

# New nonlinear hyperbolic groups

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## Abstract

We construct nonlinear hyperbolic groups which are large, torsion-free, one-ended, and admit a finite  $K(\pi, 1)$ . Our examples are built from superrigid cocompact rank one lattices via amalgamated free products and HNN extensions.

## 1 Introduction

In this note, we construct new examples of nonlinear hyperbolic groups. For us, a group is “nonlinear” if it does not admit a faithful representation into  $GL_n(F)$  for  $F$  any field. As with previous constructions, our groups are built from superrigid cocompact lattices in rank 1 Lie groups. Previous examples were quotients of such lattices, small cancellation theory was used to show that the quotients are hyperbolic, and superrigidity results were used to see that they are nonlinear (see M. Kapovich [K05, §8]). Our construction involves simple HNN extensions and free products with amalgamation, and one can prove that the resulting groups are hyperbolic using the Bestvina–Feighn Combination Theorem [BF]. Our examples are large (i.e., have finite index subgroups that surject a free group of rank two), torsion-free, one-ended, and admit a finite  $K(\pi, 1)$ .

**Theorem 1.1.** *For any  $n \geq 0$ , there exist large, torsion-free, one-ended, nonlinear hyperbolic groups that admit a finite  $K(\pi, 1)$ , have first betti number  $n$ , and surject a free group of rank  $n$ .*

We present two related constructions, both of which begin with a cocompact torsion-free lattice  $\Gamma$  in  $Sp(m, 1)$  (always with  $m \geq 2$ ) or  $F_4^{(-20)}$ . As in M. Kapovich [K05], our proofs rely crucially on Corlette’s [Cor] and Gromov–Schoen’s [GS] generalizations of the Margulis superrigidity theorem to lattices in these groups. In what follows, let  $G$  be  $Sp(m, 1)$  or  $F_4^{(-20)}$  and  $X$  be the associated rank one symmetric space, i.e., quaternionic hyperbolic  $m$ -space or the Cayley hyperbolic plane.

In our first construction, we choose elements  $\gamma_1$  and  $\gamma_2$  of  $\Gamma$  associated with primitive closed geodesics of different length in the locally symmetric space  $X/\Gamma$ . We consider the group  $\Lambda_1$  obtained by taking the HNN extension of  $\Gamma$  such that the stable letter conjugates  $\gamma_1$  to  $\gamma_2$ , i.e.,

$$\Lambda_1 = \langle \Gamma, t \mid t\gamma_1 t^{-1} = \gamma_2 \rangle.$$

We use superrigidity results to show that if  $\Lambda_1$  is linear, then it admits a faithful representation  $\rho$  into  $GL_n(\mathbb{R})$  and there is a totally geodesic embedding of  $X$  into the symmetric space  $Y_n$  of  $GL_n(\mathbb{R})$  which

is equivariant with respect to the restriction  $\rho|_\Gamma$  of  $\rho$  to  $\Gamma$ . Since the translation lengths of  $\rho(\gamma_1)$  and  $\rho(\gamma_2)$  agree in  $Y_n$  and  $f$  is totally geodesic, the translation lengths of  $\gamma_1$  and  $\gamma_2$  on  $X$  agree, which gives a contradiction. It follows that  $\Lambda_1$  is nonlinear. The Bestvina–Feighn combination theorem [BF] implies that  $\Lambda_1$  is hyperbolic, and it is clear that  $\Lambda_1$  has first betti number 1, has the same cohomological dimension as  $\Gamma$ , admits a finite  $K(\pi, 1)$ , and is torsion-free. We will see that it is easy to iterate this construction to produce examples with arbitrarily large first betti number.

