Long Time Behavior of Magnetic Field in Two Dimensions

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Abstract: As noted by Zel'dovich (1957), geometric constraints restrict the behavior of magnetic field in two dimensions. Here, tight long time bounds and decay limits of magnetic field in two dimensions with and without mean fields are proven. In addition the difference between two dimensional and three dimensional magnetic field kinematics is illustrated with an example.

1 Introduction

Magnetic fields are found throughout the universe and have significant influence on the behavior of many astrophysical objects including both the earth and sun. Thus long time magnetic evolution is clearly of importance in the astrophysical context, yet due to difficulties inherent in the governing magnetohydrodynamic (MHD) equations, little is known with rigor. In this paper long time behavior of magnetic fields in the case of two dimensional planar geometry is characterized for smooth planar fluid flow. While astrophysical systems are three dimensional and significant differences distinguish 2D and 3D MHD, an understanding of the 2D case is nevertheless useful for many models of MHD phenomena and also as a preliminary to study of long time behavior of 3D magnetic systems. Additionally, an improved understanding of the differences between 2D and 3D MHD kinematics is in and of itself useful and interesting.

In the astrophysical context, one of the most important questions regarding long time magnetic field evolution is the dynamo problem: in a diffusive system, is an initial seed magnetic field maintained over time scales longer than the diffusion time (the dynamo property) or does the magnetic field decay? This question has been answered in the negative for planar geometries with seed magnetic fields having zero mean (Zel'dovich 1957), one of relatively few rigorous results of dynamo theory. Nevertheless, a full understanding of details of the magnetic field decay history is lacking. And certainly, since a non-zero mean magnetic field can be easily seen to be conserved, the presence of a non-zero mean field provides a non-decaying source of fluctuating field. The long time effects of this source are also not entirely understood.

Kinematic dynamo theory seeks to characterize flows $\mathbf{u}(\mathbf{x}, t)$ for which mean-zero solutions $\mathbf{b}(\mathbf{x}, t)$ of the magnetic induction equation

$$\mathbf{b}_t + \mathbf{u} \cdot \nabla \mathbf{b} = \mathbf{b} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{b}, \quad \nabla \cdot \mathbf{b} = 0 \tag{1}$$

do not decay over long times (Childress & Gilbert 1995). The Zel'dovich (1957) result is a kinematic one in a strong sense – the result holds for any prescribed velocity $\mathbf{u}$ (under reasonable conditions). The same type of strong kinematic results will be presented here. This paper is not however specifically addressed to the dynamo issue. Instead the aim is to add to the general understanding of what can happen to magnetic field in the planar geometry. Zel'dovich (1957) and Zel'dovich & Ruzmaikin (1980) showed initial mean-zero fields decay using the scalar magnetic potential equation (though such field can be amplified if velocities are allowed with the unrealistic property that their flow maps are not isotopic to the identity, see Arnol'd et al. 1981, Oseledets 1993). A similar approach will be followed here though use of probabilistic techniques will result in some extra information. In addition, the probabilistic approach is useful for considering the fates of initial magnetic fields with non-zero mean. Also attention will be directed towards magnetic flux rather than magnetic energy. Flux may be more enlightening and natural in the present context and in many astrophysical problems is actually more relevant.

Finally, it should be noted that relatively little is proven about long time existence and uniqueness of solutions to the incompressible 2D MHD equations (Sulem 1977, Sermange & Temam 1983, Bardos et al. 1988, Kozono 1989, Klapper 1998, Cordoba & Marliani 2000). However, as yet there are no indications of singular behavior in these equations, even in the ideal case (Grauer & Marliani 1998). We disregard the existence question here – behavior of $\mathbf{b}(\mathbf{x}, t)$ is studied given adequately smooth velocities $\mathbf{u}(\mathbf{x}, t)$. If a velocity is provided only until a time T, bounding results to follow still apply as long as that velocity satisfies the necessary assumptions.

2 Preliminaries

We consider here solutions of (1) on the periodic planar domain $\Omega = [0, 1]^2$ for a given spatially periodic velocity $\mathbf{u}$. This domain is a convenient choice because it allows an investigation of planar MHD with and without a non-zero mean magnetic field. By Galilean invariance we can assume that $\mathbf{u}$ has zero mean. We also assume the incompressibility condition $\nabla \cdot \mathbf{u} = 0$.

