Constraints on finite-time current sheet formation at null points in two-dimensional ideal incompressible magnetohydrodynamics

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Abstract

It is shown rigorously that, under the conditions of two-dimensional ideal incompressible magnetohydrodynamics, finite-time singularity formation (including finite-time collapse to a current sheet) cannot occur at a magnetic null point of any type unless driven by a pressure singularity occurring outside a neighborhood of the null point. The proof is based on the depletion of nonlinearity at a two-dimensional magnetic null point.

1 Introduction

Formation of singularities, particularly current sheets, in magnetohydrodynamics (MHD) has been a topic of considerable interest in recent years. Current sheet formation has been extensively studied due to its proposed relevance to a number of solar phenomena. In that context it is appropriate to consider nearly ideal MHD behavior as a model and in fact as a further approximation it has been relatively common practive to rely on perfectly ideal MHD. The solutions to these ideal equations have a tendency to develop fine structures. The appearance of actual singularities, defined here to be infinities in the solution or its derivatives would naturally be significant both physically and mathematically. As a side remark, it is perhaps surprising that the mathematical theory of the existence and regularity of solutions to the incompressible MHD equations is very incomplete. The existence of smooth long-time solutions to incompressible MHD is an open question with only partial results available [1].

The central model of two dimensional (2D) current sheets and reconnection has been the collapse of a magnetic x-point (Figure 1). Here the strongly bent magnetic field in the two oblate quadrants is pulled away from the x-point by magnetic tension while the gently curved magnetic field in the two obtuse quadrants is pushed into the x-point by magnetic pressure. The resulting magnetic field tends toward the double y-point structure shown in Figure 1(c). Two questions naturally arise: what are the dynamics of this collapse, and what is the form of the resulting steady-state slightly non-ideal current sheet? The second of these two questions has received ongoing attention especially in connection with magnetic reconnection (see [2] for recent reviews) and will not be considered here. The first question of dynamics however is less explored yet still of definite physical interest. The different possible scenarios, say exponential collapse versus finite-time nonlinear collapse, may have important consequences for the physics of the problem. In addition to applications to solar coronal physics already mentioned, existence of finite-time singularities would have a profound impact on the theory of MHD turbulence. The consequences for fluid turbulence would also be interesting – 2D MHD provides a kind of intermediate model between the 2D Euler equations, known not to allow finite-time singularity formation, and the 3D Euler equations for which the question remains open. (See [3] for a recent discussion of regularity of the Euler equations.) The question of finite-time 2D incompressible MHD singularities has also arisen in the context of time-independent solutions of the 2D Euler equations [4].

In fact there has been a fair amount of speculation over the years about the rate of collapse of x-point nulls in ideal MHD (e.g., [5]), especially since static equilibrium solutions [6] and compressible dynamical solutions [7] were presented with geometries similar to that in Figure 1(c). An ongoing discussion has ensued over the dynamical accessibility of these types of solutions. We mention specifically two recent studies supporting accessibility: [8] presented numerics suggesting that the double y-point current sheet geometry might be obtainable under dissipation-free dynamics, noting that the presence of numerical dissipation clouds the issue. [9] presented an asymptotic analysis of an x-point collapse under zero-pressure dynamics; compressibility apparently plays an intrinsic role however. On the other side of the argument, [10] argues against the possibility of a finite-time x-point collapse due to the observation that in 2D the MHD equations "self-linearize" in the vicinity of a magnetic null point (see below). This point has been re-stated elsewhere as well (e.g. [11], [12]) and seems to be supported by ideal numerics [13]. The purpose of this paper is to turn this idea into a rigorous argument under the conditions of ideal incompressible 2D MHD. In Section II we introduce the MHD gradient equations which will be the central focus of the paper. In Section III, the main Section of the paper, it is proven (Theorem 2) that for smooth initial conditions there is a finite region around a 2D magnetic null point (of any type) such that unless $\boldsymbol{\nabla}\mathbf{u}$ or $\boldsymbol{\nabla}\mathbf{b}$ blow-up outside of that region, no current sheet (or any other type of singularity) can form. The key idea is the depletion of non-linearity at a null point as observed in [10]. It is to be stressed that the results reported here cover the scenario of Figure 1 as well as other types of null point singularities.