Our second construction involves amalgamated free products and produces examples with first betti number zero. Let  $\Delta = \langle \alpha, \beta \rangle$  be a malnormal, infinite index subgroup of  $\Gamma$  freely generated by  $\alpha$  and  $\beta$  that is contained in a malnormal Fuchsian subgroup of  $\Delta$ . Let  $\phi : \Delta \rightarrow \Delta$  be an isomorphism such that the ratio of the translation lengths of  $\alpha$  and  $\beta$  is different than the ratio of the translation lengths of  $\phi(\alpha)$  and  $\phi(\beta)$ . We then construct

$$\Lambda_0 = \Gamma *_\phi \Gamma$$

from two copies of  $\Gamma$  by identifying  $\Delta$  in the first copy with  $\Delta$  in the second copy via the isomorphism  $\phi$ . We argue, as before, that if  $\Lambda_0$  is linear, then there is a representation  $\rho$  of  $\Lambda_0$  into  $\mathrm{GL}_n(\mathbb{R})$  such that the restriction of  $\rho$  to each factor determines an equivariant totally geodesic embedding of  $X$  into  $Y_n$ . It follows that the ratio of the translation lengths of  $\alpha$  and  $\beta$  agrees with the ratios of the translation lengths of  $\phi(\alpha)$  and  $\phi(\beta)$ , which we have disallowed.

We regard the main advantage of our new constructions to be their relative simplicity and flexibility. For example, if one were given an explicit presentation of a superrigid lattice, one could easily write down an explicit presentation of a group of the form  $\Lambda_1$ .

The first published examples of nonlinear hyperbolic groups are due to M. Kapovich [K05]. Gromov [Gr] used small cancellation theory to show that suitable quotients of a lattice  $\Gamma$  as above are infinite hyperbolic groups (see also [Ch, D, Ol]), and then Kapovich used superrigidity results to show that any linear representation of these quotients has finite image. In particular, these examples have Property (T), since they are quotients of Property (T) groups. It follows that these groups are not large and hence are not abstractly commensurable with our examples.

The paper is organized as follows. In §2, we give the details of our constructions and show that our groups have the claimed group-theoretic properties. In §3 we recall the necessary consequences of superrigidity for lattices in  $\mathrm{Sp}(m, 1)$ ,  $m \geq 2$ , or  $\mathrm{F}_4^{(-20)}$ . The proofs of nonlinearity are given in §4.

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## 2 The constructions

In this section, we give the details of the constructions described in the introduction and establish the group-theoretic properties claimed there. Throughout this paper  $G$  will be either  $\mathrm{Sp}(m, 1)$  for  $m \geq 2$  or  $\mathrm{F}_4^{(-20)}$ , so  $G$  acts by isometries on a rank one symmetric space  $X$ , which is quaternionic hyperbolic  $m$ -space or the Cayley hyperbolic plane, respectively. Then  $\Gamma$  will always denote a torsion-free cocompact

lattice in  $G$ . In particular,  $\Gamma$  is hyperbolic, admits a finite  $K(\pi, 1)$ ,  $H^1(\Gamma, \mathbb{R}) = 0$ , and the cohomological dimension of  $\Gamma$  is the dimension of  $X$ .

We first construct the examples with nontrivial first betti number. Let  $\gamma_1$  and  $\gamma_2$  be primitive elements of  $\Gamma$  with distinct translation length. The associated geodesics in  $X/\Gamma$  are distinct, so no nontrivial power of  $\gamma_1$  is conjugate to a power of  $\gamma_2$ . Let  $\Lambda_1$  be the HNN extension of  $\Gamma$  given by

$$\Lambda_1 = \langle \Gamma, t \mid t\gamma_1 t^{-1} = \gamma_2 \rangle.$$

We may iterate this construction by choosing  $2n$  primitive elements  $\{\gamma_1, \dots, \gamma_{2n}\}$  with distinct translation lengths and defining

$$\Lambda_n = \langle \Gamma, t_1, \dots, t_n \mid t_i \gamma_{2i-1} t_i^{-1} = \gamma_{2i} \rangle.$$

to be obtained by repeated HNN extensions.

