Orders and Stability of Finite Difference Methods

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November 9, 2017

Outline

- Accuracy and stability
- Higher order methods for Poisson equation
- Methods for diffusion equation
- Von Neumann analysis

Recap

- one-sided approximation to u': $D^+u(x) \equiv \frac{u(x+h)-u(x)}{h}$
- or: $D^-u(x) \equiv \frac{u(x)-u(x-h)}{h}$
- centered approximation:

$$D^0u(x) \equiv \frac{u(x+h)-u(x-h)}{2h} = \frac{1}{2}[D^+u(x) + D^-u(x)]$$

- second order centered approximation to u'':

$$D^2u(x) \equiv \frac{u(x-h)+2u(x)+u(x+h)}{h^2} = D^+D^-u(x)$$

Local truncation error and global error

Consider a steady-state problem

$$u''(x) = f(x) \qquad for \qquad 0 < x < 1$$

with given boundary conditions
Idea: compute a grid function consisting of values $U_0, U_1, ..., U_{m+1}$, where U_j is our approximation to the solution $u(x_j)$, here $x_j = jh$ and $h = 1/(m+1)$ is the mesh width. Then use centered difference approximation D^2U we obtain a set of equations

$$\frac{1}{h^2}(U_{j-1} - 2U_j + U_{j+1}) = f(x_j) \qquad for \qquad j = 1, 2, ..., m$$

Thus we have a linear system of m equations for the m unknowns, which can be written in the form

$$AU = F \quad (1)$$

where U is the vector of unknowns $U = [U_1, U_2, ..., U_m]^T$, and

$$A = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}, F = \begin{bmatrix} f(x_1) - \alpha/h^2 \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{m-1}) \\ f(x_m) - \beta/h^2 \end{bmatrix}$$

The LTE(local truncation error) is defined as the discrepancy by replacing U_j with the true solution $u(x_j)$ in the finite difference formula. Which we denote by τ_j:

$$\tau_j = \frac{1}{h^2}[u(x_{j-1}) - 2u(x_j) + u(x_{j+1})] - f(x_j) \quad \textit{for} \quad j = 1, 2, ..., m$$

by Taylor series expansions we know that

$$\tau_j = \frac{1}{12}h^2 u''''(x_j) + O(h^4)$$

Local truncation error and global error

The global error vector E is defined by

$$E = U - \hat{U}$$

where $\hat{U}$ is the vector of true values

$$\hat{U} = [u(x_1) \quad u(x_2) \quad \dots \quad u(x_m)]^T$$

Moreover, we define τ to be the vector with components τ_j, then

$$\tau = A\hat{U} - F \quad (2)$$

Now combine (1)and(2), we obtain a relation between the local error τ and the global error E :

$$AE = -\tau \quad (3)$$

Rewrite the system (3) in the form

$$A^h E^h = -\tau^h$$

where h indicates the mesh spacing. Then solve the system gives

$$E^h = -(A^h)^{-1}\tau^h$$

and taking norms gives

$$\|E^h\| \le \|(A^h)^{-1}\| \|\tau^h\|$$

We know that $\|\tau^h\| = O(h^2)$ and we hope the same will be true of $\|E^h\|$. That is ,we need

$$\|(A^h)^{-1}\| \le C \quad for \quad small \quad h$$

Stability for BVP

Suppose a finite difference method for a linear BVP gives a sequence of matrix equations of the form $A^h E^h = -\tau^h$, where h is the mesh width. We say that the method is stable if $(A^h)^{-1}$ exists for all h sufficiently small(for $h < h_0$,say) and if there is a constant C ,independent of h, such that

$$\|(A^h)^{-1}\| \leq C \quad for \quad all \quad h < h_0$$

Return to the BVP, since A is symmetric, the 2-norm of A is equal to its *spectral radius*:

$$\|A\|_2 = \rho(A) = \max_{1 \leq p \leq m} |\lambda_p|$$

The matrix A^{-1} is also symmetric, and the eigenvalues are simply the inverses of the eigenvalues of A, so

$$\|A^{-1}\|_2 = \rho(A^{-1}) = \max_{1 \leq p \leq m} |(\lambda_p)^{-1}| = (\min_{1 \leq p \leq m} |\lambda_p|)^{-1}$$

