COUNTING PROBLEMS IN GRAPH PRODUCTS AND RELATIVELY HYPERBOLIC GROUPS

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Abstract. We study properties of generic elements of groups of isometries of hyperbolic spaces. Under general combinatorial conditions, we prove that loxodromic elements are generic (i.e. they have full density with respect to counting in balls for the word metric in the Cayley graph) and translation length grows linearly. We provide applications to a large class of relatively hyperbolic groups and graph products, including all right-angled Artin groups and right-angled Coxeter groups.

1. Introduction

Let G be a finitely generated group. One can learn a great deal about the geometric and algebraic structure of G by studying its actions on various negatively curved spaces. Indeed, Gromov's theory of hyperbolic groups [Gro87] provides the clearest illustration of this philosophy. However, weaker forms of negative curvature, ranging from relative hyperbolicity [Far98, Bow12, Osi06] to acylindrical hyperbolicity [Osi15, Bow08], apply to much larger classes of groups and still provide rather strong consequences. In all of these theories, a special role is played by the *loxodromic* (or *hyperbolic*) elements of the action, i.e. those elements which act with sink-source dynamics. In this paper, we are interested in quantifying the abundance of such isometries for the action of G on a hyperbolic space X. We emphasize that in all but the simplest situations, the natural hyperbolic spaces that arise are not locally compact. This includes actions associated to relatively hyperbolic groups [Far98], cubulated groups [KK14, Hag14], mapping class groups [MM99], and $\mathrm{Out}(F_n)$ [BF14, HM13], to name only a few. Hence, in this paper we make no assumptions of local finiteness or discreteness of the action.

Suppose that $G \curvearrowright X$ is an action by isometries on a hyperbolic space X. We address the question: *How does a typical element of G act on X?*

When G is not amenable, the word "typical" has no well defined meaning, and depends heavily on the averaging procedure: a family of finitely supported measures exhausting G. Although much is now known about measures generated from a random walk on G [Mah11, CM15, MT14, MS14], very little is known about counting with respect to balls in the word metric. This will be our main focus.

In more precise terms, fix a finite generating set S for the group G. Let B_n be the ball of radius n about 1 with respect to the word metric d determined by S. Then we call a property P *generic* if

$$\frac{\#\{g \in B_n \ : \ g \text{ has } P\}}{\# B_n} \to 1 \qquad \text{as } n \to \infty.$$

In this language, a refinement of our questions asks when the loxodromic elements of a particular action $G \curvearrowright X$ are generic with respect to a generating set S. It is important to note that genericity in the counting model depends on the generating set: a priori, sets may be generic with respect to one word metric, but not with respect to another.

The results of this paper are modeled on our previous work [GTT16], where we studied the situation where G is itself hyperbolic. Recall that $g \in G$ is loxodromic with respect to the action $G \curvearrowright X$ if and only if its translation length $\tau_X(g) = \lim d_X(x, g^n x)/n$ is strictly positive. In

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[GTT16], we prove that for any isometric action of a hyperbolic group G on a hyperbolic metric space X, loxodromic elements are generic, and translation length grows linearly. However, the genericity of loxodromic elements is in general false when the hypothesis that G is hyperbolic is dropped (see Example 1). In the present paper, we generalize this theorem to a much larger class of groups. Our general setup is discussed below, but here is a sample:

Theorem 1.1. *Suppose that either*

(1) *G is a finitely generated group which admits a geometrically finite action on a CAT(−1) space with virtually abelian parabolic subgroups and S an admissible generating set, or*
(2) *G is a right-angled Artin or Coxeter group which does not split as a direct product, and S is its standard vertex generating set.*

Then for any nonelementary isometric action $G \curvearrowright X$ on a hyperbolic metric space X there is an $L > 0$ such that

$$\frac{\#\{g \in B_n : \tau_X(g) \geq Ln\}}{\#B_n} \to 1. \tag{1}$$

In particular, loxodromic elements are generic.

In fact, our theorem applies to a more general class of relatively hyperbolic groups and graph products (see Section 2 for precise statements and definitions) and in fact to any group satisfying certain combinatorial conditions. Before moving to our general framework, we state one more result which may be of independent interest. It is a direct generalization of a theorem of Gouëzel, Mathéus, and Maucourant [GMM15] who consider the case where G is hyperbolic.

Theorem 1.2. *Let the group G and generating set S be as in Theorem 1.1, and suppose that H is any infinite index subgroup of G. Then*

$$\lim_{n\to\infty} \frac{\#(H \cap B_n)}{\#B_n} = 0.$$

That is, the proportion of elements of G of length less than n which lie in H goes to 0 as $n \to \infty$.

1.1. **General framework and results.** Our general framework is as follows. We define a **graph structure** to be a pair (G, Γ) where G is a countable group and Γ is a directed, finite graph such that:

(1) there is a labeled vertex v_0, called the *initial vertex*; for every other vertex v there exists a directed path from v_0 to v;
(2) every edge is labeled by a group element such that edges directed out of a fixed vertex have distinct labels.

By (2), there exists an *evaluation map* $\mathrm{ev}\colon E(\Gamma) \to G$ and this map extends to the set of finite paths in Γ by concatenating edge labels on the right. We denote by Ω_0 the set of finite paths starting at v_0, by $S_n \subset \Omega_0$ the set of paths of length n, and by $\#X$ the cardinality of X. A graph structure is a **geodesic combing** if the evaluation map $\mathrm{ev} : \Omega_0 \to G$ is bijective, and each path in Ω_0 evaluates to a geodesic in the associated Cayley graph. See Section 3.1 for details. We introduce the *counting measure* P^n on Ω_0 as

$$P^n(A) := \frac{\#(S_n \cap A)}{\#S_n},$$

for $A \subset \Omega_0$ The graph structure is *almost semisimple* if the number of paths of length n starting from v_0 has *pure exponential growth*, i.e. there exists $c > 0, \lambda > 1$ such that

$$c^{-1}\lambda^n \leq \#S_n \leq c\lambda^n$$

for each n. See Section 3.2 for details.

As a simple example, consider the rank $N \geq 2$ free group $G = F_N$ and fix a free basis $\{a_1, \dots, a_N\}$ of F_N. Then one has the usual geodesic combing with underlying graph Γ as follows. The graph Γ has $2N+1$ vertices, with initial vertex v_0 and other vertices labelled a_i^ϵ with $i = 1, \dots, N$, $\epsilon = \pm 1$. For each vertex $v = a_i^\epsilon$, there is a directed edge labelled a_j^η to the vertex a_j^η unless $i = j$ and $\epsilon = -\eta$. Moreover, there is a directed edge from v_0 to each of the other vertices labeled by its terminal vertex.

To state our results in fully generality, we first introduce a few dynamical properties of graph structures.

Definition 1.3. For each vertex v of Γ, we denote by L_v the set of loops based at v, and by $\Gamma_v = \mathrm{ev}(L_v)$ its image in G. We call Γ_v the *loop semigroup* associated to v.

Consider an action $G \curvearrowright X$, where X is a hyperbolic metric space. A semigroup $L < G$ is *nonelementary* if it contains two independent loxodromics. A graph structure (G, Γ) is **nonelementary** for the action $G \curvearrowright X$ if for any vertex v of maximal growth (i.e. the growth rate of Γ_v is maximal among all vertices; see Definition 3.1) the loop semigroup Γ_v is nonelementary.

We now introduce several criteria on a graph structure that guarantee it is nonelementary: we call them thickness and quasitightness. Although they may appear slightly technical, each is meant to capture 'mixing' properties of the graph structure. To help with the reader's intuition, we also illustrate each property in the case of the free group F_N.

Definition 1.4 (Thickness)**.** A graph structure is **thick** if for any vertex v of maximal growth there exists a finite set $B \subseteq G$ such that

$$G = B\Gamma_v B,$$

where the notation on the right-hand side means group multiplication between subsets of G. More generally, a graph structure is *thick relative to a subgroup* $H < G$ if for every vertex v of maximal growth there exists a finite set $B \subseteq G$ such that

$$H \subseteq B\Gamma_v B.$$

For example, the geodesic combing described above for F_N is thick. Indeed, in this case each Γ_v is maximal, and if v corresponds to, say, the generator a, then Γ_a is the set of words that end with a and do not begin with a^{-1}. From this description, it is easy to see that we may take B in the definition of thickness to be the set of words of F_N of length at most 2.

Given a path γ in Γ, we say it c–almost contains an element $w \in G$ if γ contains a subpath p such that $w = a \cdot \mathrm{ev}(p) \cdot b$ in G, with $|a|, |b| \leq c$. Here, $|a|$ denotes the word length of $a \in G$ with respect to the generating set given by edge labels. We denote as $Y_{w,c}$ the set of paths in Γ starting at the initial vertex which do not c–almost contain w. The following definition is modeled on the one found in [AL02].

Definition 1.5 (Growth quasitightness)**.** A graph structure (G, Γ) is called **growth quasitight** if there exists $c \geq 0$ such that for every $w \in G$ the set $Y_{w,c}$ has density zero with respect to P^n; that is,

$$P^n(Y_{w,c}) \to 0 \quad \text{as } n \to 0.$$

More generally, given a subgroup $H < G$ we say that (G, Γ) is *growth quasitight relative to* H if there exists a constant $c \geq 0$ such that for every $w \in H$ the set $Y_{w,c}$ has density zero.

It is also the case that the geodesic combing previously described for F_N is growth quasitight. (In fact, a similar property holds for all hyperbolic groups [AL02].) In this case, we may take $c = 0$, and so $Y_{w,0}$ is precisely the set of words that do not contain w as a subword. It is then an exercise to show that the proportion of elements of F_N with word length n that lie in $Y_{w,0}$ goes to 0 as $n \to \infty$. We will see in Example 1 a case in which growth quasitightness fails.

In the most general form, the main theorem we are going to prove is the following.

Theorem 1.6. *Let G be a countable group of isometries of a δ-hyperbolic metric space X, and let (G, Γ) be an almost semisimple graph structure which is either:*

(1) *nonelementary;*
(2) *thick relative to a nonelementary subgroup $H < G$; or*
(3) *growth quasitight relative to a nonelementary subgroup $H < G$.*

Then there exists $L > 0$ such that for every $\epsilon > 0$ one has that:

(i) *Displacement grows linearly:*
$$\frac{\#\{g \in S_n \ : \ d_X(gx, x) \geq (L - \epsilon)n\}}{\#S_n} \to 1 \qquad \text{as } n \to \infty.$$
(ii) *Translation length grows linearly:*
$$\frac{\#\{g \in S_n \ : \ \tau_X(g) \geq (L - \epsilon)n\}}{\#S_n} \to 1 \qquad \text{as } n \to \infty.$$
(iii) *As a consequence, loxodromic elements are generic:*
$$\frac{\#\{g \in S_n \ : \ g \text{ is } X-\text{loxodromic}\}}{\#S_n} \to 1 \qquad \text{as } n \to \infty.$$

If we are interested in counting with respect to balls in the Cayley graph, we get the following immediate consequence.