We now construct the examples with trivial first betti number. Suppose that  $\Gamma$  contains a maximal cocompact Fuchsian group  $\Sigma$ . Examples of  $\Gamma$  containing such a  $\Sigma$  are well-known to exist. In fact, Meyer showed that all lattices in  $\mathrm{Sp}(m, 1)$  contain such surface subgroups [Mey, Prop 8.]. Results of Long [Lo] and Bergeron [Ber, p. 113] on subgroup separability allow us to replace  $\Gamma$  with a finite index subgroup containing  $\Sigma$  so that  $\mathbb{H}^2/\Sigma$  is embedded in  $X/\Gamma$  as a totally geodesic submanifold. It follows that we can assume  $\Sigma$  is malnormal in  $\Gamma$ , since if  $\gamma \in \Gamma \setminus \Sigma$  and  $\gamma\Sigma\gamma^{-1} \cap \Sigma \neq \{1\}$ , then the immersion of  $\mathbb{H}^2/\Sigma$  into  $X/\Gamma$  cannot be an embedding.

Let  $\Delta \subset \Sigma$  be an infinite index malnormal subgroup of  $\Sigma$  that is freely generated by  $\alpha$  and  $\beta$ . For example, one may take  $\Delta$  to be identified with the fundamental group of a proper, essential subsurface of  $\mathbb{H}^2/\Sigma$  that is either an one-holed torus or a four-holed sphere. All finitely generated subgroups of a surface group are quasiconvex, so  $\Delta$  is a quasiconvex subgroup of  $\Sigma$ . Then  $\Sigma$  is quasiconvex in  $\Gamma$ , since it is the fundamental group of a totally geodesic submanifold, and hence  $\Delta$  is quasiconvex in  $\Gamma$ . Let  $\phi: \Delta \rightarrow \Delta$  be an isomorphism such that the ratio of the translation lengths of  $\alpha$  and  $\beta$  is different than the ratio of the translation lengths of  $\phi(\alpha)$  and  $\phi(\beta)$ . Let

$$\Lambda_0 = \Gamma *_\phi \Gamma.$$

be obtained from two copies of  $\Gamma$  by identifying  $\Delta$  in the first copy with  $\Delta$  in the second copy via the isomorphism  $\phi$ .

**Proposition 2.1.** *For all  $n$ , a group  $\Lambda_n$  constructed as above is hyperbolic, torsion-free, large, one-ended, has a finite  $K(\pi, 1)$ , has first betti number  $n$ , and its cohomological dimension is the dimension of  $X$ . Moreover, if  $n \geq 1$ ,  $\Lambda_n$  admits a surjective homomorphism to the free group  $F_n$  of rank  $n$ .*

*Proof.* That  $\Lambda_n$  is torsion-free, one-ended, has a finite  $K(\pi, 1)$ , has first betti number  $n$ , and has cohomological dimension equal to the dimension of  $X$  follows from standard facts about graphs of groups (see, for example, Serre [Ser, Chap. 1] or Scott–Wall [SW]). If  $n \geq 1$ , then  $\Lambda_n$  clearly surjects onto the group freely generated by  $\{t_1, \dots, t_n\}$ . The fact that each  $\Lambda_n$  is hyperbolic is a special case of the Bestvina–Feighn combination theorem [BF], which is explicitly stated in I. Kapovich [K97, Ex. 1.3] as follows:

**Theorem 2.2.**

1. *If  $A$  and  $B$  are hyperbolic groups and  $C$  is a quasiconvex subgroup of both  $A$  and  $B$  that is malnormal in either  $A$  or  $B$ , then  $A *_C B$  is hyperbolic.*

2. If  $A$  is a hyperbolic group and  $a_1$  and  $a_2$  are elements of  $A$  so that no nontrivial power of  $a_1$  is conjugate to a power of  $a_2$ , then the HNN extension

$$\langle A, t \mid ta_1t^{-1} = a_2 \rangle$$

is hyperbolic.