Stability for BVP

Actually, the m eigenvalues of A are given by

$$\lambda_p = \frac{2}{h^2}(\cos(p\pi h) - 1) \quad \textit{for} \quad p = 1, 2, ..., m$$

Then we see that the smallest eigenvalue of A is

$$\begin{aligned}\lambda_1 &= \frac{2}{h^2}(\cos(\pi h) - 1) \\ &= \frac{2}{h^2}(-\frac{1}{2}\pi^2 h^2 + \frac{1}{24}\pi^4 h^4 + O(h^6)) \\ &= -\pi^2 + O(h^2)\end{aligned}$$

So we see that the method is stable in the 2-norm. We get an error bound

$$\|E^h\|_2 \leq \|(A^h)^{-1}\|_2 \|\tau^h\|_2 \approx \frac{1}{\pi^2}\|\tau^h\|_2$$

Method of undetermined coefficients

Example: approximate u' based on $u(x), u(x-h), u(x-2h)$ of the form

$$Du(x) = au(x) + bu(x-h) + cu(x-2h)$$

Idea: determine the coefficients a, b, c to give the best possible accuracy by expanding in Taylor series and collecting terms.

$$Du(x) = (a+b+c)u(x) - (b+2c)hu'(x) + \frac{1}{2}(b+4c)h^2u''(x) - \frac{1}{6}(b+8c)h^3u'''(x) + \dots$$

If this is going to agree with u', need

$$a+b+c=0, \quad b+2c=-1/h, \quad b+4c=0$$

which gives

$$a=3/2h, \quad b=-2/h, \quad c=1/2h$$

so that the formula is

$$Du(x) = \frac{1}{2h}[3u(x) - 4u(x-h) + u(x-2h)]$$

Finite difference coefficient

Derivative	Accuracy	−4	−3	−2	−1	0	1	2	3	4
1	2				−1/2	0	1/2			
	4			1/12	−2/3	0	2/3	−1/12		
	6		−1/60	3/20	−3/4	0	3/4	−3/20	1/60	
	8	1/280	−4/105	1/5	−4/5	0	4/5	−1/5	4/105	−1/280
2	2				1	−2	1			
	4			−1/12	4/3	−5/2	4/3	−1/12		
	6		1/90	−3/20	3/2	−49/18	3/2	−3/20	1/90	
	8	−1/560	8/315	−1/5	8/5	−205/72	8/5	−1/5	8/315	−1/560
3	2			−1/2	1	0	−1	1/2		
	4		1/8	−1	13/8	0	−13/8	1	−1/8	
	6	−7/240	3/10	−169/120	61/30	0	−61/30	169/120	−3/10	7/240
4	2			1	−4	6	−4	1		
	4		−1/6	2	−13/2	28/3	−13/2	2	−1/6	
	6	7/240	−2/5	169/60	−122/15	91/8	−122/15	169/60	−2/5	7/240

Finite difference coefficient

Derivative	Accuracy	0	1	2	3	4	5	6	7	8
1	1	−1	1							
	2	−3/2	2	−1/2						
	3	−11/6	3	−3/2	1/3					
	4	−25/12	4	−3	4/3	−1/4				
	5	−137/60	5	−5	10/3	−5/4	1/5			
	6	−49/20	6	−15/2	20/3	−15/4	6/5	−1/6		
2	1	1	−2	1						
	2	2	−5	4	−1					
	3	35/12	−26/3	19/2	−14/3	11/12				
	4	15/4	−77/6	107/6	−13	61/12	−5/6			
	5	203/45	−87/5	117/4	−254/9	33/2	−27/5	137/180		
	6	469/90	−223/10	879/20	−949/18	41	−201/10	1019/180	−7/10	
3	1	−1	3	−3	1					
	2	−5/2	9	−12	7	−3/2				
	3	−17/4	71/4	−59/2	49/2	−41/4	7/4			
	4	−49/8	29	−461/8	62	−307/8	13	−15/8		