Corollary 1.7. *Let G be a group with finite generating set S. Suppose that*

(i) *there is a geodesic combing for (G, S);*
(ii) *G has pure exponential growth with respect to S; and*
(iii) *the combing for (G, S) satisfies at least one of the conditions (1), (2), (3) above.*

Then for any nonelementary action $G \curvearrowright X$ on a hyperbolic space, the set of loxodromic elements is generic with respect to S.

Note that, as we will see in detail later in Example 1, the right-angled Artin group $G = F_2 \times F_3$ with the standard generators has a geodesic combing and has pure exponential growth but loxodromic elements are not generic, so an additional dynamical condition (such as (1), (2), (3)) must be added. In fact, we will show that for graph products such as RAAGs and RACGs this condition amounts essentially to the group G not being a product. Moreover, we will prove that the three conditions are related, namely (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

2. Applications

2.1. **Hyperbolic groups.** By Cannon's theorem [Can84], a hyperbolic group G admits a geodesic combing for *any* generating set. In fact, the language recognized by the graph is defined by choosing for each $g \in G$ the smallest word (in lexicographic order) among all words of minimal length which represent g. This is called the ShortLex representative. We proved in [GTT16] that this graph structure is nonelementary, hence we can apply Theorem 1.6. This shows that Theorem 1.6 is a direct generalization of the main theorem from [GTT16].

2.2. **RAAGs, RACGs, and graph products.** Let G be a right angled Artin or Coxeter group, and let S be its standard vertex generating set. A result of Hermiller and Meier [HM95] implies that (G, S) is ShortLex automatic. In our language, (G, S) admits a geodesic combing. However, the graph Γ parameterizing this language of geodesics does not have the correct dynamical properties needed to apply Theorem 1.6. In Section 10, we modify their construction to show that when G is not a direct product, it has a graph structure with respect to the standard generators with the strongest possible dynamical properties. We then obtain the following:

Theorem 2.1. *Let G be a right-angled Artin or Coxeter group which is not virtually cyclic and does not split as a product, and consider an action of G on a hyperbolic metric space X. Then eq.* (1) *holds, and loxodromic elements are generic with respect to the standard generators.*

We note that there are many examples of actions of such G on locally infinite hyperbolic graphs. The most natural of which are the extension graph [KK14] (in the case of a RAAG) and the contact graph [Hag14] (in the case of a RACG). For both of these actions, the loxodromic isometries are all rank 1 (or Morse) elements of G [BC12, CS14] and so Theorem 2.1 implies that the rank 1 elements of these groups are generic.

Actually, Theorem 2.1 applies to all graph products of groups with geodesic combing (Theorem 10.5). We refer the reader to Section 10 for details. Let us point out that RAAGs which are products give examples of actions where loxodromics are *not* generic:

Example 1 (Nongenericity in general)**.** Denote the free group of rank n by F_n and fix a free basis as a generating set. Let $G = F_2 \times F_3$ and let X denote a Cayley graph for F_2. Give G its standard generating set; that is, the generating set consisting of a basis for F_2 and a basis for F_3. Consider the action $G \curvearrowright X$ in which the F_2 factor acts by left multiplication and the right factor acts trivially. If we denote the set of loxodromics for the action by LOX, then

$$\lim_{n\to\infty} \frac{\#\left(\mathrm{LOX} \cap B_n\right)}{\# B_n} = \frac{2}{3} \neq 1.$$

Note that in the example above G has pure exponential growth and a geodesic combing, so these two conditions are not sufficient to yield genericity of loxodromics. Moreover, the complement of LOX is a subgroup $H < G$ which has infinite index and positive density, showing that conditions are needed also in Theorem 1.2.

Let us also see that growth quasitightness fails. To construct a geodesic combing for G, let Γ_1 be a geodesic combing for F_3 and let Γ_2 be a geodesic combing for F_2 as discussed above. To obtain a geodesic combing for G, take the disjoint union of Γ_1 and Γ_2, remove the initial vertex for Γ_2 and introduce arrows from any non-initial vertex of Γ_1 to any (non-initial) vertex of Γ_2, with the appropriate labels.

Let us consider the set Ω of paths starting at the initial vertex which always stay inside Γ_1, and let $w \in F_2 \times \{1\}$. We claim that the set $Y_{w,c}$ contains Ω whenever $|w| > 2c$. Since Ω has positive density in G, the claim contradicts growth quasitightness. To prove the claim, let $\gamma \in \Omega$. If γ does not belong to $Y_{w,c}$, then there exists a, b with $|a|, |b| \leq c$ and such that $w = a \cdot \mathrm{ev}(p) \cdot b$ with p a subpath of γ. Since p is a subpath of γ, then $\mathrm{ev}(p)$ lies in $H = \{1\} \times F_3$. Consider the projection $\pi \colon F_2 \times F_3 \to F_2$. If $w = a \cdot \mathrm{ev}(p) \cdot b$, then $\pi(w) = \pi(a)\pi(b)$ hence $|w| = |\pi(w)| \leq 2c$, which proves the claim.

Moreover, as a consequence of the geodesic combing that we produce in order to prove the previous theorem, we also prove the following fine counting statement for the number of elements in a sphere with respect to the standard generating set. As far as we know, this result is also new, and it may be of independent interest.

Theorem 2.2. *Let G be a right-angled Artin group or Coxeter group which is not virtually cyclic and does not split as a product. Then there exists $\lambda > 1$, $C > 0$ such that*

$$\lim_{n\to\infty} \frac{\# S_n}{\lambda^n} = C.$$

We say that a group with a generating set with the previous property has *exact exponential growth*. This is stronger than *pure exponential growth* (where one only requires $C^{-1}\lambda^n \leq \# S_n \leq C\lambda^n$), and depends very subtly on the choice of generating set. In fact, in Theorem 11.1 we will establish this result more generally for graph products.

Note that growth functions for graph products are known by Chiswell [Chi94] and Athreya-Prasad [AP14]; however, it does not seem obvious how to use them to prove the result above.

2.3. **Relatively hyperbolic groups.** Our results also apply to a large class of relatively hyperbolic groups. Just as above, we note that there are many natural action of relatively hyperbolic groups on locally infinite hyperbolic graphs. The most famous of these is the coned-off Cayley graph introduced by Farb [Far98], where the loxodromics of G are exactly the elements not conjugate into a peripheral subgroup.

To apply our general theorem to relatively hyperbolic groups, we need two hypotheses.

First, recall that a relatively hyperbolic group $(G, \mathcal{P})$ is equipped with a compact metric space ∂G known as its *Bowditch boundary*, and such a space carries a natural *Patterson-Sullivan measure* ν, defined with respect to the word metric on $\mathrm{Cay}(G, S)$ (see Section 9.2). We call a relatively hyperbolic group G with a generating set S *doubly ergodic* if the action of G on $\partial G \times \partial G$ is ergodic with respect to the measure $\nu \times \nu$.

Second, we need a geodesic combing with respect to some generating set S. Let us call a finite generating set S *admissible* if G admits a geodesic combing with respect to S. We have the following general statement:

Theorem 2.3. *Let G be a relatively hyperbolic group with an admissible generating set S for which G is doubly ergodic. Then, for each action of G on a hyperbolic metric space X, there exists $L > 0$ such that*

$$\frac{\#\{g \in B_n : \tau_X(g) \geq Ln\}}{\#B_n} \to 1 \qquad \text{as } n \to \infty.$$

As a consequence, X-loxodromic elements are generic.

In fact, by [AC16] and [NS95], many relatively hyperbolic groups admit geodesic combings as follows. Let us call a finitely generated group G *geodesically completable* if any finite generating set S of G can be extended to a finite generating set $S' \supseteq S$ for which there exists a geodesic biautomatic structure. Antolín and Ciobanu ([AC16], Theorem 1.5) proved that whenever G is hyperbolic relative to a collection of subgroups $\mathcal{P}$ each of which is geodesically completable, then G is geodesically completable. Moreover, from automata theory ([HRR17], Theorem 5.2.7) one gets that if G admits a geodesic biautomatic structure for S, then it also admits a geodesic combing for the same S. This yields:

Proposition 2.4. *Let $(G, \mathcal{P})$ be a relatively hyperbolic group such that each parabolic subgroup $P \in \mathcal{P}$ is geodesically completable. Then every finite generating set S can be extended to a finite generating set S' which admits a geodesic combing.*

Let us note that in particular, virtually abelian groups are geodesically completable ([AC16], Proposition 10.1), hence any group hyperbolic relative to a collection of virtually abelian subgroups is geodesically completable and admits a geodesic combing. Moreover, we will prove (Proposition 9.17):

Proposition 2.5. *If a group G acts geometrically finitely on a CAT(-1) proper metric space, then G is doubly ergodic with respect to any finite generating set.*

In particular, geometrically finite Kleinian groups satisfy both hypotheses of Theorem 2.3, which establishes Theorem 1.1 (1) as a corollary of Theorem 2.3.

2.4. **Actions with strongly contracting elements.** Let us now remark that by combining our work with recent work of W. Yang one can apply our theorem in greater generality. Following [ACT15] and [Yan16], we call an element $g \in G$ *strongly contracting* for the action on $\mathrm{Cay}(G, S)$ if $n \mapsto g^n$ is a quasigeodesic and there exists $C, D \geq 0$ such that for any geodesics γ in $\mathrm{Cay}(G, S)$

whose distance from $\langle g \rangle$ is at least C, the diameter of the image of γ under the nearest point projection to $\langle g \rangle$ is bounded by D.