Part (1) immediately implies that  $\Lambda_0$  is hyperbolic, while part (2) gives that  $\Lambda_n$  is hyperbolic if  $n \geq 1$ .

To complete the proof of the proposition, it remains to show that each  $\Lambda_n$  is large. If  $n \geq 2$ , then  $\Lambda_n$  clearly surjects onto the group freely generated by  $t_1$  and  $t_2$ , so is large. Now suppose that  $n = 1$ . Since the cyclic subgroup generated by  $\gamma_i$  is separable in  $\Gamma$  (see Bergeron [Ber, p. 113]), there exist surjective homomorphisms  $\pi_1 : \Gamma \rightarrow K_1$  where  $\pi_1(\gamma_2)$  does not lie in the subgroup of the finite group  $K_1$  generated by  $\pi_1(\gamma_1)$  and  $\pi_2 : \Gamma \rightarrow K_2$  where  $\pi_2(\gamma_1)$  does not lie in the subgroup of the finite group  $K_2$  generated by  $\pi_2(\gamma_2)$ . Consider

$$\pi = \pi_1 \times \pi_2 : \Gamma \rightarrow K_1 \oplus K_2,$$

and notice that there is a surjective homomorphism onto the HNN extension

$$\Lambda_1 \rightarrow H_1 = \langle K_1 \oplus K_2, t \mid t\pi(\gamma_1)t^{-1} = \pi(\gamma_2) \rangle.$$

However, since  $\pi(\gamma_2)$  is not a power of  $\pi(\gamma_1)$  and  $\pi(\gamma_1)$  is not a power of  $\pi(\gamma_2)$ ,  $H_1$  contains a finite index subgroup isomorphic to  $F_r$  for some  $r \geq 2$ , so  $\Lambda_1$  is large. In fact, the kernel of the obvious map from  $H_1$  to  $K_1 \oplus K_2$  is free of rank at least two (see [Ser, §I.4.3]).

We now consider  $\Lambda_0$ . Similarly, since  $\Sigma$  is separable in  $\Gamma$  [Ber, p. 113] there exists a surjective homomorphism  $\pi_1 : \Gamma \rightarrow K_1$  so that  $\pi_1(\Sigma)$  is a proper subgroup of the finite group  $K_1$ . Moreover, if we choose  $\beta \in \text{Ker}(\pi_1) \setminus \Sigma$ , there exists a surjective homomorphism  $\pi_2 : \Gamma \rightarrow K_2$  so that  $K_2$  is finite and  $\pi_2(\beta)$  does not lie in  $\pi_2(\Sigma)$ . Let  $\pi_0 = \pi_1 \times \pi_2 : \Gamma \rightarrow K_1 \oplus K_2$  and consider the surjective map

$$\pi : \Lambda_0 \rightarrow H_0 = (K_1 \oplus K_2) *_{\pi_0(\Delta)} (K_1 \oplus K_2).$$

Since  $\pi_0(\Delta)$  has index greater than two in  $K_1 \oplus K_2$ ,  $H_0$  contains a finite index subgroup isomorphic to  $F_r$  for some  $r \geq 2$ , so  $\Lambda_0$  is large (again, see [Ser, §I.4.3]).  $\square$

**Remarks:** 1) I. Kapovich [K99] showed that every non-elementary hyperbolic group contains a malnormal quasiconvex subgroup which is free of rank two, so one can more generally construct nonlinear groups from any pair of superrigid rank one lattices by identifying such subgroups by an isomorphism which does not preserve ratios of translation lengths of generators. In our work, we only use the fact that  $\Delta$  is contained in a Fuchsian subgroup in our proof that  $\Lambda_0$  is large.

Kapovich [K99] further uses a malnormal quasiconvex free subgroup of a word hyperbolic group  $G$  to construct a group  $G^*$  which contains  $G$  as a non-quasiconvex subgroup. We note that  $G^*$  is a quotient of a group of the form  $\Lambda_2$ , obtained by identifying the two stable letters, so if  $G$  is a superrigid rank one lattice then  $G^*$  can be chosen to be nonlinear.