2 The Equations

We begin with the equations of incompressible ideal MHD (2D or 3D)

$$\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\nabla p^* + \mathbf{b}\cdot\boldsymbol{\nabla}\mathbf{b}, \qquad \nabla\cdot\mathbf{u} = 0 \tag{1}$$

$$\frac{\partial}{\partial t}\mathbf{b} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{b} = \mathbf{b}\cdot\boldsymbol{\nabla}\mathbf{u}, \qquad \nabla\cdot\mathbf{b} = 0 \tag{2}$$

where $\mathbf{u}$ is the fluid velocity, $\mathbf{b}$ the magnetic field, and $p^* = p + (1/2)b^2$ the modified pressure. We will assume that the initial conditions $\mathbf{u}(\mathbf{x},0)$ and $\mathbf{b}(\mathbf{x},0)$ are C^3 over all space with far field decay conditions sufficient for the convergence of all pertinent integrals (more precisely it is assumed that the initial conditions are in the Sobolev space H^s, $s > 4$, under the L_2 norm). According to the Beale-Kato-Majda (BKM) inequality for MHD [14], $\mathbf{u}$ and $\mathbf{b}$ retain their smoothness on a time interval $[0,T]$ as long as

$$\int_0^T (|\omega(t)|_\infty + |j(t)|_\infty)dt < \infty \tag{3}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is the fluid vorticity and $\mathbf{j} = \nabla\times\mathbf{b}$ is the current. In particular (3) implies that as long as $\boldsymbol{\nabla}\mathbf{u}$ and $\boldsymbol{\nabla}\mathbf{b}$ are bounded then $\mathbf{u}$ and $\mathbf{b}$ retain their original smoothness. Hence a finite-time loss of smoothness of any kind must be accompanied by a finite-time blow-up of $\boldsymbol{\nabla}\mathbf{u}$ or $\boldsymbol{\nabla}\mathbf{b}$. As a consequence, rather than studying properties of solutions of equations (1) and (2), following [15] we instead consider solutions to the gradients of those equations, namely

$$\frac{d}{dt}\boldsymbol{\nabla}\mathbf{u} = -(\boldsymbol{\nabla}\mathbf{u})^2 + (\boldsymbol{\nabla}\mathbf{b})^2 - \nabla\nabla p^* + \mathbf{b}\cdot\boldsymbol{\nabla}(\boldsymbol{\nabla}\mathbf{b}), \qquad Tr\,[\boldsymbol{\nabla}\mathbf{u}] = 0 \tag{4}$$

$$\frac{d}{dt}\boldsymbol{\nabla}\mathbf{b} = \boldsymbol{\nabla}\mathbf{u}\boldsymbol{\nabla}\mathbf{b} - \boldsymbol{\nabla}\mathbf{b}\boldsymbol{\nabla}\mathbf{u} + \mathbf{b}\cdot\boldsymbol{\nabla}(\boldsymbol{\nabla}\mathbf{u}), \qquad Tr\,[\boldsymbol{\nabla}\mathbf{b}] = 0. \tag{5}$$

Here the total, or Lagrangian, derivative d/dt is defined by $d/dt = \partial/\partial t + \mathbf{u}\cdot\boldsymbol{\nabla}$, that is, the time derivative of a quantity following the flow. Note from equation (2) that at a magnetic null point $\mathbf{a}_0$, $(d/dt)\mathbf{b}(\mathbf{a}_0) = 0$ and hence the Lagrangian trajectory $\mathbf{x}_0(t)$ defined by $(d/dt)\mathbf{x}_0 = \mathbf{u}(\mathbf{x}_0, t)$, $\mathbf{x}_0(0) = \mathbf{a}_0$ remains a null point for all t.

In two dimensions the gradient equations simplify considerably. Taking into account the divergence-free conditions $\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{b} = 0$, the 2×2 matrices $\boldsymbol{\nabla}\mathbf{u}$, $\boldsymbol{\nabla}\mathbf{b}$, and $\nabla\nabla p^*$ can be written most generally (at any point) as

$$\begin{aligned}
\boldsymbol{\nabla}\mathbf{u} &= \begin{pmatrix} s_0 & s_1-\omega \\ s_1+\omega & -s_0 \end{pmatrix} \\
\boldsymbol{\nabla}\mathbf{b} &= \begin{pmatrix} r_0 & r_1-j \\ r_1+j & -r_0 \end{pmatrix} \qquad (6)\\
\nabla\nabla p^* &= \begin{pmatrix} \sigma/2+A & B \\ B & \sigma/2-A \end{pmatrix}
\end{aligned}$$