The results presented in the following sections rely principally on two ideas. The first is a useful generalization of the Cauchy solution: when $\eta = 0$, equation (1) has solution

$$\mathbf{b}(\mathbf{x}(\boldsymbol{\alpha}, t), t) = J(\mathbf{x}(\boldsymbol{\alpha}, t))\mathbf{b}(\boldsymbol{\alpha}, 0) \tag{2}$$

(the Cauchy, or frozen field, solution) where $\mathbf{x}(\boldsymbol{\alpha}, t)$ is the trajectory satisfying

$$d\mathbf{x} = \mathbf{u}dt, \quad \mathbf{x}(0) = \boldsymbol{\alpha} \tag{3}$$

with $\boldsymbol{\alpha}$ chosen such that this trajectory arrives at location $\mathbf{x}$ at time t, and $J(\mathbf{x}(\boldsymbol{\alpha}, t))$ is the Jacobian matrix of the trajectory, i.e., J solves

$$dJ = \nabla \mathbf{u}(\mathbf{x}(\boldsymbol{\alpha}, t), t)Jdt, \quad J(\mathbf{x}(\boldsymbol{\alpha}, 0)) = Id.$$

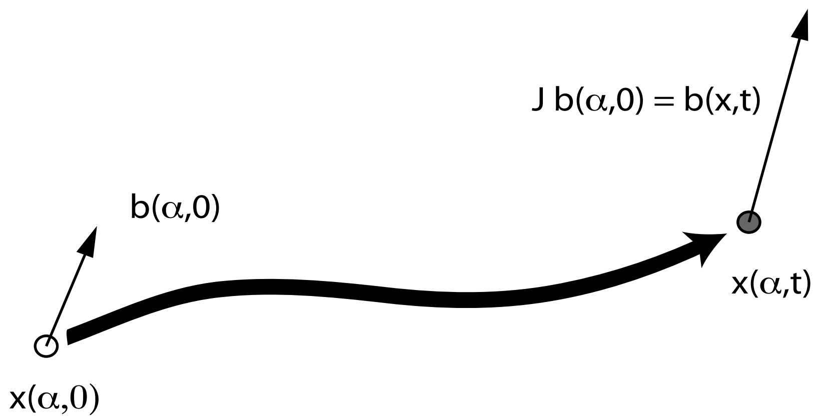

Figure 1: the Cauchy solution. Magnetic field at $\boldsymbol{\alpha}$, $t = 0$, moves along the trajectory of $\boldsymbol{\alpha}$ and is distorted according to the Jacobian matrix along that trajectory.

In words to obtain the magnetic field at a location $\mathbf{x}$ at a time t, trace the material particle at $(\mathbf{x}, t)$ backwards in time to its initial location $(\boldsymbol{\alpha}, 0)$, pick up the initial magnetic field vector there, and bring it back along the same trajectory to $(\mathbf{x}, t)$ using the flow deformation matrix J, Figure 1. Thus solving (1) is reduced to solving ODE's and in fact can be viewed as a dynamical systems problem.

When $\eta > 0$ the Cauchy solution is no longer valid. However, (2) can be generalized using a stochastic differential equation approach (Oksendal 2000, Zel'dovich et al. 1988). resulting in the averaged Cauchy solution

$$\mathbf{b}(\mathbf{x}(\boldsymbol{\alpha}, t), t) = < J(\mathbf{x}(\boldsymbol{\alpha}, t))\mathbf{b}(\boldsymbol{\alpha}, 0) > \qquad (4)$$

where the random trajectories $\mathbf{x}(t)$ are defined by integrating in backward time

$$d\mathbf{x} = \mathbf{u}(\mathbf{x}, t)dt + \sqrt{2\eta}\, d\mathbf{w}(t), \quad \mathbf{x}(t) = \mathbf{x} \qquad (5)$$

with

$$dJ = \nabla\mathbf{u}(\mathbf{x}(\boldsymbol{\alpha}, t), t)Jdt, \quad J(\mathbf{x}(\boldsymbol{\alpha}, 0)) = Id.$$

integrated in forward time (noting that the noise term is uniform in space). The noise term satisfies

$$< dw_i >= 0, \quad < dw_i dw_j >= \delta_{ij} dt \qquad (6)$$

For a proof of formula (4) see Klapper & Young (1995), Appendix B. In words, the Cauchy solution (2) based on the trajectory (3) is replaced when $\eta > 0$ by averaging over a cloud of