2) We expect that the techniques of Belegradek–Osin [BO], which also begin with quotients of superrigid lattices and employ more powerful small cancellation theoretic results, also produce large, one-ended, nonlinear hyperbolic groups (in particular, see [BO, Thm. 3.1]).

3) It is clear that one can construct infinitely many isomorphism classes of groups of the form  $\Lambda_n$ , for each  $n$ , even if one begins with a fixed superrigid lattice  $\Gamma$ . For example, if  $n \geq 1$ , it follows readily from the JSJ theory for hyperbolic groups, see Sela [Sel], that the isomorphism type of a group of the form  $\Lambda_1$  is determined, up to finite ambiguity, by the conjugacy class of the pair  $\{\gamma_1, \gamma_2\}$  in  $\Gamma$ .

### 3 Superrigidity

In this section, we record a version of the superrigidity theorem of Corlette [Cor] and Gromov–Schoen [GS] that is crafted for our purposes. In our statement  $Y_n$  will denote the symmetric space

$$Y_n = ZO(n) \backslash GL_n(\mathbb{R}) = PO(n) \backslash PGL_n(\mathbb{R})$$

associated with  $GL_n(\mathbb{R})$ , where  $Z$  denotes the center of  $GL_n(\mathbb{R})$ .

**Theorem 3.1.** *Suppose that  $\Gamma$  is a lattice in  $G$ , where  $G$  is either  $Sp(m, 1)$  or  $F_4^{(-20)}$ ,  $F$  is a field of characteristic zero, and  $\rho : \Gamma \rightarrow GL_d(F)$  is a representation with infinite image.*

1. *There exists a faithful representation  $\tau : GL_d(F) \rightarrow GL_n(\mathbb{R})$  for some  $n$  such that  $\tau \circ \rho(\Gamma)$  has noncompact Zariski closure.*
2. *If  $F = \mathbb{R}$  and  $\rho(\Gamma) \subset GL_n(\mathbb{R})$  has noncompact Zariski closure, then there exists a  $\rho$ -equivariant totally geodesic map*

$$f_\rho : X \rightarrow Y_n,$$

*where  $X = K \backslash G$  is the symmetric space associated with  $G$ .*

*Proof.* Since  $\Gamma$  is finitely generated we may assume that  $F$  is isomorphic to a subfield of  $\mathbb{C}$ . Moreover,  $GL_d(\mathbb{C})$  is a subgroup of  $GL_{2d}(\mathbb{R})$ . It follows that there exists an injective representation  $\eta : GL_d(F) \rightarrow GL_n(\mathbb{R})$  for some  $n$ , so we may assume that the original representation maps into  $GL_n(\mathbb{R})$ .

Fisher and Hitchman [FH, Thm. 3.7] then observe that the existing results on superrigidity imply that one can factor  $\rho$  as two representations

$$\rho_i : \Gamma \rightarrow GL_{n_i}(\mathbb{R}) \subseteq GL_n(\mathbb{R})$$

such that:

1. When  $\rho_1$  is nontrivial, there is a group  $G'$  locally isomorphic to  $G$ , a continuous representation  $\hat{\rho}_1 : G' \rightarrow GL_{n_1}(\mathbb{R})$ , and an embedding  $\iota : \Gamma \hookrightarrow G'$  of  $\Gamma$  as a lattice in  $G'$  such that  $\rho_1 = \hat{\rho}_1 \circ \iota$ .
2. The image of  $\rho_2$  is bounded, i.e., has compact Zariski closure.
3. The groups  $\rho_1(\Gamma)$  and  $\rho_2(\Gamma)$  commute and  $\rho(\gamma) = \rho_1(\gamma)\rho_2(\gamma)$  for all  $\gamma \in \Gamma$ .