Wenyuan Yang [Yan16] has recently announced that whenever the action $G \curvearrowright \mathrm{Cay}(G,S)$ has a strongly contracting element, G is growth quasitight and has pure exponential growth with respect to S. Combining Theorem 1.7 with Yang's result we obtain the following:

Corollary 2.6. *Let G be a group with finite generating set S. Suppose that the Cayley graph $\mathrm{Cay}(G,S)$ has a strongly contracting element and that (G,S) has a geodesic combing. Then for any nonelementary action $G \curvearrowright X$ on a hyperbolic space, the set of loxodromic elements is generic with respect to S.*

2.5. **Genericity with respect to the Markov chain.** Our approach is to deduce typical properties of elements of G from typical long term behavior of paths in the associated graph structure. As a by-product, we also obtain a general theorem about generic elements for sample paths in a Markov chain, which may be of independent interest. More precisely, an almost semisimple graph Γ defines a Markov chain on the vertices of Γ (see Section 3.3), hence it defines a Markov measure $\mathbb{P}$ on the set Ω_0 of infinite paths from the initial vertex. For such Markov chains, we prove the following:

Theorem 2.7. *Let (G,Γ) be an almost semisimple, nonelementary graph structure for $G \curvearrowright X$, and let $x \in X$. Then:*

1. *For $\mathbb{P}$-almost every sample path (w_n), the sequence $(w_n x)$ converges to a point in ∂X;*
2. *There exist finitely many constants $L_i > 0$ $(i = 1, \dots, r)$ such that for $\mathbb{P}$-almost every sample path there exists an index i such that*
$$\lim_{n\to\infty} \frac{d(w_n x, x)}{n} = L_i;$$
3. *If we denote $L := \min_{1\le i\le r} L_i$, then for each $\epsilon > 0$ one has*
$$\mathbb{P}(\tau_X(w_n) \geq n(L-\epsilon)) \to 1$$
as $n \to \infty$. As a consequence,
$$\mathbb{P}(w_n \text{ is loxodromic}) \to 1 \quad \textit{as } n \to \infty.$$

2.6. **Non-backtracking random walks.** An illustration of the previous result is given by looking at non-backtracking random walks. Let G be a group and $S = S^{-1}$ a generating set. The *non-backtracking random walk* on G is the process defined by taking g_n uniformly at random among the elements of $S \setminus \{(g_{n-1})^{-1}\}$ and considering the sample path $w_n = g_1 g_2 \dots g_n$. We prove the following, which answers a question of I. Kapovich.

Theorem 2.8. *Let G be a nonelementary group of isometries of a hyperbolic metric space X, and let S be a finite generating set. Consider the non-backtracking random walk*
$$w_n := g_1 \dots g_n$$
defined as above, and let $\mathbb{P}$ be corresponding the measure on the set Ω_0 of sample paths. Then
$$\mathbb{P}\left(w_n \text{ is loxodromic on } X\right) \to 1$$
as $n \to \infty$.

Proof. Let us consider $F = F(S)$ the free group generated by S, with its standard word metric. By composing the surjection $F \to G$ with the action on X, we can think of F as a group of isometries of X. Then F has a standard geodesic combing, whose graph Γ has only one non-trivial component, hence (by Proposition 6.3) the graph structure (F,Γ) is thick, hence nonelementary. The result then follows from Theorem 2.7. □

2.7. **Previous results.** Beginning with Gromov's influential works [Gro87, Gro93, Gro03], there is a large literature devoted to studying typical behavior in finitely generated groups. More recent developments can be found, for example, in [AO96, Arz98, BMR03, Cha95, KMSS03, KS05, KRSS07, Ol'92].

If one takes the definition of genericity with respect to random walks, instead of using counting in balls, then genericity of loxodromics has been established in many cases. In particular, the question of genericity of pseudo-Anosovs in the mapping class group goes back to at least Dunfield-Thurston [DT06], and for random walks it has been proven independently by Rivin [Riv08] and Maher [Mah11]. This relates to our setup, as a mapping class is pseudo-Anosov if and only if it acts loxodromically on the curve complex. Genericity of loxodromics for random walks on groups of isometries of hyperbolic spaces has been established with increasing level of generality in [CM15, Sis11, MT14]. Let us note that in general counting in balls and counting with random walks need not yield the same result. For instance, in Example 1, loxodromics are not generic with respect to counting the the Cayley graph despite that fact that they are typical with respect to reasonable random walks. In fact it is a very important problem to establish whether the harmonic measure for the random walk can coincide with a Patterson-Sullivan-type measure, given by taking limits of counting measures over balls. Many results in this area show that the two measures do not coincide except in particular cases (cf. [GMM15]), while an existence result of a random walk for which harmonic and PS measure coincide is due for groups of isometries of $CAT(-1)$ spaces to Connell-Muchnik [CM07].

As for counting in balls, Wiest [Wie14] recently showed that if a group G satisfies a weak automaticity condition and the action $G \curvearrowright X$ on a hyperbolic space X satisfies a strong *geodesic word hypothesis*, then the loxodromics make up a *definite proportion* of elements of the n ball. This geodesic word hypothesis essentially requires geodesics in the group G, given by the normal forms, to project to unparameterized quasigeodesics in the space X under the orbit map. In our work, on the other hand, we do not assume any nice property of the action except it being by isometries. Let us note that our theorems answer (when the hypotheses of our two approaches overlap) the open problems (3) (4) in ([Wie14], section 2.12).

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3. Background material

Since graph structures play a central role in our work, we begin by discussing some further details. The reader will notices that much of this is inspired by the theory of regular languages and automatics groups [EPC$^+$92], but we place a special focus on the graph which parameterizes the language. Thus our terminology may differ from that in the literature.

3.1. **Graph structures.** The general framework is as follows. We define a **graph structure** to be a pair (G, Γ) where G is a countable group and Γ is a directed, finite graph such that:

(1) there is a labeled vertex v_0, called the *initial vertex*; for every other vertex v there exists a directed path from v_0 to v;
(2) every edge is labeled by a group element such that edges directed out of a fixed vertex have distinct label.

Thus, there exists an **evaluation map** $\mathrm{ev}\colon E(\Gamma) \to G$ and this map extends to the set of finite paths in Γ by concatenating edge labels. Here and in what follows, the term path will always refer to a directed path. If $S = \mathrm{ev}(E(\Gamma))$, we say that (G, Γ) is a graph structure with respect to S. We denote by Ω_0 the set of finite paths starting at v_0 and by Ω the set of all finite paths. When $\mathrm{ev}(\Omega_0) = G$, we call the graph structure **surjective**; in this case $S = \mathrm{ev}(E(\Gamma))$ generates G as a semigroup. A surjective graph structure is **geodesic** if for each path $p \in \Omega$, the word length $|\mathrm{ev}(p)|_S$ is equal to the length of the path. In this case, all paths in Γ evaluate naturally to geodesic paths in the Cayley graph $\mathrm{Cay}(G, S)$. Finally, the graph structure is called **injective**, if $\mathrm{ev}\colon \Omega_0 \to G$ is injective. For example, if each path in Ω_0 labels the ShortLex geodesic representative of its evaluation (with respect to some ordering on S), then (G, Γ) is injective. A bijective, geodesic graph structure (G, Γ) with respect to S is called a **geodesic combing** of G with respect to S.

Note the evaluation map, restricted to Ω_0, factors through S^*, the set of all words in the alphabet S. The image in S^* of Ω_0 (i.e. all words which can be spelled starting at v_0) is called the language parameterized (or recognized) by Γ. This language is prefix closed by construction; an initial subword of a recognized word is also recognized. We warn the reader that references differ on the exact meaning on some of these terms. For example, Calegari–Fujiwara use the term "combing" to refer to the *language* of a bijective, geodesic graph structure rather than the graph structure itself [CF10, Cal13]. Since we will be most interested in dynamical properties of the graph parameterizing the language of geodesics, we choose to emphasize the graph structure.

3.2. **Almost semisimple graphs.** Let us summarize some of the fundamental properties about graphs and Markov chains. Much of this material appears in Calegari–Fujiwara [CF10], and we refer to that article and [GTT16] for more details and proofs.

Let Γ be a finite, directed graph with vertex set $V(\Gamma) = \{v_0, v_1, \dots, v_{r-1}\}$. The *adjacency matrix* of Γ is the $r \times r$ matrix $M = (M_{ij})$ defined so that M_{ij} is the number of edges from v_i to v_j.

Such a graph is *almost semisimple* of growth $\lambda > 1$ if the following hold:

(1) There is an *initial vertex*, which we denote as v_0;
(2) For any other vertex v, there is a (directed) path from v_0 to v;
(3) The largest modulus of the eigenvalues of M is λ, and for any eigenvalue of modulus λ, its geometric multiplicity and algebraic multiplicity coincide.

Note that by Perron-Frobenius theory λ is in fact an eigenvalue. We denote by Ω the set of all finite paths in Γ, Ω_v for the set of finite paths starting at v, and $\Omega_0 = \Omega_{v_0}$ the set of finite paths starting at v_0. For a path $g \in \Omega$, we use $[g]$ to denote its terminal vertex. Similarly, we denote as Ω^∞ the set of all infinite paths in Γ, Ω^∞_v the set of infinite paths starting at v and $\Omega^\infty_0 = \Omega^\infty_{v_0}$.

Given two vertices v_1, v_2 of a directed graph, we say that v_2 is *accessible from* v_1 and write $v_1 \to v_2$ if there is a path from v_1 to v_2, and two vertices are *mutually accessible* if $v_1 \to v_2$ and $v_2 \to v_1$. Mutual accessibility is an equivalence relation, and equivalence classes are called *irreducible components* of Γ.

For any subset $A \subseteq \Omega_0$, we define the *growth* $\lambda(A)$ of A as

$$\lambda(A) := \limsup_{n\to\infty} \sqrt[n]{\#(A \cap S_n)},$$

where $S_n \subset \Omega_0$ is the set of all paths starting at v_0 that have length n.

For each vertex v of Γ which lies in an irreducible component C, let $\mathcal{P}_v(C)$ denote the set of finite paths in Γ based at v which lie entirely in C. Moreover, for any path g from v_0 to v, we let $\mathcal{P}_g(C) = g \cdot \mathcal{P}_v(C)$ be the set of finite paths in Ω which can be written as a concatenation of g with a path contained entirely in C.

Definition 3.1. An irreducible component C of Γ is called *maximal* if for some (equivalently, any) $g \in \Omega_0$ with $[g] \in C$, the growth of $\mathcal{P}_g(C)$ equals λ. A vertex is *maximal* if it belong to a component of maximal growth. Moreover, we say a vertex v_i of Γ has *large growth* if there exists a path from v_i to a vertex in a maximal component, and it has *small growth* otherwise.

Definition 3.2. For every vertex v of Γ, the *loop semigroup* of v is the set L_v of loops in the graph Γ which begin and end at v. It is a semigroup with respect to concatenation. A loop in L_v is *primitive* if it is not the concatenation of two (non-trivial) loops in L_v.