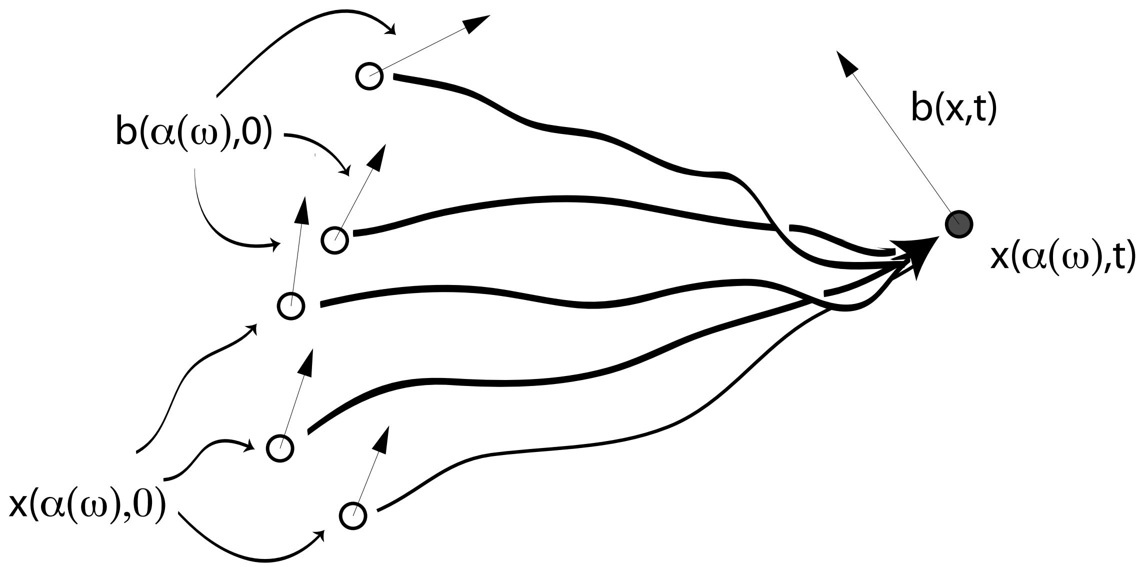

Figure 2: the noisy Cauchy solution. Magnetic field at locations $\boldsymbol{\alpha}(\omega)$, $t = 0$, moves along noisy trajectories distorting according to the respective Jacobian matrices. Upon arrival at location $\mathbf{x}$ at time t, averaging is used to obtain the actual magnetic field value.

Cauchy solutions where each trajectory of the cloud satisfies a diffusion of the form (5) for a realization of noise with properties (6) (Figure 2). The advantage of this approach is that it is possible to continue to view the $\eta > 0$ system as a dynamical system with the velocity $\mathbf{u}$ replace by $\mathbf{u}$ plus noise. In fact, we will reduce the magnetic induction equation to a scalar potential equation below, simplifying the situation somewhat.

The average in (4) is over trajectories satisfying (5). As it happens, for the corresponding scalar potential solution we will only need to average over initial points $\boldsymbol{\alpha}$. The relevant probability density $P(\boldsymbol{\alpha}, t)$, the probability density of trajectories beginning in an infinitesimal neighborhood of $\boldsymbol{\alpha}$ conditioned on arrival at $\mathbf{x}$ at time t, satisfies the diffusion equation (integrated backward in time from t to 0)

$$P_t + \mathbf{u} \cdot \nabla P = -\eta \nabla^2 P \tag{7}$$

with $P(\boldsymbol{\alpha}, t) = \delta(\boldsymbol{\alpha} - \mathbf{x})$ (Oksendal 2000).

The second important idea, just mentioned, is the reduction of (1) to a scalar equation through the magnetic potential. This can be accomplished as follows. Given a magnetic field $\mathbf{b}(\mathbf{x}, t) = (b_x, b_y)$ satisfying $\nabla \cdot \mathbf{b} = 0$, define the magnetic scalar potential $a(\mathbf{x}, t)$ by the line integral

$$a(\mathbf{x}, t) = \int_{\boldsymbol{\gamma}} \mathbf{b} \cdot \mathbf{n} \, ds \tag{8}$$

where $\boldsymbol{\gamma}$ is a smooth path from $(0,0)$ to $\mathbf{x}$, and $\mathbf{n}$, the unit normal to $\boldsymbol{\gamma}$, is chosen such that $\mathbf{n}$ and the path tangent vector, together with the unit vector $\hat{\mathbf{z}}$, form a right-handed set. For the moment we regard $\mathbf{x}$ as a point in R^2 rather than Ω. Note that $(b_x, b_y) = (\partial_y a, -\partial_x a)$. The potential a has a useful physical interpretation: the magnetic flux Φ_I through an arbitrary smooth curve $\mathbf{I}$ directed from endpoint $\mathbf{i}_0$ to endpoint $\mathbf{i}_1$ is given by

$$\Phi_I(t) = \int_I \mathbf{b} \cdot \mathbf{n} ds = \int_I \nabla \mathbf{a} \cdot d\mathbf{s} = a(\mathbf{i}_1, t) - a(\mathbf{i}_0, t).$$

One possible difficulty arises: while $a(\mathbf{x}, t)$ is well-defined on R^2, it is still possible that a, unlike $\mathbf{b}$, may not be periodic and hence may not be well-defined on Ω. To resolve this potential problem, decompose

$$\begin{aligned} b_x(x,y) &= \tilde{b}_x(x,y) + \bar{b}_x \\ b_y(x,y) &= \tilde{b}_y(x,y) + \bar{b}_y \end{aligned}$$

where $\bar{b}_x$, $\bar{b}_y$ are constants (in space and time) and $\tilde{b}_x(x,y)$, $\tilde{b}_y(x,y)$ have mean zero on Ω. Defining $A_y = \bar{b}_x$, $A_x = -\bar{b}_y$ we can then write

$$a(\mathbf{x}, t) = \tilde{a}(\mathbf{x}, t) + A_y x + A_x y = \tilde{a}(\mathbf{x}, t) + \mathbf{A} \cdot \mathbf{x} \tag{9}$$

where $\tilde{a}(\mathbf{x}, t)$ is periodic in R^2 and thus single valued in Ω. Note that the presence of a non-zero mean magnetic field is responsible for the multivaluedness of a on Ω.