where

$$\sigma = Tr[\nabla\nabla p^*] = Tr\left[(\boldsymbol{\nabla}\mathbf{b})^2 - (\boldsymbol{\nabla}\mathbf{u})^2\right] = (r_0^2 + r_1^2 - j^2) - (s_0^2 + s_1^2 - \omega^2)$$

is determined from the trace of the gradient momentum equation (4). The quantities $\omega/2$, $j/2$, $\pm\sqrt{s_0^2+s_1^2}$, and $\pm\sqrt{r_0^2+r_1^2}$ are the vorticity, current, fluid strains, and magnetic strains, respectively (and are functions of $\mathbf{x}$ and t). $A(\mathbf{x},t)$ and $B(\mathbf{x},t)$ are functions of space and time as well, determined however by an integration over space (see below).

Plugging (6) into (4) and (5) we obtain

$$\begin{aligned}
\dot{s}_0 &= -A + \mathbf{b}\cdot\nabla r_0 & \dot{r}_0 &= 2(s_1 j - \omega r_1) + \mathbf{b}\cdot\nabla s_0 \\
\dot{s}_1 &= -B + \mathbf{b}\cdot\nabla r_1 & \dot{r}_1 &= 2(\omega r_0 - s_0 j) + \mathbf{b}\cdot\nabla s_1 \qquad (7)\\
\dot{\omega}_0 &= \mathbf{b}\cdot\nabla j & \dot{j} &= 2(s_1 r_0 - s_0 r_1) + \mathbf{b}\cdot\nabla\omega.
\end{aligned}$$

The dot superscript (e.g. $\dot{s}_0$) refers to the Lagrangian time derivative. Specializing to a 2D magnetic null point $\mathbf{x}_0(t)$ where $\mathbf{b}(\mathbf{x}_0(t), t) = 0$, these equations further simplify to

$$\begin{aligned}
\dot{s}_0 &= -A & \dot{r}_0 &= 2(s_1 j - \omega r_1) \\
\dot{s}_1 &= -B & \dot{r}_1 &= 2(\omega r_0 - s_0 j) \qquad (8)\\
\dot{\omega} &= 0 & \dot{j} &= 2(s_1 r_0 - s_0 r_1).
\end{aligned}$$

The quantities A and B contain all non-local forcing effects due to the modified pressure p^*. If it happens to be the case that $A = B = 0$, then at the null point $\mathbf{x}_0$, $\boldsymbol{\nabla}\mathbf{u}$ is constant in time and (8) are a set of linear homogenous ordinary differential equations (ODE's) for $\boldsymbol{\nabla}\mathbf{b}$ and hence exhibit exponential solutions with exponential factors of the form $\exp(\pm 2\sigma_u t)$ where $\pm\sigma_u = \pm(s_0^2 + s_1^2 - \omega^2)^{1/2}$ are the eigenvalues of $\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}_0)$ [11]. If $s_0^2 + s_1^2 - \omega^2 > 0$ (strain domination) then exponential growth occurs; if $s_0^2+s_1^2-\omega^2 < 0$ (vorticity domination) then oscillatory behavior occurs. In fact this family of solutions of (8) extends to a special family of solutions of (7) over all space by setting $\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}) = \boldsymbol{\nabla}\mathbf{u}(\mathbf{x}_0)$, $\boldsymbol{\nabla}\mathbf{b}(\mathbf{x},t) = \boldsymbol{\nabla}\mathbf{b}(\mathbf{x}_0,t)$.

Integrating, we see that $\mathbf{u}$ and $\mathbf{b}$ are linear functions, i.e., $\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x}_0) + \nabla\mathbf{u}(\mathbf{x}_0)\mathbf{x}$ and $\mathbf{b}(\mathbf{x}, t) = \nabla\mathbf{b}(\mathbf{x}_0, t)\mathbf{x}$, and hence violate typical far-field conditions (like finite energy). Nevertheless these solutions are instructive because, at least in the vicinity of a null point where $\mathbf{b}$ is small, they indicate what the fields might like to do in the absence of outside forcing from A and B. In general A and B are not 0 but it might be expected that if A and B do not depend directly on $\nabla\mathbf{u}(\mathbf{x}_0)$ and $\nabla\mathbf{b}(\mathbf{x}_0)$ then (8) are a set of linear ODE's with a nonhomogeneous forcing and hence will again exhibit exponential solutions unless the forcing (i.e., pressure) blows up. This is the key idea of Theorem 2 below.

