) + u'''(\tilde{\xi}) \\
&\leq u'(x) + \frac{h^2}{12} \cdot 2\|u'''\|_\infty = u'(x) + \frac{h^2}{6}\|u'''\|_\infty \\
&\Rightarrow 2.
\end{aligned}$$

Point 3.: Consider Taylor expansion around $x \pm h$ up to the fourth order term:

$$\begin{aligned}
u(x+h) &= u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) + \frac{h^4}{24}u^{(4)}(\xi) \\
u(x-h) &= u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u'''(x) + \frac{h^4}{24}u^{(4)}(\tilde{\xi})
\end{aligned}$$

Addition of above equations and subtraction of $2u(x)$ and Division by $h^2$ yields

$$\begin{aligned}
\delta^+\delta^- u(x) &= \frac{h^2 u''(x) + \frac{h^4}{24}\left(u^{(4)}(\xi) + u^{(4)}(\tilde{\xi})\right)}{h^2} \\
\Rightarrow \delta^+\delta^- u(x) &= u''(x) + \frac{h^2}{24}\left(u^{(4)}(\xi) + u^{(4)}(\tilde{\xi})\right) \\
&\leq u''(x) + \frac{h^2}{12}\|u^{(4)}\|_\infty \\
&\Rightarrow 3.
\end{aligned}$$

□

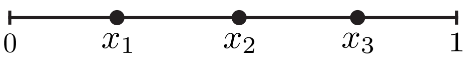
## 5.5 Setting up linear equations systems

Approximating the differential operators on a discrete computational domain leads to a finite linear equation system, which needs to be solved (numerically). We continue to consider

$$u''(x) = \Delta u(x) = f(x),$$

and approximate $\Delta \approx \Delta_h = \delta^+\delta^-$. That way we get

$$\delta^+\delta^- u(x) = f(x) + \mathcal{O}(h^2)$$




In matrix-vector notation above equation and domain can be written as

$$\delta^+\delta^- u(x) = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix} \cdot \begin{pmatrix} u_h(x_1) \\ u_h(x_2) \\ u_h(x_3) \end{pmatrix} = \begin{pmatrix} f(x_1) - \frac{1}{h^2}u(0) \\ f(x_2) \\ f(x_3) - \frac{1}{h^2}u(1) \end{pmatrix}.$$

The general case yields:

$$\frac{1}{h^2} \cdot \begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ & \ddots & & & \ddots & & \\ & & \ddots & & & \ddots & \\ 0 & 0 & \dots & 0 & 1 & -2 & 1 \\ 0 & 0 & \dots & 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ \\ \\ \vdots \\ \\ \\ u_n \end{pmatrix} = \begin{pmatrix} f_1 - \frac{u_0}{h^2} \\ f_2 \\ \\ \vdots \\ \\ f_{n-1} \\ f_n - \frac{u_n}{h^2} \end{pmatrix}.$$

Thus, we get a system

$$K_h \cdot u_h = f_h.$$

**Definition 1.** *$K_h$ is called* sparse*, if*

$$\#\{K_{i,j} \neq 0; i,j = 1 \dots n\} = \mathcal{O}(n)$$

## 5.6 Finite differences in $\mathbb{R}^2$

We now consider a two-dimensional domain $\Omega$ and $\Omega_h$.

$$\begin{aligned} \Omega &:= \{(x,y) : 0 < x < 1, 0 < y < 1\} \\ \Omega_h &:= \{(x,y) : (x,y) \in \Omega; x/h, y/h \in \mathbb{Z}\} \end{aligned}$$

with boundaries

$$\begin{aligned}\Gamma &:= \{(x,y) : x \in \{0,1\}, y \in \{0,1\}\} \\ \Gamma_h &:= \{(x,y) \in \Gamma : x/h, y/h \in \mathbb{Z}\}\end{aligned}$$

and the PDE

$$\begin{aligned}-\Delta u &= f \text{ in } \Omega \\ u &= \varphi \text{ on } \Gamma\end{aligned}$$

Finite difference discretization yields

$$-\Delta_h u_h := (-\delta_x^- \delta_x^+ - \delta_y^- \delta_y^+) u_h(x). \tag{5.19}$$

**Definition 2.** *The function $u_h$ is called* grid function *of $u$, i.e. the approximation of $u$ on the discrete domain $\Omega_h$.*