In 2 dimensions, the scalar potential equation is

$$a_t + \mathbf{u} \cdot \nabla a = \eta \nabla^2 a \tag{10}$$

the scalar advection-diffusion equation (a function of time may be added, but this extra uniform function may be disregarded here). As in the vector magnetic field equation, (10) has an explicit solution: $a(\mathbf{x}(\boldsymbol{\alpha}, t)) = a(\boldsymbol{\alpha}, 0)$ when $\eta = 0$ and $a(\mathbf{x}(\boldsymbol{\alpha}, t)) = < a(\boldsymbol{\alpha}, 0) >$ when $\eta > 0$. The averaging in the $\eta > 0$ case is again carried out over trajectories of the form (5) where the averaging density is the solution $P(\boldsymbol{\alpha}, 0)$ of (7) (Oksendal 2000).

3 Long Time Results

We now state and prove the basic results of the paper. In all cases we assume $\mathbf{u}$ is periodic with mean zero on $\Omega = [0,1]^2$, Lipschitz continuous and also bounded in time, and $\nabla \cdot \mathbf{u} = 0$. We consider the magnetic field history of a periodic seed field $\mathbf{b}(\mathbf{x}, 0)$ under evolution according to (1). The initial magnetic field is assumed to be smooth and to satisy $\nabla \cdot \mathbf{b}(\mathbf{x}, 0) = 0$. Let $\mathbf{I}$ be any smooth directed curve in R^2. Let $\mathbf{i}_0$ and $\mathbf{i}_1$ be the starting and ending points of $\mathbf{I}$ respectively ($\mathbf{i}_0 = \mathbf{i}_1$ is possible). Denote unit vectors in the x and y directions by $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ respectively. We call $\mathbf{I}$ a flux curve if $\mathbf{i}_1 - \mathbf{i}_0 \in \Omega$. We denote the flux through $\mathbf{I}$ at time t by $\Phi_{\mathbf{I}}(t)$.

Theorem 1: suppose $\mathbf{b}(\mathbf{x}, 0)$ has zero mean and $\eta = 0$. Then the flux through any fixed flux curve $\mathbf{I}$ is bounded for all time. Furthermore $\max_{\mathbf{I}} |\Phi_{\mathbf{I}}|$ is constant in time.

Theorem 2: suppose $\mathbf{b}(\mathbf{x},0)$ has zero mean and $\eta > 0$. Then the flux through any fixed flux curve $\mathbf{I}$ decays to zero exponentially with rate at least η. Furthermore $\max_{\mathbf{I}} |\Phi_{\mathbf{I}}(t)|$ is monotonically decreasing in time.

Theorem 2 (and its proof to follow) are close to the Zel'dovich (1957) result. One difference is the attention to magnetic flux rather than magnetic energy. Magnetic energy can in fact grow significantly for some time before decay. The monotone decay of maximum flux is one indication of the advantage of flux over energy in the present context.

Theorem 3: suppose $\mathbf{b}(\mathbf{x},0)$ has non-zero mean and $\eta = 0$. Then the flux through any fixed flux curve $\mathbf{I}$ grows at most linearly in time.

Note that linear growth can in fact be obtained. For example, choose $\mathbf{u}(\mathbf{x},t) = (0, \sin 2\pi x)$ and $\mathbf{b}(\mathbf{x},0) = (1,0)$. Then $a(\mathbf{x},0) = y$ and the magnetic flux through the flux curve $\mathbf{I}(s) = (s,0)$, $0 \le s \le 1/4$, is $\Phi_{\mathbf{I}} = a(1/4,0,t) - a(0,0,t) = a(1/4,-t,0) - a(0,0,0) = -t$.

Theorem 4: suppose $\mathbf{b}(\mathbf{x},0)$ has non-zero mean and $\eta > 0$. Then the flux through any fixed flux curve $\mathbf{I}$ can be bounded by $C\eta^{-1}$ for some constant C. C depends on $\mathbf{u}$ but not η or the choice of flux curve.