Let Γ be an almost semisimple graph of growth $\lambda > 1$. Then there exist constants $c > 0$ and $\lambda_1 < \lambda$ such that ([GTT16], Lemma 2.3):

(1) For any vertex v of large growth and any $n \geq 0$,
$$c^{-1}\lambda^n \leq \#\{\text{paths from } v \text{ of length } n\} \leq c\lambda^n$$
(2) For any vertex v of small growth and any $n \geq 0$,
$$\#\{\text{paths from } v \text{ of length } n\} \leq c\lambda_1^n$$
(3) If v belongs to the maximal component C, then for any $n \geq 0$
$$c^{-1}\lambda^n \leq \#\{\text{paths in } \mathcal{P}_v(C) \text{ of length } n\} \leq c\lambda^n$$
and also ([GTT16], Lemma 6.5)
$$c^{-1}\lambda^n \leq \#\{\text{paths in } L_v \text{ of length } n\} \leq c\lambda^n$$

3.3. **Markov chains.** Given an almost semisimple graph Γ of growth λ with edge set $E(\Gamma)$, one constructs a Markov chain on the vertices of Γ as follows. Let us define for each i the quantity
$$\rho_i := \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N} \frac{(M^n 1)_i}{\lambda^n},$$
where 1 is the vector all of whose coordinates equal 1, and w_i denotes the i^{th} coordinate of the vector w. Note that $\rho_i > 0$ if and only if v_i has large growth. Then if v_i has large growth, we set the probability $\mu(v_i \to v_j)$ of going from v_i to v_j as
$$\mu(v_i \to v_j) = \frac{M_{ij}\rho_j}{\lambda\rho_i}, \tag{2}$$
and if v_i has small growth, we set $\mu(v_i \to v_j) = 0$ for $i \neq j$ and $\mu(v_i \to v_i) = 1$.

Now, for each vertex v the measure μ induces measures $\mathbb{P}^n_v$ on the space Ω_v of finite paths starting at v, simply by setting
$$\mathbb{P}^n_v(\gamma) = \mu(e_1)\cdots\mu(e_n)$$
for each path $\gamma = e_1 \dots e_n$ of length n starting at v, and $\mathbb{P}^n_v(\gamma) = 0$ otherwise. Similarly, we define a measure $\mathbb{P}_v$ on the space Ω^∞_v of infinite paths starting at v by setting the measure $\mathbb{P}_v$ of the (cylinder) set of all infinite paths starting with γ equal to $\mathbb{P}^n_v(\gamma)$, where $n = |\gamma|$. The most important cases for us will be the measures on the set of (finite and infinite, respectively) paths starting at v_0, which we will denote as $\mathbb{P}^n = \mathbb{P}^n_{v_0}$ and $\mathbb{P} = \mathbb{P}_{v_0}$. Each measure $\mathbb{P}_v$ defines a Markov chain on the space $V(\Gamma)$, and we consider for each n the random variable
$$w_n : \Omega^\infty \to \Omega$$
$$w_n((e_1, \dots, e_n, \dots)) = e_1 \dots e_n$$
defined as the concatenation of the first n edges of the infinite path.

In order to compare the n-step distribution for the Markov chain to the counting measure, let us denote as Ω_{LG} the set of paths from v_0 ending at a vertex of large growth. Then we note ([GTT16], Lemma 3.4) that there exists $c > 1$ such that, for each $A \subseteq \Omega_0$,
$$c^{-1}\, \mathbb{P}^n(A) \leq P^n(A \cap \Omega_{LG}) \leq c\, \mathbb{P}^n(A). \tag{3}$$

It turns out (see [GTT16], Lemma 3.3) that, with respect to this choice of measure, a vertex v belongs to a maximal irreducible component of Γ if and only if it is *recurrent*, i.e. :

(1) there is a path from v_0 to v of positive probability; and

(2) whenever there is a path from v to another vertex w of positive probability, there is also a path from w to v of positive probability.

For this reason, maximal components will also be called *recurrent components*.

It is easy to see that for almost every path of the Markov chain there exists one recurrent component C such that the path lies completely in C from some time on, and visits each vertex of C infinitely many times. Thus, for each recurrent component C, we let Ω_C be the set of all infinite paths from the initial vertex which enter C and remain inside C forever, and denote as $\mathbb{P}_C$ the conditional probability of $\mathbb{P}$ on Ω_C.

Moreover, for each recurrent vertex v the distribution of return times decays exponentially: There is a $c > 1$ such that

$$\mathbb{P}_v\left(\tau_v^+ = n\right) \le ce^{-n/c} \tag{4}$$

where $\tau_v^+ = \min\{n \ge 1 \ : \ [w_n] = v\}$ denotes the first return time to vertex v.

We will associate to each recurrent vertex of the Markov chain a random walk, and use previous results on random walks to prove statements about the asymptotic behavior of the Markov chain.

For each sample path $\omega \in \Omega^\infty$, let us define $n(k, v, \omega)$ as the k^{th} time the path ω lies at the vertex v. In formulas,

$$n(k, v, \omega) := \begin{cases} 0 & \text{if } k = 0 \\ \min\left(h > n(k-1, v, \omega) \ : \ [w_h] = v\right) & \text{if } k \ge 1 \end{cases}$$

To simplify notation, we will write $n(k, v)$ instead of $n(k, v, \omega)$ when the sample path ω is fixed.

We now define the *first return measure* μ_v on the set of primitive loops by setting, for each primitive loop $\gamma = e_1 \dots e_n$ with edges $e_1, \dots, e_n$,

$$\mu_v(\gamma) = \mu(e_1) \dots \mu(e_n).$$

Extend μ_v to the entire loop semigroup L_v by setting $\mu_v(\gamma) = 0$ if $\gamma \in L_v$ is not primitive. Note that almost every path starting at v visits v infinitely many times, so it can be decomposed as the infinite concatenation of primitive loops; moreover, $\mu_v(\gamma)$ equals the probability that the first loop in this decomposition equals γ. Hence, μ_v is a probability measure.

By equation (4), for every recurrent vertex v, the first return measure μ_v has finite exponential moment, i.e. there exists a constant $\alpha > 0$ such that

$$\int_{L_v} e^{\alpha|\gamma|} \, d\mu_v(\gamma) < \infty. \tag{5}$$

3.4. **Hyperbolic spaces.** In this paper, X will always be a geodesic metric space. Such a space is called δ*-hyperbolic* for some $\delta \ge 0$ if for every geodesic triangle in X, each side is contained within the δ–neighborhood of the other two sides. Given $x, y, z \in X$, their *Gromov product* is defined as $(y, z)_x := \frac{1}{2}(d(x, y) + d(x, z) - d(y, z))$. Each hyperbolic space has a well-defined *Gromov boundary* ∂X, and we refer the reader to [BH09, Section III.H.3], [GdlH90], or [KB02, Section 2] for definitions and properties.

If g is an isometry of X, its *translation length* is defined as

$$\tau_X(g) := \lim_{n\to\infty} \frac{d(g^n x, x)}{n}$$

where the limit does not depend on the choice of $x \in X$. In order to estimate the translation length, we will use the following well-known lemma; see for example [MT14, Proposition 5.8].

Lemma 3.3. *There exists a constant c, which depends only on δ, such that for any isometry g of a δ-hyperbolic space X and any $x \in X$ with $d(x, gx) \ge 2(gx, g^{-1}x)_x + c$, the translation length of g is given by*

$$\tau_X(g) = d(x, gx) - 2(gx, g^{-1}x)_x + O(\delta).$$

An isometry g of X is *loxodromic* if it has positive translation length; in that case, it has precisely two fixed points on ∂X. We say two loxodromic elements are *independent* if their fixed point sets are disjoint. A semigroup (or a group) $G < \text{Isom } X$ is *nonelementary* if it contains two independent loxodromics. We will use the following criterion.

Proposition 3.4 ([DSU14, Proposition 7.3.1])**.** *Let L be a semigroup of isometries of a hyperbolic metric space X. If the limit set $\Lambda_L \subset \partial X$ of L on the boundary of X is nonempty and L does not have a finite orbit in ∂X, then L is nonelementary.*

Finally, we turn to the definition and basic properties of shadows in the δ-hyperbolic space X. For $x, y \in X$, the *shadow in X around y based at x* is

$$S_x(y, R) = \{z \in X : (y, z)_x \geq d(x, y) - R\},$$

where $R > 0$. The *distance parameter* of $S_x(y, R)$ is by definition the number $r = d(x, y) - R$, which up to an additive constant depending only on δ, measures the distance from x to $S_x(y, R)$. Indeed, $z \in S_x(y, R)$ if and only if any geodesic $[x, z]$ 2δ–fellow travels any geodesic $[x, y]$ for distance $r + O(\delta)$. The following observation is well-known.

Lemma 3.5. *For each $D \geq 0$, and each x, y in a metric space, we have*

$$N_D(S_x(y, R)) \subseteq S_x(y, R + D).$$

3.5. **Random walks.** A probability measure μ on G is said to be *nonelementary* with respect to the action $G \curvearrowright X$ if the semigroup generated by the support of μ is nonelementary.

We will need the fact that a random walk on G whose increments are distributed according to a nonelementary measure μ almost surely converge to the boundary of X and has positive drift in X.

Theorem 3.6 ([MT14, Theorems 1.1, 1.2])**.** *Let G be a countable group which acts by isometries on a hyperbolic space X, and let μ be a nonelementary probability distribution on G. Fix $x \in X$, and let (u_n) be the sample path of a random walk with independent increments with distribution μ. Then:*

(1) *almost every sample path $(u_n x)$ converges to a point in the boundary of ∂X, and the resulting hitting measure ν is nonatomic;*
(2) *moreover, if μ has finite first moment, then there is a constant $L > 0$ such that for almost every sample path*

$$\lim_{n \to \infty} \frac{d(x, u_n x)}{n} = L > 0.$$

The constant $L > 0$ in Theorem 3.6 is called the *drift* of the random walk (u_n). Let us remark, as suggested in [GST17, Remark 4], we do not need to assume that X is separable (see also [MT18, Lemma 2.6]).

4. Behavior of generic sample paths for the Markov chain

Let G be a group with a nonelementary action $G \curvearrowright X$ on a hyperbolic space X. In this section we assume that G has a graph structure (G, Γ) which is almost semisimple and nonelementary.

4.1. **Convergence to the boundary of X.** Here we show that almost every sample path for the Markov chain converges to the boundary of X. Since we are *assuming* that the graph structure is nonelementary, the exact same proof as in ([GTT16], Theorem 6.8) yields the following.

Theorem 4.1. *For $\mathbb{P}$-almost every path (w_n) in the Markov chain, the projection $(w_n x)$ to the space converges to a point in the boundary ∂X.*

As a consequence, we have for every vertex v of large growth a well-defined *harmonic measure* ν_v^X, namely the hitting measure for the Markov chain on ∂X: for each (Borel) $A \subseteq \partial X$ we define

$$\nu_v^X(A) := \mathbb{P}_v(\lim_{n\to\infty} w_n x \in A).$$

Theorem 4.1 together with the Markov property implies a decomposition result for the harmonic measures ν_v^X. Indeed, if $\mathcal{R}$ is the set of recurrent vertices of Γ, then we have:

$$\nu_v^X = \sum_{w\in\mathcal{R}} \sum_{\gamma : v\to w} \mu(\gamma)\mathrm{ev}(\gamma)_* \nu_w^X \tag{6}$$

Here, the sum is over all finite paths from v to w which only meet a recurrent vertex at their terminal endpoint. Note that if v is recurrent, then ν_v^X is the harmonic measure for the random walk on G generated by the first return measure μ_v, as defined in Section 3.3 (see also [GTT16], Lemma 4.2).