If  $\rho_1$  is nontrivial, the continuous embedding  $\hat{\rho}_1 : G' \rightarrow GL_{n_1}(\mathbb{R})$  determines a totally geodesic embedding of  $X$  into  $Y_{n_1}$ . Since  $\rho_1$  and  $\rho_2$  commute, this is a  $\rho$ -equivariant map.

When  $\rho_1$  is trivial, we follow arguments in the proof of [K05, Thm. 8.1]. Note that our use of [FH, Thm. 3.7] allows us to know beforehand that the solvable radical considered in [K05] is trivial. As in [K05], the fact that  $\Gamma$  has Property (T) allows us to conclude that we may conjugate  $\rho$  so that  $\rho(\Gamma) \subseteq GL_n(k)$  for some number field  $k$ . Given an element  $\sigma \in \text{Aut}(k/\mathbb{Q})$ , we can choose an extension of  $\sigma$  to an element of  $\text{Aut}(\mathbb{C}/\mathbb{Q})$ , which we continue to denote by  $\sigma$ . Applying  $\sigma$  to matrix entries induces an embedding  $\tau_\sigma : GL_n(F) \rightarrow GL_n(\mathbb{C})$ .

Following the adelic argument in [K05], if  $\rho(\Gamma)$  were bounded for every valuation of  $k$  then  $\rho(\Gamma)$  would be finite, which is a contradiction. Moreover,  $\rho(\Gamma)$  must be bounded for every nonarchimedean valuation by nonarchimedean superrigidity [GS]. Consequently, there exists  $\sigma \in \text{Aut}(k/\mathbb{Q})$  such that  $\tau_\sigma(\rho(\Gamma))$  has noncompact Zariski closure in  $GL_n(\mathbb{R})$  or  $GL_{2n}(\mathbb{R})$ , according to whether  $\sigma(k) \otimes_{\mathbb{Q}} \mathbb{R}$  is  $\mathbb{R}$  or  $\mathbb{C}$ . Applying the previous argument to  $\tau_\sigma \circ \rho$ , there is a  $(\tau_\sigma \circ \rho)$ -equivariant totally geodesic embedding of  $X$  into  $Y_n$  or  $Y_{2n}$ , accordingly. This completes the sketch of the proof.  $\square$

M. Kapovich [K05] also points out that superrigidity rules out faithful representations of  $\Gamma$  into linear groups of fields of positive characteristic. Briefly, one shows that the image of  $\rho$  lies in  $\mathrm{GL}_n(k)$  where  $k$  is a finite extension of  $\mathbb{F}_p(x_1, \dots, x_n)$ . Then, applying Gromov–Schoen superrigidity [GS] to each valuation of  $k$  associated with some  $x_i^{\pm 1}$ , one sees that  $\rho(\Gamma)$  is bounded in each field associated with such a valuation on  $k$ , as all valuations on  $k$  are nonarchimedean. It follows that  $\rho(\Gamma)$  is bounded and hence finite. Thus we have:

**Proposition 3.2.** *If  $\Gamma$  is a lattice in either  $Sp(m, 1)$  or  $F_4^{(-20)}$  and  $F$  is a field of characteristic  $p > 0$ , then there does not exist a faithful representation of  $\Gamma$  into  $\mathrm{GL}_n(F)$  for any  $n$ .*

## 4 Proofs of nonlinearity

To complete the proof of Theorem 1.1 it remains to prove:

**Theorem 4.1.** *Groups of the form  $\Lambda_n$  constructed in Section 2 are nonlinear.*

*Proof.* We begin with a group of the form

$$\Lambda_1 = \langle \Gamma, t \mid t\gamma_1 t^{-1} = \gamma_2 \rangle$$

constructed in Section 2, where  $\Gamma$  is a cocompact lattice in  $G$  and  $G$  is either  $Sp(m, 1)$  or  $F_4^{(-20)}$ . Recall that  $X$  is the symmetric space associated with  $G$  and that  $\gamma_1$  and  $\gamma_2$  are assumed to have different translation lengths on  $X$ .