Extrapolation method

Another approach is to use one second order accurate method on two different grids, with spacing h and $h/2$, and then to extrapolate in h to obtain a better approximation on the coarse gird. Denote the coarse grid solution by $\quad U_j \approx u(jh), \quad j = 1, 2, ..., m$
and the fine grid solution by $\quad V_j \approx u(jh/2), \quad j = 1, 2, ..., 2m+1$
By Taylor series we know

$$U_j - u(jh) = C_2 h^2 + C_4 h^4 + C_6 h^6 + \dots$$

$$V_{2j} - u(jh) = C_2 (\frac{h}{2})^2 + C_4 (\frac{h}{2})^4 + C_6 (\frac{h}{2})^6 + \dots$$

The extrapolated value is chosen to be

$$\bar{U}_j = \frac{1}{3}(4V_{2j} - U_j)$$

so that the h^2 term of the errors cancels out and we get fourth order accuracy:

$$\bar{U}_j - u(jh) = \frac{1}{3}(\frac{1}{4} - 1)C_4 h^4 + O(h^6)$$

Deferred corrections

Consider the problem

$$u''(x) = f(x) \qquad \textit{for} \qquad 0 < x < 1$$

We know

$$AE = -\tau$$

$$\tau_j = \frac{1}{12}h^2 u''''(x_j) + O(h^4)$$

Note that for this problem we have

$$u''''(x) = f''(x)$$

so the LTE can be estimated directly from the given function $f(x)$. Modify the righthand side of the original problem by setting

$$\bar{F}_j = f(x_j) + \frac{1}{12}h^2 f''(x_j)$$

Now solving $AU = \bar{F}$ will give a fourth order accurate solution.

Methods for diffusion equations

Consider the problem

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2}$$

one nature discretization would be

$$\frac{U_i^{n+1} - U_i^n}{k} = \frac{a}{h^2}(U_{i-1}^n - 2U_i^n + U_{i+1}^n)$$

This uses standard centered difference in space and a forward difference in time, sometimes called FTCS.
Another method , BTCS, using backward difference in time is

$$\frac{U_i^{n} - U_i^{n-1}}{k} = \frac{a}{h^2}(U_{i-1}^n - 2U_i^n + U_{i+1}^n)$$

Methods for diffusion equations

Another much more useful method is the *Crank-Nicolson* method

$$\frac{U_i^{n+1} - U_i^n}{k} = \frac{a}{2}(D^2 U_i^n + D^2 U_i^{n+1})$$

$$= \frac{a}{2h^2}(U_{i-1}^n - 2U_i^n + U_{i+1}^n + U_{i-1}^{n+} - 2U_i^{n+1} + U_{i+1}^{n+1})$$

Using Taylor series, we will find that the truncation error of FTCS and BTCS are $O(h^2 + k)$, that is,
second order accurate in space and first order accurate in time. While the Crank-Nicolson method is second order accurate in both space and time.

Von Neumann analysis

In practice, von Neumann analysis often will give proper stability restrictions more easily, especially for MOL formulation. The von Neumann approach is based on Fourier analysis(see [2]). Von Neumann analysis is based on this fact the grid function $W_j = e^{ijh\xi}$ is an *eigenfunction* of any translation-invariant FD operator. For example, for $D^0 W_j = \frac{1}{2h}(W_{j+1} - W_{j-1})$, we obtain

$$\begin{aligned} D^0 W_j &= \frac{1}{2h}(e^{i(j+1)h\xi} - e^{i(j-1)h\xi}) \\ &= \frac{1}{2h}(e^{ih\xi} - e^{-ih\xi})e^{ijh\xi} \\ &= \frac{i}{h} sin(h\xi) e^{ijh\xi} \\ &= \frac{i}{h} sin(h\xi) W_j \end{aligned}$$

so W is an "eigengridfunction" of the operator D^0.