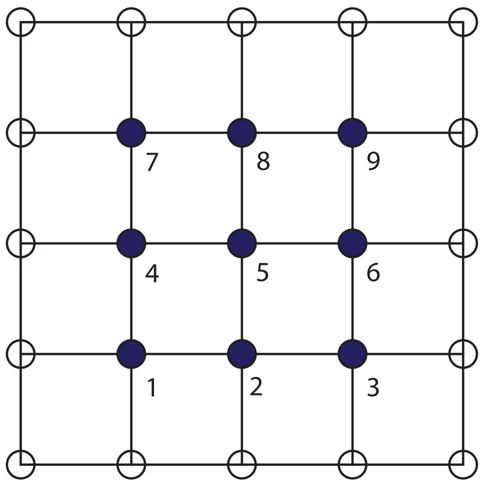
Application of the discrete Laplace operator $\Delta_h$ on the grid function $u_h$ produces

$$\begin{aligned}-\Delta_h u_h &= (-\delta_x^- \delta_x^+ - \delta_y^- \delta_y^+) u_h(x) \\ &= -\frac{1}{h^2}\left(u(x-h,y) + u(x+h,y) + u(x,y-h) + u(x,y+h) - 4u(x,y)\right).\end{aligned}$$

The function $u$ is evaluated at 5 grid points. Therefore, above equation is often referred to as *five point stencil.*

## Matrix-vector notation in $\mathbb{R}^2$

In the one-dimensional case there exists a natural grid node order, which defined the matrix structure. In $\mathbb{R}^2$ we need to choose a node ordering.



### Lexicographic ordering

Above lexicographic ordering follows a line by line numbering and yields the system

$$\frac{1}{h^2}\begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix} \cdot u_h = \tilde{f}_h \tag{5.20}$$

where $\tilde{f}_h$ denotes the right hand side $f$ with the entries of the boundary conditions. The above matrix has a block diagonal structure of the following form:

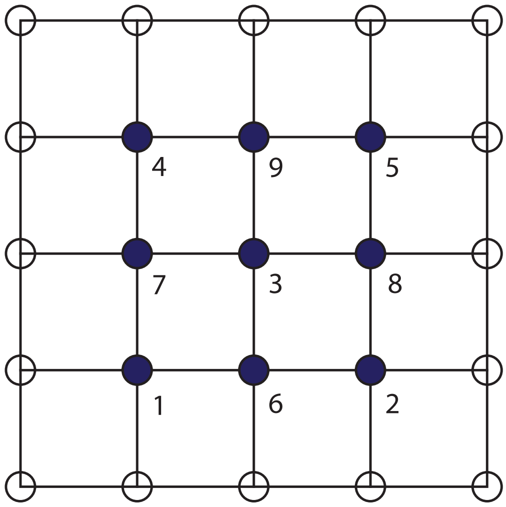
$$K_h = \frac{1}{h^2}\begin{pmatrix} D & -I & 0 & 0 \\ -I & D & -I & 0 \\ & & \ddots & \\ 0 & 0 & -I & D \end{pmatrix} \tag{5.21}$$

with

$$\begin{aligned} D &= \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 \\ -1 & 4 & -1 & 0 & \cdots \\ & & \ddots & & \\ & & & \ddots & \\ 0 & \cdots & 0 & -1 & 4 \end{pmatrix} \\ -I &= \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & \cdots & 0 & 0 & -1 \end{pmatrix} \end{aligned}$$

### Checker board numbering

Checker board numbering follows a black/white ordering: This leads to the following




system:

$$\frac{1}{h^2}\begin{pmatrix} 4 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 4 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 4 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & 0 & 4 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & 4 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 & 0 & 0 & 4 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 4 \end{pmatrix} \cdot u_h = \tilde{f}_h \tag{5.22}$$

and a matrix with the following structure:

$$K_h = \begin{pmatrix} D_1 & L \\ L^T & D_2 \end{pmatrix}. \tag{5.23}$$

$D_i$, $L$ and $L^T$ are the submatrices visible in above linear system.

**Note 1.** *The node ordering does not change the algebraic properties of the system, but can have technical implications (such as better structure for parallel computing).*

## 5.7 Convergence properties of the finite difference method

We now want to know whether the method we designed by approximating the differential operators leads to a convergent method, i.e. with $h \to 0$ we get $u_h \to u$. To show convergence we will show that the method is *stable* and *consistent*, and then show that these two properties lead to convergence. Consider

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u|_\Gamma &= \varphi \end{aligned}$$

and

$$\begin{aligned} -\Delta_h u_h &= f_h \text{ in } \Omega_h \\ u_h|_{\Gamma_h} &= \varphi_h \end{aligned}$$

**Note 2.** *The continuous and discrete solutions exist in different spaces. To compare both, they need to be defined in the same space.*

**Definition 3.** *(Restriction operator)*

$$\begin{aligned} R_h : C(\bar{\Omega}) &\longrightarrow U_h \\ u &\mapsto R_h u \end{aligned}$$

*with* $(R_h u)(\vec{x}) = u(\vec{x})$ *for all* $\vec{x} \in \bar{\Omega}_h$ *defines a* restriction *of the solution in* $\bar{\Omega}$ *onto* $\bar{\Omega}_h$.