Lemma 4.2. *For any v of large growth, the measure ν_v^X is non-atomic.*

Proof. Since the random walk measures ν_w^X are non-atomic, so are the measures $\mathrm{ev}(\gamma)_*\nu_w^X$ for each γ, hence by equation (6) the measure ν_v^X is also non-atomic as it is a linear combination of non-atomic measures. □

4.2. **Positive drift along geodesics.** Next we show that almost every sample path has a well-defined and positive drift in X.

Theorem 4.3. *For $\mathbb{P}$-almost every sample path $\omega = (w_n)$ there exists a recurrent component $C = C(\omega)$ for which we have*

$$\lim_{n\to\infty} \frac{d(w_n x, x)}{n} = L_C,$$

where $L_C > 0$ depends only on C.

Since Γ is finite, this gives at most finitely many potential drifts for the Markov chain.

Proof. Let v be a recurrent vertex. Since the graph structure is nonelementary, the loop semigroup Γ_v is nonelementary, hence the random walk given by the return times to v has positive drift. More precisely, from Theorem 3.6, there exists a constant $\ell_v > 0$ such that for almost every sample path which enters v,

$$\lim_{k\to\infty} \frac{d(w_{n(k,v)}x, x)}{k} = \ell_v.$$

Morever, as the distribution of return times has finite exponential moment, for almost every sample path the limit

$$T_v := \lim_{k\to\infty} \frac{n(k, v, \omega)}{k}$$

exists. These two facts imply

$$\lim_{k\to\infty} \frac{d(w_{n(k,v)}x, x)}{n(k, v)} = \frac{\ell_v}{T_v}.$$

Now, almost every infinite path visits every vertex of some recurrent component infinitely often. Thus, for each recurrent vertex v_i which belongs to a component C, there exists a constant $L_i > 0$ such that for $\mathbb{P}_C$-almost every path (w_n), there is a limit

$$L_i = \lim_{k\to\infty} \frac{d(w_{n(k,v_i)}x, x)}{n(k, v_i)}.$$

Let C be a maximal component, and $v_1, \dots, v_k$ its vertices. Our goal now is to prove that $L_1 = L_2 = \dots L_k$. Let us pick a path $\omega \in \Omega_0$ such that the limit L_i above exists for each $i = 1, \dots, k$, and define $A_i = \{n(k, v_i), k \in \mathbb{N}\}$, and the equivalence relation $i \sim j$ if $L_i = L_j$. Since $w_{n(k,v_i)}$ and $w_{n(k,v_i)+1}$ differ by one generator, $d(w_{n(k,v_i)}x, w_{n(k,v_i)+1}x)$ is uniformly bounded, hence

$$\lim_{k\to\infty} \frac{d(w_{n(k,v_i)+1}x, x)}{n(k, v_i) + 1} = \lim_{k\to\infty} \frac{d(w_{n(k,v_i)}x, x)}{n(k, v_i)} = L_i$$

so the equivalence relation satisfies the hypothesis of ([GTT16], Lemma 6.9), hence there is a unique limit $L_C = L_i$ so that

$$\lim_{n\to\infty} \frac{d(w_n x, x)}{n} = L_C.$$

□

Corollary 4.4. *For every vertex v of large growth, and for $\mathbb{P}_v$-almost every sample path (w_n) there exists a recurrent component C accessible from v such that we have*

$$\lim_{n\to\infty} \frac{d(w_n x, x)}{n} = L_C.$$

Proof. By Theorem 4.3, for $\mathbb{P}$-almost every path which passes through v, the drift equals L_C for some recurrent component C. Let g_0 be a path from v_0 to v of positive probability. Then for any $\omega = (w_n) \in \Omega_v^\infty$, the path $(\widetilde{w}_n) = g_0 \cdot \omega$ belongs to Ω_0^∞, and moreover $\widetilde{w}_{n+k} = w_{n+k}(g_0 \cdot \omega) = g_0 \cdot w_n(\omega)$ where $k = |g_0|$. Hence

$$d(w_n x, x) - d(x, g_0 x) \le d(\widetilde{w}_{n+k}x, x) = d(w_n x, g_0 x) \le d(w_n x, x) + d(x, g_0 x)$$

and so by Theorem 4.3

$$\lim_{n\to\infty} \frac{d(w_n x, x)}{n} = \lim_{n\to\infty} \frac{d(\widetilde{w}_{n+k}x, x)}{n} = L_C$$

as required. □

For application in Section 5, we will need the following convergence in measure statement. Let us denote

$$L := \min_{C \text{ recurrent}} L_C > 0 \tag{7}$$

the smallest drift.

Corollary 4.5. *For any $\epsilon > 0$, and for any v of large growth,*

$$\mathbb{P}_v \left(\frac{d(x, w_n x)}{n} \le L - \epsilon \right) \to 0.$$

Proof. By the theorem, the sequence of random variables $X_n = \frac{d(w_n x, x)}{n}$ converges almost surely to a function X_∞ with the finitely many values $L_1, \dots, L_r$. Moreover, for every n the variable X_n is bounded above by the Lipschitz constant of the orbit map $G \to X$. Thus, X_n converges to X_∞ in L^1, yielding the claim. □

4.3. **Decay of shadows for $\mathbb{P}$.** For any shadow S, we denote its closure in $X \cup \partial X$ by $\overline{S}$. Since the harmonic measures ν_v^X for the Markov chain are nonatomic (by Lemma 4.2), we get by the same proof as in [GTT16] the following decay of shadows results.

Proposition 4.6 ([GTT16], Proposition 6.19)**.** *There exists a function $p : \mathbb{R} \to [0, 1]$ with $p(r) \to 0$ as $r \to \infty$, such that for each vertex v and any shadow $S_x(gx, R)$ we have*

$$\mathbb{P}_v\Big(\exists n \ge 0 \ : \ w_n x \in S_x(gx, R)\Big) \le p(r),$$

where $r = d(x, gx) - R$ is the distance parameter of the shadow.

5. Generic elements with respect to the counting measure

We now use the results about generic paths in the Markov chain to obtain results about generic paths with respect to the counting measure.

5.1. Genericity of positive drift.

The first result is that the drift is positive along generic paths:

Theorem 5.1. *Let (G, Γ) be an almost semisimple, nonelementary graph structure, and $L > 0$ be the smallest drift as given by eq. (7). Then for every $\epsilon > 0$ one has*

$$\frac{\#\{g \in S_n \ : \ d(gx, x) \geq (L - \epsilon) \ |g|\}}{\#S_n} \to 1 \qquad \text{as } n \to \infty.$$

The result follows from Corollary 4.5 similarly as in the proof of ([GTT16], Theorem 5.1).

Proof. Let A_L denote the set of paths

$$A_L := \{g \in \Omega_0 \ : \ d(gx, x) \leq L|g|\}.$$

We know by Corollary 4.5 that for any $L' < L$ one has

$$\mathbb{P}^n(A_{L'}) \to 0 \qquad \text{as } n \to \infty.$$

Let us recall that Ω_{LG} denotes the set of paths starting at v_0 and ending at a vertex of large growth, and for each path g of length n let us denote as $\widehat{g}$ the prefix of g of length $n - \lfloor \log n \rfloor$. Then we observe that

$$P^n(A_{L-\epsilon}) \leq \frac{\#\{g \in S_n \ : \ \widehat{g} \notin \Omega_{LG}\}}{\#S_n} + \frac{\#\{g \in S_n \cap A_{L-\epsilon} \ : \ \widehat{g} \in \Omega_{LG}\}}{\#S_n}$$

and it is easy to see that the first term tends to 0 (see [GTT16, Proposition 2.5]). Now, by writing $g = \widehat{g}h$ with $|h| = \lfloor \log |g| \rfloor$ we have that $d(gx, x) \leq (L - \epsilon)|g|$ implies

$$d(\widehat{g}x, x) \leq d(gx, x) + d(\widehat{g}x, gx) \leq (L - \epsilon)|g| + d(x, hx)$$

hence, there exists C such that

$$d(\widehat{g}x, x) \leq (L - \epsilon)|g| + C\lfloor \log |g| \rfloor \leq L'|\widehat{g}|$$

for any $L - \epsilon < L' < L$ whenever $|g|$ is sufficiently large. This proves the inclusion

$$\{g \in S_n \cap A_{L-\epsilon} \ : \ \widehat{g} \in \Omega_{LG}\} \subseteq \{g \in S_n \ : \ \widehat{g} \in A_{L'} \cap \Omega_{LG}\}$$

and by Lemma 3.2 (1)

$$\#\{g \in S_n \ : \ \widehat{g} \in A_{L'} \cap \Omega_{LG}\} \leq c\lambda^{\lfloor \log n \rfloor} \#(S_{n - \lfloor \log n \rfloor} \cap A_{L'} \cap \Omega_{LG}) \leq$$

hence by equation (3) and considering the size of $S_{n - \lfloor \log n \rfloor}$

$$\leq c_1 \lambda^{\lfloor \log n \rfloor} \mathbb{P}^{n - \lfloor \log n \rfloor}(A_{L'}) \#S_{n - \lfloor \log n \rfloor} \leq c_2 \lambda^n \mathbb{P}^{n - \lfloor \log n \rfloor}(A_{L'}).$$

Finally, using that $\mathbb{P}^{n - \lfloor \log n \rfloor}(A_{L'}) \to 0$ we get

$$\limsup_{n \to \infty} \frac{\#\{g \in S_n \cap A_{L-\epsilon} \ : \ \widehat{g} \in \Omega_{LG}\}}{\#S_n} \leq \limsup_{n \to \infty} c_3 \mathbb{P}^{n - \lfloor \log n \rfloor}(A_{L'}) = 0$$

which proves the claim. □

5.2. **Decay of shadows for the counting measure.** For $g \in G$, we set

$$S_x^\Gamma(gx, R) = \{h \in \Omega_0 : hx \in S_x(gx, R)\},$$

where as usual, $S_x(gx, R)$ is the shadow in X around gx centered at the basepoint $x \in X$ and $hx = \mathrm{ev}(h)x$. We will need the following decay property for $S_x^\Gamma(gx, R) \subseteq \Omega_0$.

Proposition 5.2. *There is a function $\rho : \mathbb{R} \to [0, 1]$ with $\rho(r) \to 0$ as $r \to \infty$ such that for every $n \geq 0$*

$$P^n(S_x^\Gamma(gx, R)) \leq \rho\,(d(x, gx) - R)\,.$$

We start with the following lemma in basic calculus.