See Cattaneo & Vainshtein (1991), Nunez (1997) for a discussion of the nature of $\mathbf{b}(\mathbf{x},t)$ in this case. Tightness of the scaling η^{-1} is demonstrated by the same example as follows the statement of Theorem 3 above except with $\eta > 0$ in which case $\Phi_{\mathbf{I}}(t) = (4\pi^2\eta)^{-1}(e^{-4\pi^2\eta t} - 1)$. As a remark, it is worth noting the useful idea of flux expulsion (Weiss 1966, Rhines & Young 1983) – eddies of closed streamlines in steady flows will quickly wind up and then dissipate magnetic field lines, thereby expelling magnetic field to the eddy boundaries. As a result, in the presence of a non-zero mean field, magnetic field may form thin diffusive layers of size $O(\eta^{1/2})$ with $O(1)$ flux, in essence an increase in flux from $t = 0$ of magnitude $O(\eta^{-1/2})$. However, shear by open velocity streamlines can increase flux by an $O(\eta^{-1})$ amount. The series of papers Childress (1979), Childress & Soward (1989), Soward & Childress (1990) presents an thorough analysis of a representative family of flows.

Proof of Theorem 1: by the requirements on $\mathbf{b}(\mathbf{x},0)$, we observe that A_x and A_y are zero so $a(\boldsymbol{\alpha},0)$ is bounded. Choose a flux-curve $\mathbf{I}$ and a time $t \ge 0$. Let $\boldsymbol{\alpha}_0$, $\boldsymbol{\alpha}_1$ be such that $\mathbf{x}(\boldsymbol{\alpha}_0,t) = \mathbf{i}_0$, $\mathbf{x}(\boldsymbol{\alpha}_1,t) = \mathbf{i}_1$ under time integration according to (3). Then

$$|\Phi_I(t)| = |a(\boldsymbol{\alpha}_1,0) - a(\boldsymbol{\alpha}_0,0)| \le \max_{\boldsymbol{\alpha}}(a(\boldsymbol{\alpha},0)) - \min_{\boldsymbol{\alpha}}(a(\boldsymbol{\alpha},0)) \tag{11}$$

and thus the flux through any cross-section is bounded independently of time. A flux curve with flux equal in magnitude to the righthand side of (11) can be obtained at any time t by choosing any flux curve $\mathbf{I}$ such that $\mathbf{i}_1 = \mathbf{x}(\boldsymbol{\alpha}_{\max},t)$, $\mathbf{i}_0 = \mathbf{x}(\boldsymbol{\alpha}_{\min},t)$ where $\max_{\boldsymbol{\alpha}} a(\boldsymbol{\alpha},0)$ is attained at $\boldsymbol{\alpha}_{\max}$ and where $\min_{\boldsymbol{\alpha}} a(\boldsymbol{\alpha},0)$ is attained at $\boldsymbol{\alpha}_{\min}$.

Proof of Theorem 2: again, $a(\boldsymbol{\alpha},0)$ is periodic and bounded. When $\eta > 0$

$$|\Phi_I(t)| = <a(\boldsymbol{\alpha}_1,0) - a(\boldsymbol{\alpha}_0,0)> = <a(\boldsymbol{\alpha}_1,0)> - <a(\boldsymbol{\alpha}_0,0)>$$

where $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_0$ are such that $x(\boldsymbol{\alpha}_1, t) = \mathbf{i}_1$ and $x(\boldsymbol{\alpha}_0, t) = \mathbf{i}_0$ under a realization of (5). Setting τ to be negative time, the probability distribution $P_k(\boldsymbol{\alpha}, \tau)$, $k = 0, 1$, used in the averaging satisfies (7),

$$P_{k,\tau} + \mathbf{u} \cdot \nabla P_k = -\eta \nabla^2 P_k, \tag{12}$$

to be solved on Ω for $P_k(\boldsymbol{\alpha}, 0)$ from initial time $\tau = t$ with initial conditions $P_1(\boldsymbol{\alpha}, t) = \delta(\mathbf{i}_1 - \boldsymbol{\alpha})$, $P_0(\boldsymbol{\alpha}, t) = \delta(\mathbf{i}_0 - \boldsymbol{\alpha})$. Note that the solutions P_k exist, are unique and smooth for $0 \leq \tau < t$ (Ito 1992). Also, since $\nabla \cdot \mathbf{u} = 0$, (12) implies that

$$\int_\Omega P_k d\mathbf{x} = 1$$

and

$$\frac{1}{2} \frac{d}{d\tau} \int_\Omega P_k^2 d\mathbf{x} = \eta \int_\Omega (\nabla P_k)^2 d\mathbf{x} \tag{13}$$

Writing $P_k(x, y) = 1 + \tilde{P}_k(x, y)$ for $\tau < t$ where $\tilde{P}_k$ has mean zero, we note the inequality

$$\int_\Omega (\nabla \tilde{P}_k)^2 d\mathbf{x} \geq \int_\Omega \tilde{P}_k^2 d\mathbf{x}$$

(this can be checked by expansion in a Fourier series for example). Hence, using (13), $\|\tilde{P}_k|_{\tau=0}\|_2 \to 0$ exponentially as t increases so that $\Phi_I(t) \to 0$ exponentially fast with rate at least η.

