Addendum to ‘Property (FA) and lattices in $\mathrm{SU}(2,1)$’

Matthew Stover[*]
University of Michigan
stoverm@umich.edu

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Two proofs in [10] contain errors and the statement of [10, Thm. 1] requires correction. The purpose of this note is to correct these. Fortunately, the corrections allow us to extend [10, Thm. 3] to higher dimensions.

We now describe the organization of this addendum. The first section corrects the statement of [10, Thm. 1]. The second section corrects [10, Prop. 19], which applies to [10, Thm. 3] and [10, Cor. 4]. This correction requires an extra hypothesis that, conjecturally, is unnecessary. Finally, §3 gives a corrected proof of [10, Prop. 20]. The corrected proof also implies Theorem 3.1 below, which generalizes [10, Thm. 3] to $\mathrm{SU}(n,1)$ for $n \geq 4$ (independent from the corrections to [10, Thm. 3]).

1 The statement of Theorem 1

In [10, Thm. 1], it is shown that $\mathrm{PU}(2,1;\mathcal{O}_3)$ has Property (FA). This proof is correct. See [11] for a similar proof that $\mathrm{PU}(2,1;\mathbb{Z}[i])$ also has (FA). However, it is then assumed that the lift of $\mathrm{PU}(2,1;\mathcal{O}_3)$ to $\mathrm{SU}(2,1)$ is $\mathrm{SU}(2,1;\mathcal{O}_3)$. This is not the case. The correct statement of [10, Thm. 1] is the following.

Theorem 1.1. *Let Γ be the lift of $\mathrm{PU}(2,1;\mathcal{O}_3)$ to $\mathrm{SU}(2,1)$. Then $\mathrm{PU}(2,1;\mathcal{O}_3)$ and Γ have Property (FA). The groups $\mathrm{SU}(2,1;\mathcal{O}_3)$ and $\mathrm{PSU}(2,1;\mathcal{O}_3)$ do not.*

Proof. To show that $\mathrm{SU}(2,1;\mathcal{O}_3)$ has (FA), [10] appeals to an exact sequence

$$1 \to \mathbb{Z}/3\mathbb{Z} \to \mathrm{SU}(2,1;\mathcal{O}_3) \to \mathrm{PU}(2,1;\mathcal{O}_3) \to 1. \tag{1}$$

However, the image $\mathrm{PSU}(2,1;\mathcal{O}_3)$ of $\mathrm{SU}(2,1;\mathcal{O}_3)$ in $\mathrm{PU}(2,1)$ is a proper subgroup of $\mathrm{PU}(2,1;\mathcal{O}_3)$. More precisely,

$$[\mathrm{PU}(2,1;\mathcal{O}_3) : \mathrm{PSU}(2,1;\mathcal{O}_3)] = 3 \tag{2}$$

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because $\mathrm{PSU}(2,1;\mathcal{O}_3)$ does not contain the image in $\mathrm{PU}(2,1)$ of the element

$$n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{U}(2,1;\mathcal{O}_3). \tag{3}$$

Therefore, the lift Γ of $\mathrm{PU}(2,1;\mathcal{O}_3)$ to $\mathrm{SU}(2,1)$ is generated by $\mathrm{SU}(2,1;\mathcal{O}_3)$ and $\overline{\zeta}_3^{1/3} n$. This is a maximal lattice in $\mathrm{SU}(2,1)$, and it has Property (FA) by the argument in [10].

In fact, $\mathrm{SU}(2,1;\mathcal{O}_3)$ and $\mathrm{PSU}(2,1;\mathcal{O}_3)$ do not have Property (FA), as we now prove. An explicit homomorphism of $\mathrm{PSU}(2,1;\mathcal{O}_3)$ onto the infinite dihedral group D_∞ can be constructed as follows.

Using the presentation of Falbel–Parker [4] used in [10] and GAP [5], there is a unique normal index three subgroup of $\mathrm{PU}(2,1;\mathcal{O}_3)$. This is necessarily $\mathrm{PSU}(2,1;\mathcal{O}_3)$. It has presentation

$$\langle a, b, c \ : \ a^4, a^{-2}c^2, b^2a^2, c^{-1}abc^{-1}a^{-1}b^{-1} \rangle. \tag{4}$$

Presenting D_∞ as

$$\langle x, y \ : \ x^2, y^2 \rangle, \tag{5}$$

the following determines a surjective representation $\rho : \mathrm{PSU}(2,1;\mathcal{O}_3) \to \mathrm{D}_\infty$:

$$\rho(a) = x \qquad \rho(b) = y \qquad \rho(c) = y. \tag{6}$$

Therefore $\mathrm{PSU}(2,1;\mathcal{O}_3)$, and thus $\mathrm{SU}(2,1;\mathcal{O}_3)$, does not have Property (FA). $\square$

2 Lattices of second type

The proof of Proposition 19 from [10] is not correct because Equation (22) does not imply Equation (23). The purpose of this section is to give a corrected statement and proof. At the end of the section, we give adjusted statements of [10, Thm. 3] and [10, Cor. 4] and discuss their relation to a conjectural picture of torsion in the cohomology of arithmetic groups. We first recall the statement of the proposition.