Von Neumann analysis

Example: consider FTCS. Set

$$U_j^n = e^{ijh\xi}$$

Then we expect that

$$U_j^{n+1} = g(\xi)e^{ijh\xi}$$

where $g(\xi)$ is the *growth factor* for this wave number. Inserting these expressions into the method gives

$$g(\xi)e^{ijh\xi} = e^{ijh\xi} + \frac{ak}{h^2}(e^{i(j-1)h\xi} - 2e^{ijh\xi} + e^{i(j+1)h\xi})$$

$$= e^{ijh\xi}[1 + \frac{ak}{h^2}(e^{-ih\xi} - 2 + e^{ih\xi})]$$

hence

$$g(\xi) = 1 + 2\frac{ak}{h^2}(\cos(\xi h) - 1)$$

Von Neumann analysis

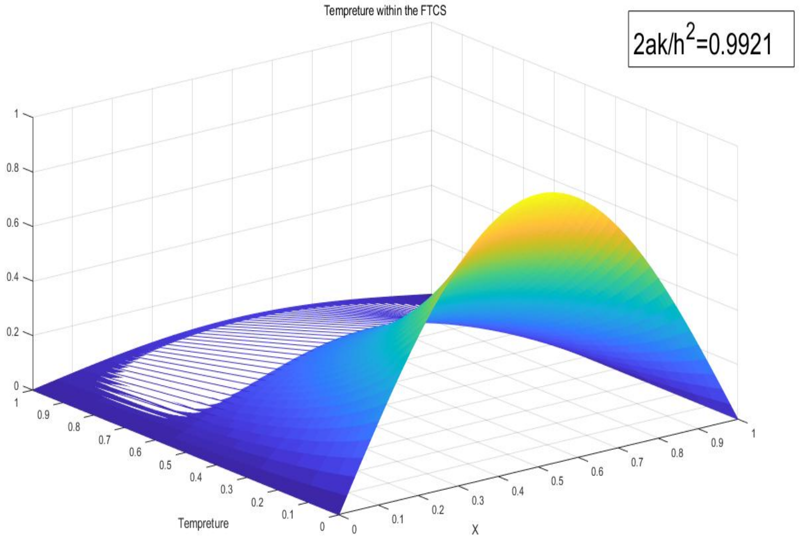
Since $-1 \leq \cos(\xi h) \leq 1$, we see that

$$1 - 4\frac{ak}{h^2} \leq g(\xi) \leq 1$$

for all ξ . We can guarantee that $|g(\xi)| \leq 1$ for all ξ if we require

$$2\frac{ak}{h^2} \leq 1$$

This stability restriction can be interpreted as “The maximum allowed time step is the diffusion time across a cell of width h”.


Tempreture within the FTCS
2ak/h^2=0.9921
Tempreture
X
1
0.8
0.6
0.4
0.2
0
1
0.9
0.8
0.7
0.6
0.5
0.4
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0.1
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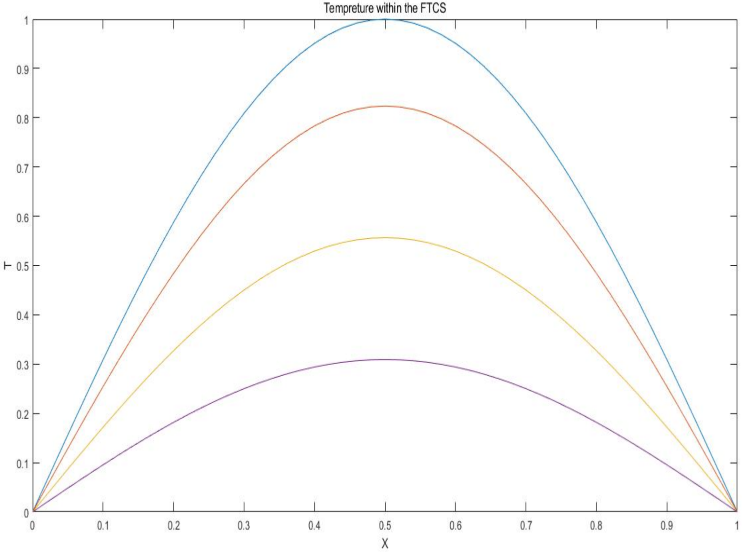
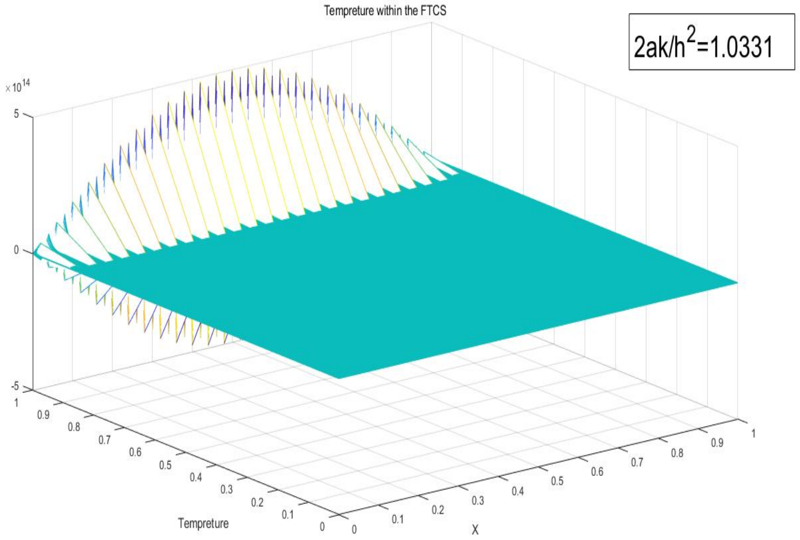
0.8
0.9
1