The statement about monotone decay of maximum flux follows immediately from applying standard maximum principles (Evans 1998) to the maximum and minimum values of $a(\mathbf{x}, t)$ and from the fact that $a(\mathbf{x}, 0)$ has zero mean.

Proof of Theorem 3: in the case that the mean magnetic field is not zero, the scalar potential $\tilde{a}(\mathbf{x}, t) + \mathbf{A} \cdot \mathbf{x}$ is no longer bounded. Choose a flux curve $\mathbf{I}$ and a time t. Find points $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_1$ such that $\mathbf{x}(\boldsymbol{\alpha}_0, t) = \mathbf{i}_0$, $\mathbf{x}(\boldsymbol{\alpha}_1, t) = \mathbf{i}_1$, under (3). Then

$$\begin{aligned} |\Phi_I(t)| &= |a(\mathbf{i}_1, t) - a(\mathbf{i}_0, t)| \\ &= |a(\boldsymbol{\alpha}_1, 0) - a(\boldsymbol{\alpha}_0, 0)| \\ &\leq |\max_{\boldsymbol{\alpha}} \tilde{a}(\boldsymbol{\alpha}, 0) + \mathbf{A} \cdot \boldsymbol{\alpha}_1 - \min_{\boldsymbol{\alpha}} \tilde{a}(\boldsymbol{\alpha}, 0) - \mathbf{A} \cdot \boldsymbol{\alpha}_0| \\ &\leq \max_{\boldsymbol{\alpha}} \tilde{a}(\boldsymbol{\alpha}, 0) - \min_{\boldsymbol{\alpha}} \tilde{a}(\boldsymbol{\alpha}, 0) + C(1 + Kt) \end{aligned}$$

where $C = |\mathbf{i}_1 - \mathbf{i}_0||\mathbf{A}|$ and K is the Lipschitz coefficient for $\mathbf{u}$. The idea here is that linearity of (10) allows us to divide a into a piece arising from the mean zero initial field and a piece from the constant initial field. It is the latter contribution that may result in linear flux growth, i.e., flux generation comes from the mean field only.

Proof of Theorem 4: when $\eta > 0$ we again take advantage of linearity of (10) to divide the potential $a(\mathbf{x}, t)$ into 2 parts a_1 and a_2, one with periodic, zero mean, initial potential $a_1(\mathbf{x}, 0)$ and one with linear initial potential $a_2(\mathbf{x}, 0) = \mathbf{A} \cdot \mathbf{x}$. The piece a_1 vanishes exponentially as in the proof of Theorem 2 and will thus be neglected. The remaining linear initial conditions lead for a given flux curve $\mathbf{I}$ to

$$\begin{aligned}\Phi_I(t) &= < a(\boldsymbol{\alpha}_1, 0) - a(\boldsymbol{\alpha}_0, 0) > \\ &= < a(\boldsymbol{\alpha}_1, 0) > - < a(\boldsymbol{\alpha}_0, 0) > \\ &= \mathbf{A} \cdot \int_{R^2} \boldsymbol{\alpha}(\hat{P}_1 - \hat{P}_0) d\boldsymbol{\alpha}\end{aligned}$$

where $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}_1$ are as in the proof of Theorem 2 and $\hat{P}_k$, $k = 0, 1$, is the solution to (12) at $\tau = 0$ on R^2. The distribution P_k of trajectories of (5) on the periodic domain $\Omega = [0, 1]^2$ can be constructed from $\hat{P}_k$ using the formula $P_k(x, y, \tau) = \sum \hat{P}_k(x + j, y + l, \tau)$ where the sum is over all integers j and l. Integrating, then, the first moment of (12) over R^2 we see that

$$\int_{R^2} \mathbf{x}\hat{P}_{k,\tau} d\mathbf{x} + \int_{R^2} \mathbf{x}\mathbf{u} \cdot \nabla \hat{P}_k d\mathbf{x} = -\eta \int_{R^2} \mathbf{x} \nabla^2 \hat{P}_k d\mathbf{x}$$

Integrating by parts and using the fact that $\hat{P}_k, \nabla \hat{P}_k \to 0$ exponentially at ∞, it then follows that

$$\frac{\partial}{\partial \tau} \int_{R^2} \mathbf{x}\hat{P}_k d\mathbf{x} = \int_{R^2} \mathbf{u}\hat{P}_k d\mathbf{x} \tag{14}$$

As $\mathbf{u}$ is periodic then (14) can be written

$$\frac{\partial}{\partial \tau} \int_{R^2} \mathbf{x}\hat{P}_k d\mathbf{x} = \int_{\Omega} \mathbf{u} P_k d\mathbf{x}$$

and so

$$\int_{R^2} \boldsymbol{\alpha}\hat{P}_k|_{\tau=0} d\boldsymbol{\alpha} = \int_t^0 \int_{\Omega} \mathbf{u} P_k d\mathbf{x} d\tau + \mathbf{i}_k$$