Proposition 2.1 ([10] Prop. 19)**.** *If $\Gamma < \mathrm{SU}(2,1)$ is a congruence arithmetic lattice of second type, then Γ admits no homomorphism onto the infinite dihedral group.*

We briefly recall the construction of Γ. Let ℓ/k be a totally imaginary extension of a totally real number field and D a cyclic division algebra of degree three with center ℓ and involution τ of the second kind, i.e., such that τ restricted to ℓ is the Galois involution of ℓ/k. For $h \in D$ a τ-hermitian element, the elements $x \in D^\times$ of norm one such that $\tau(x)hx = h$ determine a k-algebraic group $\mathcal{G}$. We assume that h is chosen such that $\mathcal{G}_\mathbb{R}$ is, up to compact factors, $\mathrm{SU}(2,1)$. Then

$$\Gamma = \mathcal{G}_k \cap K_f, \tag{7}$$

where K_f is an open compact subgroup of $\mathcal{G}_{\mathbb{A}_f}$, where $\mathbb{A}_f$ denotes the finite adèles of k.

We replace Proposition 2.1 with the following.

Proposition 2.2. *Let* $\Gamma < \mathrm{SU}(2,1)$ *be a congruence arithmetic lattice of second type, and suppose that there is a torsion free congruence subgroup* $\Gamma' < \Gamma$ *such that* $\mathrm{H}^1(\Gamma'; \mathbb{F}_2)$ *is trivial. Then* Γ *does not admit a homomorphism onto the infinite dihedral group.*

Proof. As in [10], we can assume that Γ is torsion free. Let $\rho : \Gamma \to \mathrm{D}_\infty$ be a nontrivial representation. This produces a nontrivial element of $\mathrm{H}^1(\Gamma; \mathbb{F}_2)$, which determines a nontrivial element of $\mathrm{H}^1(M; \mathbb{F}_2)$, where $M = \mathbf{H}^2_{\mathbb{C}}/\Gamma$ and $\mathbb{F}_2$ is the field of two elements. (Since Γ is torsion free, M is a manifold and thus a $\mathrm{K}(\Gamma, 1)$.)

Let $M' \to M$ be a congruence covering with fundamental group $\Gamma' < \Gamma$. Then, as explained in [10], $\rho|_{\Gamma'}$ has infinite dihedral image in $\rho(\mathrm{D}_\infty)$. In particular, we see that $\mathrm{H}^1(M'; \mathbb{F}_2)$ is nontrivial for every congruence covering of M. This proves the proposition. □

This leads to the following corrected statement of Theorem 3 and Corollary 4 in [10], which we combine here as a single statement.

Theorem 2.3. *Let* $\Gamma < \mathrm{SU}(2,1)$ *be a congruence arithmetic lattice of second type, and suppose that* Γ *contains a torsion free congruence subgroup* Γ' *such that* $\mathrm{H}^1(\Gamma'; \mathbb{F}_2)$ *is trivial. Then* Γ *has Property (FA).*

We now briefly describe how the full statement of Theorem 3 in [10] follows from a conjectural picture of the cohomology of complex hyperbolic manifolds coming from congruence arithmetic lattices of second type. The above argument implies that any $\rho : \Gamma \to \mathrm{D}_\infty$ determines an element order two in $\mathrm{H}^1(M'; \mathbb{Z})$ for all congruence coverings M' of M. Let M_∞ be the inverse limit over all congruence subgroups Γ_K of the corresponding quotients of complex hyperbolic space. Its cohomology is the inverse limit of the cohomology of the Γ_K (see [9]). Thus ρ determines a nontrivial element of $\mathrm{H}^1(M_\infty; \mathbb{Z})$, which is necessarily of finite order. Indeed, and element of infinite order in $\mathrm{H}^1(M_\infty; \mathbb{Z})$ determines a nontrivial element of $\mathrm{H}^1(M_\infty; \mathbb{Q})$, which contradicts [8, Thm. 15.3.1]. Recent work of Boyer [1] and Emerton–Gee [3] suggests that $\mathrm{H}^1(M_\infty; \mathbb{Z})$ should be torsion free for all primes ℓ. See in particular the discussion on page 4 of [3]. This conjecture, along with Proposition 2.2, implies Proposition 2.1, and thus proves Theorem 3 and Corollary 4 in [10].

In the next section, we give an unconditional proof of the analogous result for lattices of second type in $\mathrm{SU}(n,1)$ for all $n \geq 4$.