Lemma 5.3. *Let $p : \mathbb{R} \to [0, 1]$ be a function with $p(x) = 1$ if $x < 0$ and $p(x) \to 0$ as $x \to \infty$. For each $\alpha > 0$ and each $C > 0$, there exists a function $\widetilde{p} : \mathbb{R} \to \mathbb{R}^+$ such that:*

1. *$\widetilde{p}(x) = 1$ for $x < 0$;*
2. *$\widetilde{p}(x) \geq p(x)$ for each $x \in \mathbb{R}^+$;*
3. *$\widetilde{p}(x + mC) \geq \widetilde{p}(x)e^{-\alpha m}$ for each $x \in \mathbb{R}^+$, $m \in \mathbb{N}$:*
4. *$\widetilde{p}(x) \to 0$ as $x \to \infty$.*

Proof. Begin by setting $m_k = \sup\{p(x) : x \geq kC\}$, for $k \geq 0$. For $x < 0$, define $\widetilde{p}(x) = p(x) = 1$. Otherwise, $x \in [Ck, C(k + 1)]$ for $k \geq 0$ an integer, and define

$$\widetilde{p}(x) = \max_{0 \leq i \leq k}\{m_i \cdot e^{-\alpha(k-i)}\}.$$

Item (1) follows by definition. For (2), we have that for $x \in [Ck, C(k + 1)]$, $\widetilde{p}(x) \geq m_k \geq p(x)$. For (3), we again suppose that $x \in [Ck, C(k + 1)]$ so that $x + mC \in [C(k + m), C(k + m + 1)]$. Then

$$\begin{aligned}
\widetilde{p}(x + mC) &= \max_{0 \leq i \leq k+m}\{m_i \cdot e^{-\alpha(k+m-i)}\} \\
&= \max_{0 \leq i \leq k+m}\{m_i \cdot e^{-\alpha(k-i)} \cdot e^{-\alpha m}\} \\
&\geq \max_{0 \leq i \leq k}\{m_i \cdot e^{-\alpha(k-i)} \cdot e^{-\alpha m}\} \\
&= \widetilde{p}(x)e^{-\alpha m}.
\end{aligned}$$

Finally, if the max in the definition of $\widetilde{p}(x)$ occurs for $i \leq k/2$, then $\widetilde{p}(x) \leq m_0 e^{-\alpha k/2}$. If $i \geq k/2$, then $\widetilde{p}(x) \leq m_{k/2}$. Hence, $\widetilde{p}(x) \leq m_0 e^{-\alpha k/2} + m_{k/2} \to 0$ as $x \to \infty$, completing the proof of (4). □

Proof of Proposition 5.2. Pick a path $h \in \Omega_0$ of length n in $S_x^\Gamma(gx, R)$, and let $\widehat{h}$ denote the longest subpath of h starting at the initial vertex and which ends in a vertex of large growth. Let us write $h = \widehat{h}l$ where l is the second part of the path. Note that we have

$$d(hx, \widehat{h}x) = d(\widehat{h}lx, \widehat{h}x) = d(lx, x) \leq kC$$

where $k := |l|$ and C is the Lipschitz constant of the orbit map, hence by Lemma 3.5

$$\widehat{h} \in S_x^\Gamma(gx, R' + kC)$$

where $R' = R + D$ and $D = O(\delta)$. Note that for each element $\widehat{h}$ there are at most $c\lambda_1^k$ choices of the continuation l, hence

$$\#(S_n \cap S_x^\Gamma(gx, R)) \leq c\sum_{k=0}^{n} \lambda_1^k \#(S_{n-k} \cap S_x^\Gamma(gx, R' + kC) \cap \Omega_{LG}) \leq$$

and by using eq. (3) and Proposition 4.6

$$\leq c_1 \sum_{k=0}^{n} \lambda_1^k \lambda^{n-k} \, \mathbb{P}^{n-k}(S_x^\Gamma(gx, R' + kC)) \leq c_1 \lambda^n \sum_{k=0}^{n} (\lambda_1/\lambda)^k \; p(d(x, gx) - R' - kC)$$

Now, by Lemma 5.3 we can replace p by $\widetilde{p}$, choosing α so that $e^\alpha \lambda_1 < \lambda$, thus getting

$$\widetilde{p}(d(x, gx) - R' - kC) \leq e^{\alpha k} \widetilde{p}(d(x, gx) - R')$$

Thus, the previous estimate becomes

$$P^n(S_x^\Gamma(gx, R)) \leq c_2 \sum_{k=0}^{n} (e^\alpha \lambda_1/\lambda)^k \; \widetilde{p}(d(x, gx) - R') \leq c_3 \; \widetilde{p}(d(x, gx) - R')$$

which proves the lemma if one sets $\rho(r) := \min\{c_3 \; \widetilde{p}(r - D), 1\}$. □

5.3. **Genericity of loxodromics.** We now use the previous counting results to prove that loxodromic elements are generic with respect to the counting measure.

The strategy is to apply the formula of Lemma 3.3 to show that translation length grows linearly as function of the length of the path: in order to do so, one needs to show that the distance $d(gx, x)$ is large (as we did in Theorem 5.1) and, on the other hand, the Gromov product $(gx, g^{-1}x)_x$ is not too large. The trick to do this is to split the path g in two subpaths of roughly the same length, and show that the first and second half of the paths are almost independent.

To define this precisely, for each n let us denote $n_1 = \lfloor \frac{n}{2} \rfloor$ and $n_2 = n - n_1$. For each path $g \in \Omega$, we define its *initial part* $i(g)$ to be the subpath given by the first n_1 edges of g, and its *terminal part* $t(g)$ to be the subpath given by the last n_2 edges of g. With this definition, $g = i(g) \cdot t(g)$ and $|i(g)| = n_1$, $|t(g)| = n_2$. Moreover, we define the random variables $i_n, t_n : \Omega^\infty \to \Omega$ by $i_n(w) = i(w_n)$ and $t_n(w) = t(w_n)$. Note that by definition $i_n = w_{n_1}$ and by the Markov property we have for each paths $g, h \in \Omega$:

$$\mathbb{P}(i_n = g \text{ and } t_n = h) = \mathbb{P}(w_{n_1} = g)\mathbb{P}_v(w_{n_2} = h)$$

where $v = [g]$. In the next lemma, we use the notation $C(\omega)$ to refer to the recurrent component to which the sample path $\omega = (w_n)$ eventually belongs, as in Theorem 4.3.

Lemma 5.4. *For any $\epsilon > 0$ we have*

$$\mathbb{P}\left(\lim_{n \to \infty} \left| \frac{d(x, t_n(\omega)x)}{n} - \frac{L_{C(\omega)}}{2} \right| > \epsilon \right) \to 0$$

as $n \to \infty$.

Proof. Note that by definition $t_n(\omega) = w_{n_2}(T^{n_1}\omega)$, where $T : \Omega^\infty \to \Omega^\infty$ is the shift in the space of infinite paths. Note that for every $A \subseteq \Omega^\infty$ by the Markov property we have

$$\mathbb{P}(T^{-n}A) = \sum_{v \in V} \mathbb{P}([w_n] = v)\mathbb{P}_v(A)$$

Let us define the function

$$S_n(\omega, \omega') := \left| \frac{d(x, w_n(\omega)x)}{n} - L_{C(\omega')} \right|.$$

Note that from Corollary 4.4 for every vertex v of large growth and every $\epsilon > 0$

$$\mathbb{P}_v(S_n(\omega, \omega) \geq \epsilon) \to 0$$

Moreover, for every n, if the path $(e_1, \dots, e_n, \dots)$ lies entirely in the component C from some point on, then the same is true for the shifted path $(e_{n+1}, e_{n+2}, \dots)$, i.e. $C(T^n\omega) = C(\omega)$ almost surely, and so

$$S_n(\omega, T^k\omega') = S_n(\omega, \omega') \qquad \text{for all } n, k$$

hence

$$\mathbb{P}(S_{n_2}(T^{n_1}\omega, \omega) > \epsilon) = \mathbb{P}(S_{n_2}(T^{n_1}\omega, T^{n_1}\omega) > \epsilon) =$$
$$= \sum_{v \in V} \mathbb{P}([w_{n_1}] = v])\mathbb{P}_v(S_{n_2}(\omega, \omega) > \epsilon) \leq \sum_{v \in V \cap \Omega_{LG}} \mathbb{P}_v(S_{n_2}(\omega, \omega) > \epsilon)$$

and the right-hand side tends to 0 by Corollary 4.4, proving the claim. □

We now show that $i(g)$ and $t(g)^{-1}$ generically do not fellow travel. For the argument, let $S_n(v)$ denote the set of paths in Ω which start at v and have length n.

Lemma 5.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be any function such that $f(n) \to +\infty$ as $n \to +\infty$. Then*

$$P^n\Big(g \in \Omega_0 \ : \ (i(g)x, t(g)^{-1}x)_x \geq f(n)\Big) \to 0$$

as $n \to \infty$.

Proof. We compute

$$P^n\left(g \ : \ (i(g)x, t(g)^{-1}x)_x \geq f(n)\right) = \frac{\#\{g \in S_{n_1}, [g] = v, h \in S_{n_2}(v) \ : \ (gx, h^{-1}x)_x \geq f(n)\}}{\#S_n} \leq$$

and by fixing v and forgetting the requirement that $[g] = v$ we have

$$\leq \sum_{v \in V} \frac{\#\{g \in S_{n_1}, h \in S_{n_2}(v) \ : \ (gx, h^{-1}x)_x \geq f(n)\}}{\#S_n} \leq$$

then by fixing a value of h

$$\leq \sum_{v \in V} \frac{1}{\#S_n} \sum_{h \in S_{n_2}(v)} \# \left\{g \in S_{n_1} \ : \ gx \in S_x(h^{-1}x, d(x, h^{-1}x) - f(n))\right\} \leq$$

hence from decay of shadows (Proposition 5.2) follows that

$$\leq \sum_{v \in V} \sum_{h \in S_{n_2}(v)} \frac{\rho(f(n))\#S_{n_1}}{\#S_n} \leq \frac{\#V\#S_{n_1}\#S_{n_2}\rho(f(n))}{\#S_n} \leq c\rho(f(n)) \to 0.$$

□

Once we have shown that $i(g)$ and $t(g)^{-1}$ are almost independent, we still need to show that also g and g^{-1} are almost independent. In order to do so, we note that $i(g)$ is the beginning of g while $t(g)^{-1}$ is the beginning of g^{-1}, and then we use the following trick from hyperbolic geometry. See e.g. [TT15].