Note that

$$\left| \int_{\Omega} \mathbf{u} P_k d\mathbf{x} \right| = \left| \int_{\Omega} \mathbf{u} \tilde{P}_k d\mathbf{x} \right| \le \|\mathbf{u}\|_2 \|\tilde{P}_k\|_2$$

where $\tilde{P}_k$ is defined as in the proof of Theorem 2. Recall also from the proof of Theorem 2 that $\|\tilde{P}_k\|_2$ decays at rate (at least) η. Thus given $\epsilon > 0$ there is a constant C depending on $\mathbf{u}$ but independent of $\mathbf{I}$ and η such that for $\hat{t} = C\eta^{-1}$ and for $t \ge \hat{t}$,

$$\left| \int_t^0 \int_{R^2} \mathbf{u}\hat{P}_k d\mathbf{x} d\tau \right| \le \left| \int_t^{t-\hat{t}} \int_{R^2} \mathbf{u}\hat{P}_k d\mathbf{x} d\tau \right| + \frac{1}{2}\epsilon.$$

Also

$$\left| \int_t^{t-\hat{t}} \int_{R^2} \mathbf{u}\hat{P}_k d\mathbf{x} d\tau \right| \le \hat{t} \|\mathbf{u}\|_\infty.$$

Thus

$$|\Phi_{\mathbf{I}}(t)| \le |\Phi_{\mathbf{I}}(0)| + 2C\eta^{-1} \|\mathbf{u}\|_\infty |\mathbf{A}| + \epsilon.$$

As a remark, note that the time scale for attaining maximum flux is at most $O(\eta^{-1})$.

4 Example and Discussion

Three dimensional MHD is markedly more complex than 2D MHD, not only in respect to 3D dynamics (arising from, for example, the Lorentz force $(\nabla \times \mathbf{b}) \times \mathbf{b}$), but also in respect to 3D kinematics. The importance of the extra dimension to kinematics however is of special significance. In this Section, we extend methods of the previous Sections to a class of 3D MHD systems to explore the consequences of the added dimension.

In particular we now consider 3D systems in the periodic domain $\Omega = [0,1]^3$ of the form

$$\begin{aligned} \mathbf{u}(\mathbf{x},t) &= (u(x,y,t), v(x,y,t), w(x,y,t)) \\ \mathbf{b}(\mathbf{x},0) &= (b_x(x,y), b_y(x,y), 0)e^{i\beta z}. \end{aligned} \tag{15}$$

It is believed that such systems are capable of generating magnetic flux without bound for some choices of $\mathbf{u}$ in both the $\eta = 0$ and $\eta > 0$ cases (Childress & Gilbert 1995). To proceed we immediately face the difficulty that, in general, the 3D magnetic potential is a vector field. However in the case (15), $\mathbf{a}(\mathbf{x},0) = (0,0,a(x,y)e^{i\beta z})$ so that at time $t=0$ the potential can still be regarded as a scalar and hence we may apparently proceed as before. So the question arises: what has changed from 2D?

To illustrate the difference, we again consider the growth rate of flux through a section, this time of the two dimensional form $I = \hat{\mathbf{I}} \times [-\Delta z/2, \Delta z/2]$ where $\hat{\mathbf{I}}$ is a flux curve in the x-y plane. Choose a preferred normal vector $\mathbf{n}$ on I. Then when $\eta = 0$, using conservation of flux in the first step,

$$\begin{aligned} \Phi_I(t) &= \int_{I(t)} \mathbf{b}(\mathbf{x},t) \cdot \mathbf{n} d\mathbf{x} \\ &= \int_{I(0)} \mathbf{b}(\mathbf{x},0) \cdot \mathbf{n}(0) d\mathbf{x} \\ &= \int_{I(0)} \nabla \times \mathbf{a}(\mathbf{x},0) \cdot \mathbf{n}(0) d\mathbf{x} \\ &= \oint_{\partial I(0)} \mathbf{a}(\mathbf{x},0) \cdot d\mathbf{s} \end{aligned}$$

Evaluating, we obtain

$$\begin{aligned} \Phi_I(t) &= \frac{2\sin(\beta\Delta z/2)}{\beta}(e^{i\beta z_1} a(\boldsymbol{\alpha}_1, 0) - e^{i\beta z_0} a(\boldsymbol{\alpha}_0, 0)) \\ &\quad + 2\sin(\beta \Delta z/2) \int_{(\hat{\mathbf{I}}(0),z)} a(\mathbf{x},0) e^{i\beta z} \hat{\mathbf{I}}_s \cdot \hat{\mathbf{z}} ds \end{aligned} \tag{16}$$

The integral is that part of the boundary integral around $\mathbf{I}(0)$ which traverses the preimages of the sides $(\hat{\mathbf{I}}(t), \Delta z/2)$ and $(\hat{\mathbf{I}}(t), -\Delta z/2)$.