Tempreture within the FTCS
T
X
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Tempreture within the FTCS
2ak/h^2=1.0331
×10^14
5
0
-5
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Tempreture
0
0.1
0.2
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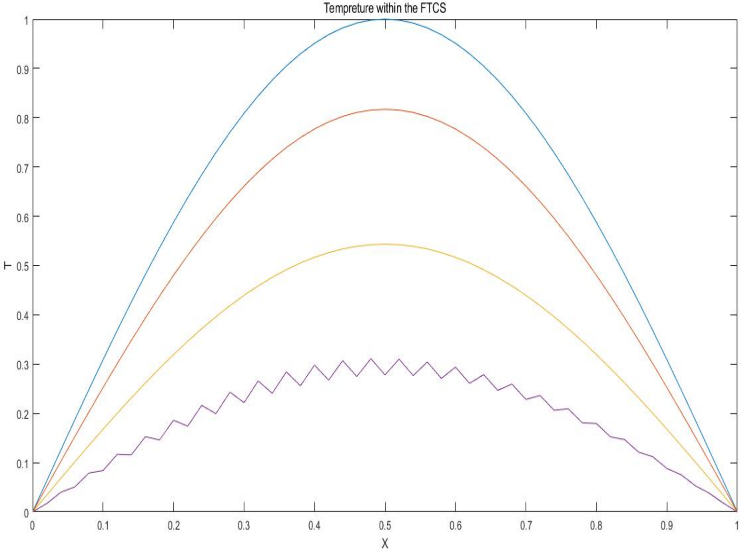
0.9
1
X



Tempreture within the FTCS
T
X
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Von Neumann analysis

Example: consider Crank-Nicolson. Substitute the grid functions into the method

$$g(\xi) = 1 + \frac{ak}{2h^2}(e^{-ih\xi} - 2 + e^{ih\xi})(1 + g(\xi))$$

and hence

$$g(\xi) = \frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z}$$

where

$$z = \frac{2ak}{h^2}(\cos(\xi h) - 1)$$

since $z \leq 0$ for all ξ , we see that $|g(\xi)| \leq 1$ and the method is unconditionally stable, i.e., stable for any choice of k and h.

Method of lines discretization

Idea: first discretize in space alone, which gives a large system of ODEs with each component of the system corresponding to the solution at some grid point, as a function of time. This system of ODEs is also often called a *semidiscrete method*, since we have discretized in space but not yet in time.

- The second order accurate spatial difference operator could be replaced by a higher order method.
- The time stepping could be done by using Runge-Kutta-Chebyshev method, which works for mildly stiff problems with real eigenvalues.
- The time stepping could be done using the exponential time differencing(ETD) methods described in [1].

References

[1] Randall J. LeVeque (2007)
Finite Difference Methods for Ordinary and Partial Differential Equations

[2] J. C. Strikwerda (2004)
Finite Difference Schemes and Partial Differential Equations

Thank you!