### Convergence

The discrete solution $u_h \in U_h$ converges to $u$, if

$$\|u_h - R_h u_h\|_h \longrightarrow 0 \tag{5.24}$$

**Definition 4.** $u_h - R_h u$ *is called the* discretization error *of the discretization method.*

**Definition 5.** *The discretization* $K_h$ *is called* stable, *if*

$$\sup_{h \in H} \|K_h^{-1}\|_\infty \leq C < \infty \tag{5.25}$$

**Note 3.** *Consider the two systems*

$$\begin{aligned} K_h(u_h) &= f_h \\ K_h(\tilde{u}_h) &= f_h + \epsilon \end{aligned}$$

*Then*

$$\begin{aligned} u_h &= K_h^{-1}(f_h) \\ \tilde{u}_h &= K_h^{-1}(f_h + \epsilon) \\ \Rightarrow \|\tilde{u}_h - u_h\| &\leq C \cdot \|\epsilon\| \end{aligned}$$

*Thus, stability means, that small changes on the right hand side produce finite changes in the solution.*

**Note 4.** *The 5 point stencil to the Poisson equation has the property:*

$$\|K_h^{-1}\|_\infty \leq \frac{1}{8},$$

*thus is stable.*

### Consistency

Let $K_h u_h = f_h$ be the discretization of $Ku = f$. Let $K$ be a differential operator of order $m$. In addition, let $R_h$ and $\tilde{R}_h$ be restriction operators for $u$ and $f$, respectively. The discretization $(K_h, R_h, \tilde{R}_h)$ of the differential operator $K$ has *consistency order* $k$ with respect to $\|.\|_\infty$, if

$$\|K_h R_h u - \tilde{R}_h K u\|_\infty \leq C \cdot h^k \cdot \|u\|_{C^{k+m}(\bar{\Omega})} \quad \forall u \in C^{k+m}(\bar{\Omega}) \tag{5.26}$$

**Example 1.** *Let $R_h = \tilde{R}_h$ be given by*

$$(R_h u)(\vec{x}) = u(\vec{x}) \quad \forall \vec{x} \in \Omega_h.$$

*The $(K_h, R_h, \tilde{R}_h) = (\Delta_h, R_h, R_h)$ is consistent of order 2 with respect to $\|.\|_\infty$.*

**Proof.** One can show that

$$(\partial^- \partial^+ u)(x) = u''(x) + h^2 R, \quad |R| \leq \frac{1}{12}\|u\|_{C^4(\bar{\Omega})}$$

In $\mathbb{R}^2$ we can use this in $x$ and $y$ direction. Taylor expansion yields

$$\begin{aligned} -\Delta_h R u(x,y) &= -\Delta u(x,y) + h^2(R_x + R_y) \\ \text{with } |R_x| &\leq \frac{1}{12}\|u^{(4)}\|_{C^0(\bar{\Omega})} \leq \frac{1}{12}\|u\|_{C^4(\bar{\Omega})} \end{aligned}$$

Analogously it follows: $|R_y| \leq \frac{1}{12}\|u\|_{C^4(\bar{\Omega})}$.
$\Rightarrow C = \frac{1}{6}$ with $\|K_h R_h u - R_h K u\| \leq C \cdot h^2 \|u\|_{C^4(\bar{\Omega})}$

□

**Theorem 1.** *(Convergence)*
*Let the discretization $(K_h, R_h, \tilde{R}_h)$ be consistent of order $k$. Let $K_h$ be stable with respect to $\|.\|_\infty$. Then the method is* convergent *of order $k$, if $u \in C^{k+m}(\bar{\Omega})$. Here, $m$ denotes the order of the differential operator $K_h$.*

**Proof.** Consider $w_h := u_h - R_h u$. Our goal is to show that

$$w_h \to 0 \text{ for } h \to 0 \Rightarrow u_h \to u \text{ for } h \to 0.$$

Then

$$K_h w_h = K_h u_h - K_h R_h u = f_h - K_h R_h u = \tilde{R}_h f - K_h R_h u = \tilde{R}_h R u - K_h R_h u.$$

$$\begin{aligned} \Rightarrow w_h &= K_h^{-1}(\tilde{R}_h K u - K_h R_h u) \\ \Rightarrow \|w_h\|_\infty &= \|u_h - R_h u\|_\infty \leq \|K_h^{-1}\|_\infty \cdot \|\tilde{R}_h K u - K_h R_h u\ \|_\infty \\ \Rightarrow \|u_h - R_h u\|_\infty &\leq C h^k \|u\|_{C^{k+m}(\bar{\Omega})} \end{aligned}$$

□