Suppose that  $F$  is a field and  $\eta : \Lambda_1 \rightarrow \mathrm{GL}_d(F)$  is a faithful representation. Applying Proposition 3.2 to the restriction  $\rho = \eta|_{\Gamma}$  of  $\eta$  to  $\Gamma$ , we conclude that  $F$  has characteristic zero. Theorem 3.1 implies that there exists a faithful representation  $\tau_{\sigma} : \mathrm{GL}_d(F) \rightarrow \mathrm{GL}_n(\mathbb{R})$ , for some  $n$  and a  $(\tau_{\sigma} \circ \rho)$ -equivariant embedding  $f$  of  $X$  into  $Y_n$ , where  $Y_n$  is the symmetric space associated with  $\mathrm{GL}_n(\mathbb{R})$ .

Since  $\tau_{\sigma}(\rho(\gamma_1))$  is conjugate to  $\tau_{\sigma}(\rho(\gamma_2))$  in  $\tau_{\sigma}(\eta(\Lambda_1))$ , and hence in  $\mathrm{GL}_n(\mathbb{R})$ , they have the same translation length on  $Y_n$ . However, since  $f$  is a  $\rho$ -equivariant totally geodesic embedding, this implies that  $\gamma_1$  and  $\gamma_2$  have the same translation length in  $X$ , which is a contradiction, hence  $\Lambda_1$  is nonlinear. Notice that if  $n \geq 2$ , then any group of the form  $\Lambda_n$  constructed in Section 2 contains a subgroup of the form  $\Lambda_1$ , so  $\Lambda_n$  is also nonlinear.

Now suppose we have a group of the form

$$\Lambda_0 = \langle \Gamma_1, \Gamma_2 \mid \alpha_1 = \phi(\alpha)_2, \beta_1 = \phi(\beta)_2 \rangle$$

where each  $\Gamma_i$  is a copy of  $\Gamma$ ,  $\Delta = \langle \alpha, \beta \rangle$  is a subgroup of  $\Gamma$  freely generated by  $\alpha$  and  $\beta$ ,  $\Delta_i$  is the copy of  $\Delta$  in  $\Gamma_i$  and if  $\delta \in \Delta$ , then  $\delta_i$  is the copy of  $\delta$  in  $\Delta_i$ . Moreover,  $\phi$  is an automorphism of  $\Delta$  so that the ratio of the translation lengths of  $\alpha$  and  $\beta$  on  $X$  differs from the ratio of translation lengths of  $\phi(\alpha)$  and  $\phi(\beta)$  on  $X$ .

Suppose that  $F$  is a field and  $\eta : \Lambda_0 \rightarrow \mathrm{GL}_d(F)$  is a faithful representation. We again apply Proposition 3.2 to conclude that  $F$  has characteristic zero, Theorem 3.1 implies that there exists a faithful representation  $\tau_{\sigma} : \mathrm{GL}_d(F) \rightarrow \mathrm{GL}_n(\mathbb{R})$ , for some  $n$  and a  $(\tau_{\sigma} \circ \rho_1)$ -equivariant embedding  $f$  of  $X$  into  $Y_n$ , where  $Y_n$  is the symmetric space associated with  $\mathrm{GL}_n(\mathbb{R})$ . Let  $\rho_2 = \eta|_{\Gamma_2}$ . Since  $\tau_{\sigma}(\rho_1(\Delta_1)) = \tau_{\sigma}(\rho_2(\Delta_2))$  has noncompact Zariski closure, Theorem 3.1 implies that there exists a  $(\tau_{\sigma} \circ \rho_2)$ -equivariant embedding  $g$  of  $X$  into  $Y_n$ . Notice that  $\tau_{\sigma}(\rho_1(\alpha_1)) = \tau_{\sigma}(\rho_2(\phi(\alpha)_2))$  and that  $\tau_{\sigma}(\rho_1(\beta_1)) = \tau_{\sigma}(\rho_2(\phi(\beta)_2))$ .