3 Fake projective planes and lattices of second type in higher dimensions

The purpose of this section is two-fold. Most important is to give a correct proof of [10, Prop. 20]. The proof there uses Klingler's archimedean superrigidity theorem for fundamental groups of fake projective planes [6]. For this theorem to apply, the representation must have Zariski-dense image, and the representation given in [10] does not. Fortunately, the proof below is quite elementary. The same methods will prove the following theorem, which generalizes [10, Thm. 1].

Theorem 3.1. *Let $\Gamma < \mathrm{SU}(n,1)$, $n \geq 4$, be a congruence arithmetic lattice constructed via a division algebra of degree $n+1$ over a CM field. Then Γ has Property (FA).*

See [2] for the appropriate generalizations of the definitions from [10] and §2. The key observation is the following lemma (cf. [6, Lem. 2.14]).

Lemma 3.2. *Let Γ be the fundamental group of a compact Kähler manifold M for which $\mathrm{h}^{1,0}(M) = 0$ and $\mathrm{H}^2(M;\mathbb{C})$ is generated by the Kähler class. Then there is no representation of Γ onto the infinite dihedral group D_∞.*

Proof. Given $\rho : \Gamma \to \mathrm{D}_\infty$, there exists $\Lambda < \Gamma$ of index two so that $\rho(\Lambda) \cong \mathbb{Z}$. Thus the corresponding double cover N of M supports a nontrivial holomorphic 1-form α. Let g be the Deck transformation for the covering.

Since $\mathrm{H}^{1,0}(M) = \{0\}$, g acts on $\mathrm{H}^{1,0}(N)$ by -1. In particular, $\alpha \wedge \overline{\alpha}$ is nonzero and g-invariant, and thus projects to a nontrivial $(1,1)$-form on M. This projection is necessarily a nonzero multiple of the Kähler class on M. Since the square of the Kähler class under the cup product is nonzero, this implies, as in [6, Lem. 2.14], that

$$(\alpha \wedge \overline{\alpha}) \wedge (\alpha \wedge \overline{\alpha}) \neq 0. \tag{8}$$

This is a contradiction. □

Then [10, Prop. 20] follows immediately. We now prove Theorem 3.1.

Proof of Theorem 3.1. As in [10], we may assume Γ is torsion free. We must show that

1. $\mathbf{H}^n_{\mathbb{C}}/\Gamma$ does not admit a holomorphic mapping onto a compact Riemann surface and

2. Γ does not admit a homomorphism onto $\mathbb{Z}$ or D_∞.

We need the following consequence of Clozel's generalization of the results of Rogawski and Blasius–Rogawski applied in [10].

Theorem 3.3 ([2] Thm. 3.2 and Thm. 3.4)**.** *Let Γ be as in Theorem 3.1. Then $\mathrm{h}^{1,0}(\mathbf{H}^2_{\mathbb{C}}/\Gamma;\mathbb{C}) = 0$ and $\mathrm{H}^2(\mathbf{H}^2_{\mathbb{C}}/\Gamma;\mathbb{C})$ is generated by the Kähler class.*

More specifically, Clozel proves that the possible cohomology classes which do not come from powers of the Kähler form are in dimension $i \in I_a$, where a divides $n+1$ and

$$I_a = \{i \in [n-a+1, n+a-1] \ : \ i \equiv n-a+1 \pmod 2\}. \tag{9}$$

For $i = 1$ to appear requires $a = 1$ and $n = 2$. For $i = 2$ to appear forces $n = 3$ and $a = 2$ or $n = 2$ and $a = 1$.

Therefore if $n \geq 4$, there is no homomorphism from Γ to $\mathbb{Z}$, and there is no homomorphism onto D_∞ by Lemma 3.2. To see that there is no holomorphic map of $\mathbf{H}^n_{\mathbb{C}}/\Gamma$ onto a compact Riemann surface, let $f : \mathbf{H}^n_{\mathbb{C}} \to \Sigma$ be such a mapping. Then

$$f^*([\Sigma]) \in \mathrm{H}^2(\mathbf{H}^n_{\mathbb{C}}/\Gamma) \tag{10}$$

is a non-zero class, where $[\Sigma] \in \mathrm{H}^2(\Sigma)$ is the canonical class. By Theorem 3.3, it is a non-zero multiple of the Kähler class ω on $\mathbf{H}^n_{\mathbb{C}}/\Gamma$. As in Lemma 3.2, this is impossible since

$$f^*([\Sigma]) \wedge f^*([\Sigma]) = 0. \tag{11}$$

This completes the proof. □

Remark. For $n = 3$, [2] does not imply $\mathrm{H}^{1,1}$ is generated by the Kähler class, though it does imply $\mathrm{h}^{1,0} = 0$. Thus, if the Picard number is 1, then Theorem 3.1 still applies. According to [7, Thm. 1.4.5], this is indeed the case.

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