Before proceeding to the main Theorem however, we note the following geometrical and topological constraint on the magnetic field in the vicinity of a magnetic null:

Theorem 1 *[16, 17] At a two-dimensional magnetic null point $\mathbf{x}_0(t)$ under the equations of ideal incompressible MHD, the eigenvalues of $\nabla\mathbf{b}(\mathbf{x}_0)$ are constant in time.*

Proof: the eigenvalues of $\nabla\mathbf{b}(\mathbf{x}_0)$ are $\pm\sqrt{r_0^2 + r_1^2 - j^2}$. Using the right-hand 3 equations of (8), a short calculation shows that $(d/dt)(r_0^2 + r_1^2 - j^2) = 0$. $\Box$

Remark: this Theorem can also be shown to be true for 3D null points; in addition it can be extended easily to account for a compressible velocity. A consequence of Theorem 1 is a significant restriction on the form of current and magnetic field at and in the vicinity of magnetic null points prior to (finite or infinite time) singularity formation. In particular this is a stronger statement than the conservation of topological type of the null point (e.g. x-point, o-point, etc.). If $\nabla\mathbf{b}(\mathbf{x}_0)$ becomes infinite then in order to satisfy Theorem 1 it must do so in the form of a current sheet. For instance if coordinates are chosen so that the magnetic strain axes are inclined at $\pi/4$ radians from horizontal (i.e., coordinates are chosen so that $r_0 = 0$, which is always possible), then $r_1^2 - j^2$ is a constant and a singularity at time T in $\nabla\mathbf{b}(\mathbf{x}_0)$ will take the form of something like

$$\lim_{t\to T} \nabla\mathbf{b}(\mathbf{x}_0, t) = \lim_{t\to T} \begin{pmatrix} 0 & r_1 - j \\ r_1 + j & 0 \end{pmatrix} = \begin{pmatrix} 0 & \infty \\ 0 & 0 \end{pmatrix},$$

a classic current sheet.

3 Regularity Near 2D Null Points

Clearly in order to characterize the smoothness of solutions of equations (8) it will be necessary to closely consider the pressure derived terms A and B and hence the pressure Hessian $\nabla\nabla p^*$. To begin we remark as before that by taking the trace of equation (4) we obtain the equation

$$\nabla^2 p^* = \sigma \tag{9}$$

where $\sigma = Tr\,[(\nabla\mathbf{b})^2 - (\nabla\mathbf{u})^2] = (r_0^2 + r_1^2 - j^2) - (s_0^2 + s_1^2 - \omega^2)$. Equation (9) has solution

$$p^*(\mathbf{x}) = \int\int K(r)\sigma(\mathbf{x}')\mathbf{dx}'$$

where $r = |\mathbf{x} - \mathbf{x}'|$ and in 2 dimensions $K(r) = (1/4\pi)\ln r^2$. Now decompose

$$\nabla\nabla p^*(\mathbf{x}_0) = \mathbf{L} + \mathbf{F}$$

where (with $\sigma_0 = \sigma(\mathbf{x}_0)$)

$$\begin{aligned} \mathbf{L} &= \sigma_0 \int\int \nabla\nabla K \mathbf{dx}' \\ \mathbf{F} &= \int\int (\sigma(\mathbf{x}') - \sigma_0)\nabla\nabla K \mathbf{dx}' \end{aligned}$$

are symmetric 2×2 matrices containing the "self-induced" and outer parts of $\nabla\nabla p^*$ respectively. A standard calculation shows that for a given smooth function $f(\mathbf{x})$

$$\begin{aligned} \int\int f(\mathbf{x}')\nabla\nabla K(\mathbf{x},\mathbf{x}')\mathbf{dx}' &= \left[\frac{1}{2}\int\int f(\mathbf{x}')\delta(\mathbf{x}-\mathbf{x}')\mathbf{dx}'\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &+ \lim_{\epsilon\to 0}\left[\frac{1}{2\pi}\int_0^{2\pi}\int_\epsilon^\infty \begin{pmatrix} -\cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \frac{f(\mathbf{x}) - f(\mathbf{x}')}{r} dr d\theta\right] \end{aligned}$$

so that, using the form of $\nabla\nabla p^*$ in (6),

$$\begin{aligned} \mathbf{L} &= \begin{pmatrix} \sigma_0/2 & 0 \\ 0 & \sigma_0/2 \end{pmatrix} \\ \mathbf{F} &= \begin{pmatrix} A & B \\ B & -A \end{pmatrix}. \end{aligned}$$