The non-integral contribution in (16) reduces to the 2D expression for flux when $\beta = 0$. As in 2D, if $\mathbf{b}(\mathbf{x},0)$ has mean zero then this contribution is bounded regardless of the length of $\hat{\mathbf{I}}(0)$. The integral contribution however is new to three dimensions in an essential way. It measures contribution of z-phase shifted flux. As the length of $\hat{\mathbf{I}}(0)$ may grow, for example exponentially fast if $\mathbf{u}$ is chaotic, this integral may possibly grow as well.

In the case $\eta > 0$ the flux is an average once again. The boundary contribution to (16) behaves as in the 2D case but again the integral contribution may be able to grow without bound. In both instances $\eta = 0$ and $\eta > 0$, evaluation of the integral is apparently difficult.

In two dimensions, generation of new magnetic flux is not possible without a non-decaying source, in particular a mean magnetic field, as is illustrated in Figure 3. When $\eta = 0$ the flux through $\mathbf{I}$ at time t is given by the flux through $\mathbf{I}(0)$ at time 0. The flux through $\mathbf{I}(0)$ at $t = 0$ is determined by the geometry of $\mathbf{I}(0)$.

How then can flux be increased? Length of $\mathbf{I}(0)$ alone does not matter; due to $\nabla \cdot \mathbf{b} = 0$ the endpoints of $\mathbf{I}(0)$ alone determine the flux. Roughly speaking, in 2D if a field line crosses $\mathbf{I}(0)$ in the one direction then it is likely to cross $\mathbf{I}(0)$ in the other direction somewhere else. In the case $\eta > 0$, flux is determined by an average of curves of the form in Figure 3 and flux increase is thus even more difficult. In fact the only way to generate new flux is for the endpoints of $\mathbf{I}(0)$ to move away from each other across a non-zero mean field. This sort of lengthening of $\mathbf{I}(0)$ is at best linear for bounded flow, unlike the sort illustrated in Figure 3 that can, for example, be exponential even for fairly simple flows. Hence even in the presence of a non-zero mean field there are limits to magnetic flux production.

In three dimensions, these geometrical restrictions disappear. Instead a flux curve or, more properly, a thin flux sheet can move out of the plane and it is no longer to be expected that a field line crossing the flux surface in one direction will cross back somewhere else. (This idea was utilized in Bayly and Childress 1988). Hence it becomes possible to generate significant new flux in a confined region even without a non-zero mean field by using the kind of curve stretching shown in Figure 3.

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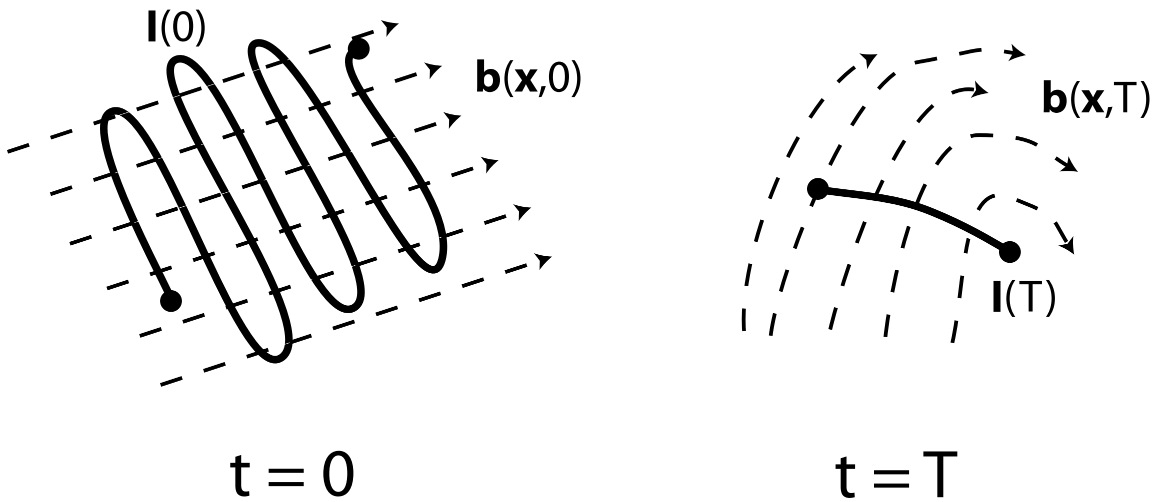

Figure 3: flux curve $\mathbf{I}$ at initial time $t = 0$ and final time $t = T$. As $\mathbf{I}$ is pulled back in time it is distorted and lengthened, but flux through $\mathbf{I}(0)$ is not necessarily large despite its length.

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