Lemma 5.6 (Fellow traveling is contagious)**.** *Let X be a δ–hyperbolic space with basepoint x and let that $A \geq 0$. If a, b, c, d are points of X with $(a, b)_x \geq A$, $(c, d)_x \geq A$, and $(a, c)_x \leq A - 3\delta$. Then $(b, d)_x - 2\delta \leq (a, c)_x \leq (b, d)_x + 2\delta$.*

In order to apply Lemma 5.6, we need to check that the first half of g (which is $i(g)$) and the first half of g^{-1} (which is $t(g)^{-1}$) generically do not fellow travel.

Lemma 5.7. *For each $\eta > 0$, the probability*

$$P^n\left(g \in \Omega_0 \ : \ (t(g)^{-1}x, g^{-1}x)_x \leq \frac{n(L - \eta)}{2}\right) \to 0$$

and

$$P^n\left(g \in \Omega_0 \ : \ (i(g)x, gx)_x \leq \frac{n(L - \eta)}{2}\right) \to 0$$

as $n \to \infty$.

Proof. Consider the set $B_L := \{g \in \Omega_0 \ : \ d(x, gx) - d(x, i(g)x) \leq \frac{|g|L}{2}\}$. We know by Theorem 4.3 for $\mathbb{P}$-almost every sample path we have

$$\lim_{n\to\infty} \frac{d(x, w_n x) - d(x, w_{n_1} x)}{n} = \frac{L_C}{2} \geq \frac{L}{2}$$

hence for any $L' < L$ one has $\mathbb{P}^n(B_{L'}) \to 0$ as $n \to \infty$. Hence, as in the proof of Theorem 5.1 we get for any $\epsilon > 0$,

$$P^n\left(d(x, gx) - d(x, i(g)x) \geq \frac{n(L-\epsilon)}{2}\right) \to 1 \tag{8}$$

Finally, by writing out the Gromov product, the triangle inequality and the fact that the action is isometric we get

$$(t(g)^{-1}x, g^{-1}x)_x \geq d(x, g^{-1}x) - d(t(g)^{-1}x, g^{-1}x) = d(x, gx) - d(x, i(g)x)$$

which combined with (8) proves the first half of the claim.

The second claim follows analogously. Namely, from Theorem 4.3 and Lemma 5.4, we have for any $\epsilon > 0$

$$\mathbb{P}\left(d(x, w_n(\omega)x) - d(x, t_n(\omega)x) \leq \frac{n(L_{C(\omega)} - \epsilon)}{2}\right) \to 0$$

which then implies as before

$$P^n\left(d(x, gx) - d(x, t(g)x) \geq \frac{n(L-\epsilon)}{2}\right) \to 1$$

and to conclude we use that

$$(i(g)x, gx)_x \geq d(x, gx) - d(i(g)x, gx) = d(x, gx) - d(x, t(g)x).$$

□

We now use Lemma 5.6 (fellow traveling is contagious) to show that the Gromov products $(gx, g^{-1}x)_x$ do not grow too fast with respect to our counting measures.

Proposition 5.8. *Let $f : \mathbb{N} \to \mathbb{R}$ be a function such that $f(n) \to +\infty$ as $n \to \infty$. Then*

$$P^n\Big((gx, g^{-1}x)_x \leq f(n)\Big) \to 1$$

as $n \to \infty$.

Proof. Define

$$f_1(n) = \min\left\{f(n) - 2\delta, \frac{n(L-\eta)}{2} - 3\delta\right\}$$

It is easy to see that $f_1(n) \to \infty$ as $n \to \infty$. By Lemma 5.6, if we know that:

1. $(i(g)x, gx)_x \geq n(L-\eta)/2$,
2. $(t(g)^{-1}x, g^{-1}x)_x \geq n(L-\eta)/2$, and
3. $(i(g)x, t(g)^{-1}x)_x \leq f_1(n) \leq n(L-\eta)/2 - 3\delta$,

then

$$(gx, g^{-1}x)_x \leq (i(g)x, t(g)^{-1}x)_x + 2\delta \leq f_1(n) + 2\delta.$$

Using Lemmas 5.5 and 5.7, the probability that conditions (1),(2), (3) hold tends to 1, hence we have

$$P^n((gx, g^{-1}x)_x \leq f(n)) \to 1$$

as $n \to \infty$. □

Finally, we put together the previous estimates and use Lemma 3.3 to prove that translation length grows linearly and loxodromic elements are generic.

Theorem 5.9 (Linear growth of translation length)**.** *Let (G, Γ) be an almost semisimple, nonelementary graph structure, and L the smallest drift given by eq.* (7)*. Then for any $\epsilon > 0$ we have*

$$\frac{\#\{g \in S_n : \tau_X(g) \geq n(L-\epsilon)\}}{\#S_n} \to 1,$$

as $n \to \infty$. As a consequence,

$$\frac{\#\{g \in S_n : g \text{ is } X-\text{loxodromic}\}}{\#S_n} \to 1,$$

as $n \to \infty$.

Proof. If we set $f(n) = \eta n$ with $\eta > 0$, then by Proposition 5.8 and Theorem 5.1 the events $(gx, g^{-1}x)_x \leq \eta n$ and $d(x, gx) \geq n(L - \eta)$ occur with probability (P^n) which tends to 1, hence by Lemma 3.3

$$P^n\Big(\tau_X(g) \geq n(L-3\eta)\Big) \geq P^n\Big(d(x,gx) - 2(gx, g^{-1}x)_x + O(\delta) \geq n(L-3\eta)\Big)$$

which approaches 1 as $n \to \infty$. This implies the statement if we choose $\epsilon > 3\eta$. The second statements follows immediately since elements with positive translation length are loxodromic. □

5.4. **Genericity of loxodromics for the Markov chain.** We now remark that a very similar proof yields that loxodromics are generic for $\mathbb{P}$-almost every sample path of the Markov chain. More precisely, we have the following (which is a reformulation of Theorem 2.7):

Theorem 5.10. *Let (G, Γ) be an almost semisimple, nonelementary graph structure, and let L be the smallest drift. Then for every $\epsilon > 0$, one has*

$$\mathbb{P}\Big(\tau_X(w_n) \geq n(L-\epsilon)\Big) \to 1,$$

as $n \to \infty$. As a consequence,

$$\mathbb{P}\Big(w_n \text{ is loxodromic on } X\Big) \to 1$$

as $n \to \infty$.

Proof. The proof is very similar to the proof of Theorem 5.9, so we will just sketch it. First, by using the Markov property we establish that

$$\lim_{n\to\infty} \mathbb{P}\big((i_n x, t_n^{-1}x)_x \geq g(n)\big) = 0$$

for any choice of function $g : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\to+\infty} g(n) = +\infty$. Then, by using positivity of the drift as in the proof of Lemma 5.7 we prove that for each $\epsilon > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}\big((w_n^{-1}x, t_n^{-1}x)_x \leq n(L-\epsilon)/2\big) = 0$$

and

$$\lim_{n\to\infty} \mathbb{P}\big((i_n x, w_n x)_x \leq n(L-\epsilon)/2\big) = 0$$

From the previous three facts, using Lemma 5.6 one proves:

$$\lim_{n\to\infty} \mathbb{P}\Big((w_n x, w_n^{-1}x)_x \geq f(n)\Big) = 0$$

for any $f : \mathbb{N} \to \mathbb{R}$ such that $\lim_{n\to+\infty} f(n) = +\infty$. The theorem then follows immediately from this fact and Corollary 4.5, applying the formula of Lemma 3.3. □

6. Thick graph structures

We begin by recalling the definition of a thick graph structure.

Definition 6.1. A graph structure (G, Γ) is *thick* if for every vertex v of maximal growth there exists a finite set $B \subseteq G$ such that

$$G = B\Gamma_v B \tag{9}$$

where Γ_v is the loop semigroup of v.

In greater generality, if $H < G$ is a subgroup, we say that the graph structure (G, Γ) is *thick relative to H* if for any vertex v of maximal growth there exists a finite set $B \subseteq G$ such that

$$H \subseteq B\Gamma_v B. \tag{10}$$

6.1. The case of only one non-trivial component.

We say that a component C is *non-trivial* if there is at least one closed path of positive length entirely contained in C.

Proposition 6.2. *If a graph structure (G, Γ) has only one non-trivial component, then it is thick.*

Proof. Let C be the unique maximal component of Γ. Every finite path γ in the graph can be written as $\gamma = h_1 g h_2$, where h_1 is a path from the initial vertex to C, g is a path entirely in C, and h_1 is a path going out of C. By assumption, the lengths of h_1 and h_2 are uniformly bounded. Fix some vertex v of C and let s be a shortest path from v to the last vertex of h. Further, let t be a shortest path from the last vertex of g to v. Then one can write

$$\gamma = h_1 g h_2 = h_1 s^{-1}(sgt)t^{-1}h_2$$

where $h_1 s^{-1}$ and $t^{-1}h_2$ vary in a finite set, and $sgt \in \Gamma_v$. Hence $G = B\Gamma_v B$ with B a finite set. □

6.2. Thick implies nonelementary.

Proposition 6.3. *Fix an action $G \curvearrowright X$ of G on a hyperbolic metric space X. Let (G, Γ) be an almost semisimple graph structure, and $H < G$ a nonelementary subgroup. If (G, Γ) is thick relative to H, then it is nonelementary. That is, for any maximal vertex v the action of the loop semigroup Γ_v on X is nonelementary.*

Proof. Since the action of H is nonelementary, there exists a free subgroup $F \subseteq H$ of rank 2 which quasi-isometrically embeds in X. Hence, the orbit map $F \to X$ extends to an embedding $\partial F \to \partial X$, and we identify ∂F with its image. Thickness implies $F \subseteq B\Gamma_v B$, and taking limit sets in ∂X we see that

$$\partial F \subset \bigcup_{b \in B} b \cdot \Lambda_{\Gamma_v} \subset X,$$

from which we conclude that Λ_{Γ_v} is infinite. To complete the proof that Γ_v is nonelementary, it suffices to show that Γ_v does not have a fixed point on ∂X (Proposition 3.4). Suppose toward a contradiction that $p \in \partial X$ is such a fixed point.