Heuristically, $\mathbf{L}$ contains local contributions to $\nabla\nabla p^*$ from the linear solution discussed above while $\mathbf{F}$ contains the non-linear corrections due to the nonlinearity of the $\mathbf{u}$ and $\mathbf{b}$ vector fields. Close to the null point $\mathbf{x}_0$, $\mathbf{u}$ and $\mathbf{b}$ can be well approximated by the first two terms of their Taylor series and the contribution $\sigma - \sigma_0$ to $\mathbf{F}$ is small (the integrand of $\mathbf{F}$ is in fact regular at $r = 0$). In this sense $\mathbf{F}$ contains the non-local contributions to $\nabla\nabla p^*$, and the numbers A and B contain the non-local forcing on $\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}_0)$ and $\nabla\mathbf{b}(\mathbf{x}_0)$.

Theorem 2 *Consider 2D ideal incompressible MHD. Suppose* $\mathbf{b}(\mathbf{x}_0) = 0$. *There exists a length* $\delta(t) > 0$, $t \in [0, T]$, *such that if* $\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}, 0)$ *and* $\boldsymbol{\nabla}\mathbf{b}(\mathbf{x}, 0)$ *have continuous derivatives,* $\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}, t)$ *and* $\boldsymbol{\nabla}\mathbf{b}(\mathbf{x}, t)$ *satisfy the far field conditions and are bounded for* $|\mathbf{x} - \mathbf{x}_0| > \delta(t)$, $t \in [0, T]$, *then* $\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}, t)$ *and* $\boldsymbol{\nabla}\mathbf{b}(\mathbf{x}, t)$ *are bounded for all* $\mathbf{x}$ *when* $t \in [0, T]$.

Proof: For a given $\epsilon > 0$ and a finite time T we choose a function $\delta_1(t) \geq 0$, $t \in [0, T]$, $\delta_1(t)$ as large as possible so that if $|\mathbf{x} - \mathbf{x}_0| \leq \delta_1(t)$ then

$$\begin{aligned} |\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}, t) - \boldsymbol{\nabla}\mathbf{u}(\mathbf{x}_0(t), t)| + |\boldsymbol{\nabla}\mathbf{b}(\mathbf{x}, t) - \boldsymbol{\nabla}\mathbf{b}(\mathbf{x}_0(t), t)| &\leq \epsilon \\ |\nabla\sigma(\mathbf{x}, t) - \nabla\sigma(\mathbf{x}_0(t), t)| &\leq \epsilon \end{aligned}$$

Set $\delta(t) = \min(\delta_1(t), (|\nabla\sigma(\mathbf{x}_0, t)| + \epsilon)^{-1}, 1)$ for $t \in [0, T]$ (note that the possibility $\delta(t) = 0$ has not yet been excluded). Roughly speaking, $\delta(t)$ is the *linear length-scale* of the null point, i.e., the length-scale over which $\mathbf{u} \approx \mathbf{u}_0 + \boldsymbol{\nabla}\mathbf{u}_0 \cdot \mathbf{x}$ and $\mathbf{b} \approx \boldsymbol{\nabla}\mathbf{b}_0 \cdot \mathbf{x}$. Define $B_\delta(t) = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| \leq \delta(t)\}$. Since by assumption $\boldsymbol{\nabla}\mathbf{u}$ and $\boldsymbol{\nabla}\mathbf{b}$ are bounded outside of B_δ, then according to (3), the BKM inequality for MHD, a loss of smoothness of any kind can only occur if $\boldsymbol{\nabla}\mathbf{u}$ or $\boldsymbol{\nabla}\mathbf{b}$ become singular inside B_δ. Hence $\delta(t) > 0$ strictly is a consequence of $\boldsymbol{\nabla}\mathbf{u}(\mathbf{x},t)$ and $\boldsymbol{\nabla}\mathbf{b}(\mathbf{x},t)$ remaining finite.