Since  $f$  and  $g$  are equivariant totally geodesic embeddings, there exist positive constants  $c_1$  and  $c_2$  so that if  $\gamma \in \Gamma$ , then the ratio of the translation length of  $\tau_{\sigma}(\rho_i(\gamma_i))$  on  $Y_n$  and the translation length

of  $\gamma$  on  $X$  is  $c_i$ . Indeed, the metrics on  $f(X)$  and  $g(X)$  differ by a scalar multiple. It follows that the ratio of the translation lengths of  $\alpha$  and  $\beta$  on  $X$  agrees with the ratio of the translation lengths of  $\phi(\alpha)$  and  $\phi(\beta)$  on  $X$ . However, this contradicts our assumptions, so  $\Lambda_0$  is nonlinear.  $\square$

## References

- [BO] M. Belegradek and D. Osin, Rips construction and Kazhdan property (T), *Groups, Geom., Dyn.* **2** (2008) 1–12.
- [Ber] N. Bergeron, Premier nombre de Betti et spectre du laplacian de certain variétés hyperboliques. *L’Enseignement Math.* **46** (2000), 109–137.
- [BF] M. Bestvina and M. Feighn, A combination theorem for negatively curved groups. *J. Differential Geom.* **35** (1992), no. 1, 85–101.
- [Ch] C. Champetier, “Petite simplification dans les groupes hyperboliques,” *Ann. Fac. Sci. Toulouse Math.* **3**(1994), 161–221.
- [Cor] K. Corlette, Archimedean superrigidity and hyperbolic geometry. *Ann. of Math. (2)* **135** (1992), no. 1, 165–182.
- [D] T. Delzant, Sous-groupes distingués et quotients des groupes hyperboliques. *Duke Math. J.* **83** (1996), no. 3, 661–682.
- [FH] D. Fisher and T. Hitchman, Strengthening Kazhdan’s Property (T) by Bochner methods. *Geom. Dedicata* **160** (2012), 333–364.
- [Gr] M. Gromov, Hyperbolic groups. *Essays in group theory*, 75–263, Math. Sci. Res. Inst. Publ., **8**, Springer 1987.
- [GS] M. Gromov and R. Schoen, Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. *Inst. Hautes Études Sci. Publ. Math.* **76** (1992), 165–246.
- [K97] I. Kapovich, Quasiconvexity and amalgams. *Internat. J. Algebra Comput.* **7** (1997), no. 6, 771–811.
- [K99] I. Kapovich, A non-quasiconvexity embedding theorem for hyperbolic groups, *Math. Proc. Camb. Phil. Soc.* **127**(1999), 461–486.
- [K05] M. Kapovich, Representations of polygons of finite groups. *Geom. Topol.* **9** (2005), 1915–1951.
- [Lo] D. D. Long, Immersions and embeddings of totally geodesic surfaces. *Bull. London Math. Soc.* **19** (1987), no. 5, 481–484.
- [Mey] J. Meyer, Totally geodesic spectra of quaternionic hyperbolic orbifolds. arXiv:1505.03643.
- [Ol] A.Y. Ol’shanskii, “On residualing homomorphisms and  $G$ -subgroups of hyperbolic groups,” *Int. J. Alg. Comp.* **3**(1993), 365–409.

- [SW] P. Scott and T. Wall, “Topological methods in group theory,” in *Homological Group Theory*, L.M.S. Lecture Note Series vol. **36**, Camb, Univ. Press, 1979, 137–203.
- [Sel] Z. Sela, “Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups. II,” *G.A.F.A.* **7**(1997), 561–593.
- [Ser] J.-P. Serre, *Trees*. Springer Monographs in Mathematics. Springer-Verlag, 2003.