Solutions of (8) [and hence $\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}_0,t)$ and $\boldsymbol{\nabla}\mathbf{b}(\mathbf{x}_0,t)$] are easily estimated via a Gronwall inequality to remain bounded for $t \in [0,T]$ if $A(\mathbf{x}_0,t)$ and $B(\mathbf{x}_0,t)$ are bounded for $t \in [0,T]$. Further, if $\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}_0,t)$ and $\boldsymbol{\nabla}\mathbf{b}(\mathbf{x}_0,t)$ are bounded then $\boldsymbol{\nabla}\mathbf{u}(\mathbf{x},t)$ and $\boldsymbol{\nabla}\mathbf{b}(\mathbf{x},t)$ are also bounded for $\mathbf{x} \in B_\delta$, $t \in [0,T]$ and hence are bounded for all $\mathbf{x}$ for $t \in [0,T]$, completing the proof.

$A(\mathbf{x}_0,t)$ and $B(\mathbf{x}_0,t)$ can be shown to be bounded in the following manner. Using the region B_δ, divide the integral expression for the matrix $\mathbf{F}$ into inner and outer parts. Inside the approximately linear region B_δ,

$$\left\| \int\int_{B_\delta} (\sigma - \sigma_0)\nabla\nabla K \mathbf{dx}' \right\| \leq C\delta \max_{B_\delta} |\nabla\sigma(\mathbf{x})| \leq C$$

for some constant C. In the outer non-linear region $B'_\delta = R^2 - B_\delta$,

$$\int\int_{B'_\delta} (\sigma - \sigma_0)\nabla\nabla K \mathbf{dx}' = \int\int_{B'_\delta} \sigma\nabla\nabla K \mathbf{dx}' - \sigma_0 \int\int_{B'_\delta} \nabla\nabla K \mathbf{dx}' = \int\int_{B'_\delta} \sigma\nabla\nabla K \mathbf{dx}'. \quad (10)$$

Since $\boldsymbol{\nabla}\mathbf{u}(\mathbf{x},t)$ and $\boldsymbol{\nabla}\mathbf{b}(\mathbf{x},t)$ are bounded and satisfy the far-field decay conditions in B'_δ then so does σ and hence $\| \int\int_{B'_\delta} (\sigma - \sigma_0)\nabla\nabla K \mathbf{dx}' \| \leq C'$ for some some constant C'. Hence the matrix $\mathbf{F}$ is bounded on $t \in [0,T]$ and equations (8) have bounded solutions on $t \in [0,T]$. Thus no singularity forms at the null point which in turn implies that no singularity forms in B_δ further implying that $\delta(t) > 0$ for $t \in [0.T]$. □

Remark: the assumption that $\boldsymbol{\nabla}\mathbf{u}$ and $\boldsymbol{\nabla}\mathbf{b}$ are bounded outside the linear region B_δ is considerably stronger than necessary. It is sufficient for instance that the integral in (10) be finite. If $\pm\lambda_u(\mathbf{x},t)$ and $\pm\lambda_b(\mathbf{x},t)$ are the eigenvalues of $\boldsymbol{\nabla}\mathbf{u}(\mathbf{x},t)$ and $\boldsymbol{\nabla}\mathbf{b}(\mathbf{x},t)$ respectively then $\sigma(\mathbf{x},t) = \lambda_b^2 - \lambda_u^2$ so

$$\left\| \int\int_{B'_\delta} \sigma\nabla\nabla K \mathbf{dx}' \right\| \leq \frac{C}{\delta} \int\int_{B'_\delta} (\|\boldsymbol{\nabla}\mathbf{u}\|_2^2 + \|\boldsymbol{\nabla}\mathbf{b}\|_2^2)\mathbf{dx}',$$

where $\|\cdot\|_2$ is the pointwise matrix 2-norm, (some care is needed here to determine that $\delta > 0$) and (10) is bounded if $\boldsymbol{\nabla}\mathbf{u}$ and $\boldsymbol{\nabla}\mathbf{b}$ have pointwise norms with finite integral.

As a special case, Theorem 2 excludes a finite-time collapse of the type diagrammed in Figure 1 as long as non-local singularities do not form:

Corollary *Under the initial smoothness conditions and the boundedness assumptions of Theorem 2, an initially isolated magnetic null point remains isolated for all finite time, and no current singularity forms there.*

Proof: Suppose $\mathbf{b}(\mathbf{x}_0, 0) = 0$ and $\mathbf{b}(\mathbf{x}, 0) \neq 0$ for $\mathbf{x}$ sufficiently close to $\mathbf{x}_0$. The frozen field property then implies that unless a loss of smoothness in $\mathbf{b}$ of some type occurs, there is a neighborhood $B(t)$ of $\mathbf{x}_0$ such that $\mathbf{b}(\mathbf{x}, t) \neq 0$ for $\mathbf{x} \in B$, $\mathbf{x} \neq \mathbf{x}_0$ (that is, a non-null Lagrangian point cannot become a null point). According to the BKM inequality for MHD a loss of smoothness of any kind must be accompanied by unbounded behavior of $\nabla \mathbf{u}$ or $\nabla \mathbf{b}$. However under the assumptions of Theorem 2 this cannot occur. The fact that no current singularity forms at $\mathbf{x}_0$ is a restatement of Theorem 2. □

4 Discussion

The aim of this paper is to analyze the dynamics of 2D incompressible ideal MHD in the neighborhood of a magnetic null point. Based partially on the observation that the local (to a 2D null point) MHD dynamics are nearly linear, significant restrictions on the magnetic field structure and rate of change of that structure are present. The main result proven is that a finite-time singularity (including one of the type diagrammed in Figure 1) cannot occur unless driven by singular events in the non-local, non-linear parts of the fluid. In fact the argument of the proof indicates that current sheet formation at a null point can only be driven by non-local singularities in the modified pressure. Theorem 2 does not rule out the possibility of formation of singularities at the null point and simultaneously somewhere else in the flow, yet it would seem at the least to require a global mechanism much more elaborate than those proposed to date. Further if such a global event were to be possible, it is not evident why a null point in particular should take a part in it. This conclusion complicates efforts to model nearly ideal current sheets by ideal ones. It would still seem entirely possible, however, that models of ideal current sheets, while not reflecting true incompressible ideal dynamics, still are good models of the actual MHD dynamics in the more physically relevant situation of singular limits of vanishing viscosity and dissipation.

We remark again that the results of this paper are rather general, applying to any type of 2D null and any type of singularity, including the most commonly considered type, namely x-point collapse to a current sheet. The proofs rest on the BKM inequality for incompressible MHD and the depletion of non-linearity that occurs at 2D null points. The main restrictions are incompressibility and two-dimensionality. Extension of the procedures in this paper to three dimensions (3D) would seem to be difficult at this time (although see [16]); for example even if the magnetic field is identically zero (a very degenerate magnetic null!) the question of the rate of singularity formation in the remaining fluid is very difficult. On the other hand it may be possible that the second restriction, namely incompressibility, can be relaxed if an extension of the BKM inequality exists. The possibility of finite-time singularity formation away from null points in 2D ideal incompressible MHD remains an important open question.

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Figure 1: magnetic field separatrices at a 2D x-point. Field lines are represented by thick lines, fluid velocity lines by thin lines. In (a) and (b) the x-point is strained by the velocity. In (c) the x-point collapses to a pair of y-points and a current sheet.

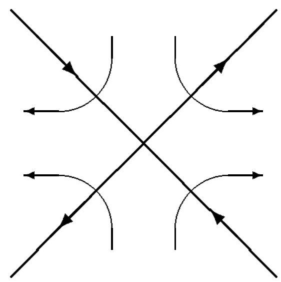

Figure 1(a): $t = 0$

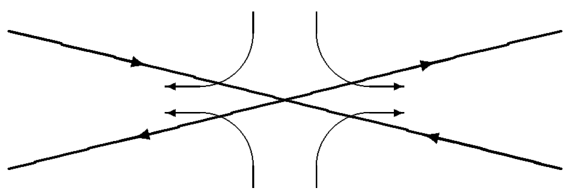

Figure 1(b): $t > 0$

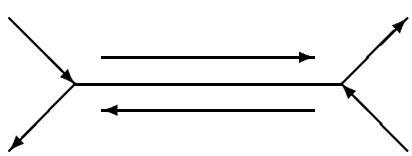

Figure 1(c): $t =?$