Let us write $F = \langle f, g \rangle$ where f, g are free generators of F, and consider the sequence of elements $h_{i,j} = f^i g^j$ in F. For each i, j there are $a_{i,j}, c_{i,j} \in B$ such that $h_{i,j} = a_{i,j} l_{i,j} c_{i,j}$ for some $l_{i,j}$ in Γ_v. Since B is finite, we may pass to a subsequence and assume that $a_{i,j} = a$ and $c_{i,j} = c$ for all i, j. Then $l_{i,j} = a^{-1} h_{i,j} c^{-1}$ fixes the point p for all i and so

$$h_{i,j}(c^{-1}(p)) = a(p)$$

for all i, j. Hence $h_{i_0,j_0}^{-1} h_{i,j} = g^{-i_0} f^{i-i_0} g^j$ is a sequence of elements of F which fix the point $q = c^{-1}(p) \in \partial F \subset \partial X$. Since F is a free group, this implies that $g^{-i_0} f^{i-i_0} g^j$ agree up to powers for infinitely many i, j, a clear contradiction. □

From Proposition 6.3 and Theorem 5.9 we get:

Theorem 6.4. *Let $G \curvearrowright X$ be a nonelementary action of a countable group on a hyperbolic metric space. Suppose that G has an almost semisimple graph structure Γ which is thick with respect to a nonelementary subgroup H. Then loxodromic elements are generic:*

$$\lim_{n\to\infty} \frac{\#\{g \in S_n \ : \ g \text{ is loxodromic on } X\}}{\# S_n} = 1$$

In fact, the translation length generically grows linearly: there exists $L > 0$ such that

$$\lim_{n\to\infty} \frac{\#\{g \in S_n \ : \ \tau_X(g) \geq Ln\}}{\# S_n} = 1.$$

7. Relative growth quasitightness

Fix a graph structure (G, Γ). In practice, we will often show that the graph structure is thick by establishing the property of growth quasitightness. This property was introduced in [AL02] and and further studied in [Yan16]. Our notion of quasitightness depends on the particular graph structure.

Given a path γ in Γ, we say it c–almost contains an element $w \in G$ if γ contains a subpath p such that $w = a \cdot \text{ev}(p) \cdot b$ in G, with $|a|, |b| \leq c$. We denote as $Y_{w,c}$ the set of paths in Γ starting at the initial vertex which do not c–almost contain w.

Definition 7.1. A graph structure (G, Γ) is called *growth quasitight* if there exists $c > 0$ such that for every $w \in G$ the set $Y_{w,c}$ has density zero with respect to P^n; that is,

$$P^n(Y_{w,c}) \to 0 \qquad \text{as } n \to 0.$$

More generally, given a subgroup $H < G$ we say that (G, Γ) is *growth quasitight relative to H* if there exists a constant $c > 0$ such that for every $w \in H$ the set $Y_{w,c}$ has density zero.

7.1. Growth quasitight implies thick.

Proposition 7.2. *Let (G, Γ) be an almost semisimple graph structure, and $H < G$ a subgroup. If (G, Γ) is growth quasitight relative to H, then it is thick relative to H.*

Proof. Let C be a component of maximal growth, let v a vertex in C, and let γ be some path from the initial vertex to v. Denote the length of γ by d. Let $w \in H$. By growth quasitightness plus maximal growth, there is a path of the form $\gamma\gamma_1$, which c–almost contains w and where γ_1 is entirely contained in C. Since γ has length d, the path γ_1 $(c + d)$–almost contains w; that is,

$$\gamma_1 = p_1 p_2 p_3$$

where $\text{ev}(p_2) = awb$ for $|a|, |b| \leq c + d$. Let q_1 be a shortest path from v to the initial vertex of p_2 and q_2 be a shortest path from the terminal vertex of p_2 to v. Then

$$w = (a^{-1}\text{ev}(q_1)^{-1}) \cdot \text{ev}(q_1 p_2 q_2) \cdot (\text{ev}(q_2)^{-1} b^{-1}),$$

where $(a^{-1}\text{ev}(q_1)^{-1})$ and $(\text{ev}(q_2)^{-1} b^{-1})$ vary in a finite set B. Since $\text{ev}(q_1 p_2 q_2) \in \Gamma_v$, this completes the proof. □

Combining Proposition 7.2 with Theorem 6.4 we get:

Theorem 7.3. *Let (G, Γ) be an almost semisimple graph structure which is growth quasitight with respect to a nonelementary subgroup H. Then loxodromic elements are generic.*

This completes the proof of Theorem 1.6.

8. Infinite index subgroups have zero density

In this section, we prove that in our general setup a subgroup $H < G$ of infinite index has zero density with respect to counting. Combined with what we are going to prove in Section 9 and 10, this immediately implies Theorem 1.2 in the introduction. Recall that $\mathrm{ev} : \Omega_0 \to G$ is the evaluation map for paths starting at v_0.

Theorem 8.1. *Let (G, Γ) be an injective, almost semisimple, thick graph structure. Let $H < G$ be an infinite index subgroup. Then*

$$P^n(\{p \in \Omega_0 : \mathrm{ev}(p) \in H\}) \to 0,$$

as $n \to \infty$. That is, the proportion of paths starting at v_0 and spelling elements of H goes to 0 as the length of the path goes to ∞.

The proof is an adaptation of ([GMM15], Theorem 4.3) to the non-hyperbolic case. We will consider an extension Γ_H of $\Gamma = (V, E)$ defined as follows. The vertex set of Γ_H is $V \times H \setminus G$. For any edge $\sigma : x \to y$ in Γ there is an edge in Γ_H from (x, Hg) to (y, Hgg') where $g' = \mathrm{ev}(\sigma)$.

Lemma 8.2. *Let C be a component of maximal growth of Γ. For any $v_1 \in C$ and $g_1 \in G$ there are infinitely many $Hg \in H \setminus G$ such that (v_1, Hg) can be reached from (v_1, Hg_1) by a path contained in $C \times H \setminus G$.*

Proof. Suppose not, so that the only points of $H \setminus G$ that can be reached in this manner are $\{Hz : z \in T\}$ where T is a set of size D. Consider $w \in G$. By thickness, there exists a finite set $B \subseteq G$ and some path γ lying in C, starting and ending at v_1 such that

$$\mathrm{ev}(\gamma) = g_2 w g_3$$

where g_2, g_3 lie in B. Then γ lifts to a path in Γ_H from (v_1, Hg_1) to $(v_1, Hg_1\mathrm{ev}(\gamma))$. By assumption, this implies $Hg_1\mathrm{ev}(\gamma) = Hz$ for some $z \in T$. Thus, there is an $h \in H$ with $g_2 w g_3 = \mathrm{ev}(\gamma) = g_1^{-1} h z$ and hence $w \subset B^{-1} g_1^{-1} H T B^{-1}$. Thus there is a finite subset $\Upsilon = B^{-1} g_1^{-1} \cup T B^{-1}$ with $G = \Upsilon H \Upsilon$, so by Neumann's theorem [Neu54] H must be of finite index, giving a contradiction. $\square$

The following general result about Markov chains is Lemma 4.4 of [GMM15].

Lemma 8.3. *Let X_n be a Markov chain on a countable set V, and m a stationary measure. Let $\widetilde{V}$ be the set of points $x \in V$ such that $\sum_{y:x\to y} m(y) = \infty$ where $x \to y$ means there is a positive probability path from x to y. Then for all $x \in V$ and $x' \in \widetilde{V}$ we have $P_x(X_n = x') \to 0$.*

Combining Lemmas 8.3 and 8.2 we obtain:

Corollary 8.4. *For any $x_1, x_2 \in \Gamma$ lying in a maximal component C and $g_1, g_2 \in G$, the number of paths of length n in Γ_H from (x_1, Hg_1) to (x_2, Hg_2) is $o(\lambda^n)$.*

Proof. The Markov chain μ on Γ restricts to a Markov chain μ_C on C, which in turn lifts to a Markov chain $\mu_{C,H}$ on the induced graph C_H on the vertex set $C \times H \setminus G$ of Γ_H (obtained by assigning to an edge the transition probability of its projection to C). A $\mu_{C,H}$ stationary measure $\widetilde{m}$ on C_H is given by taking the product of the stationary measure m on C and the counting measure on $H \setminus \Gamma$. Any vertex $v \in C$ has positive m measure and all lifts of v in C_H have equal positive $\widetilde{m}$ measure. Thus, Lemma 8.2 implies $\sum_{y:x\to y} \widetilde{m}(y) = \infty$. The corollary now follows by applying Lemma 8.3 to the chain $\mu_{C,H}$. $\square$

Note, paths of length n in Γ_H from (x_1, Hg_1) to (x_2, Hg_2) are in bijection with paths of length n in Γ beginning at x_1, ending at x_2, and evaluating to elements of $g_1^{-1} H g_2$. Thus, we obtain:

Corollary 8.5. *For any $x_1, x_2 \in \Gamma$ lying in a maximal component C and $g_1, g_2 \in G$, the number of paths of length n in Γ beginning at x_1, ending at x_2, and evaluating to elements of $g_1^{-1} H g_2$ is $o(\lambda^n)$.*

We now complete the proof of Theorem 8.1. Given $k > 0$, let $P_{n,k}$ (resp. $Q_{n,k}$) be the set of paths $p \in \Omega_0$ of length n which spend time at most k (resp. more than k) in non-maximal components.

Note that there is a $\eta < \lambda$ with $|Q_{n,k}| \leq \eta^k \lambda^{n-k}$ for all n and k. Now, consider a path γ in $P_{n,k} \cap \mathrm{ev}^{-1} H$. We can decompose it as $\gamma = \gamma_1 \gamma_2 \gamma_3$ where γ_1 and γ_3 have length adding up to at most k and γ_2 is contained in a maximal component C. Since a path in $P_{n,k}$ spends at most time k in nonmaximal components, there are only D^k possibilities for γ_1 and γ_3, where D depends only on the graph. Now, given a path γ in $\mathrm{ev}^{-1} H$, once γ_1 and γ_3 are fixed, by Corollary 8.5 there are at most $f_k(n)$ possibilities for γ_2, where for each fixed k $f_k(n)/\lambda^n \to 0$ as $n \to \infty$. Thus, for all $k < n$ we have $|P_{n,k} \cap \mathrm{ev}^{-1} H| \leq D^k f(n)$ and so

$$P^n(\mathrm{ev}^{-1} H) \leq C'' \lambda^{-n} (|P_{n,k} \cap \mathrm{ev}^{-1} H| + |Q_{n,k}|) \leq C'' D^k \lambda^{-n} f(n) + C''(\eta/\lambda)^k.$$

Fixing k we see that

$$\limsup_{n \to \infty} P^n(\mathrm{ev}^{-1} H) \leq C''(\eta/\lambda)^k.$$

As this is true for arbitrary k, we get $\lim_{n \to \infty} P^n(\mathrm{ev}^{-1} H) = 0$, as claimed.

9. Application to relatively hyperbolic groups

In this section, we show how our main theorem applies to a large class of relatively hyperbolic groups.

Let G be a finitely generated group, and $\mathcal{P}$ be a collection of subgroups. Following [Bow12], let us recall that G is *hyperbolic relative to* $\mathcal{P}$ if there is a compactum M on which G acts geometrically finitely, and the maximal parabolic subgroups are the conjugates of elements of $\mathcal{P}$. Such a compactum M is then unique up to G-equivariant homeomorphisms, and it is called the *Bowditch boundary* of G. We will denote it as ∂G.

More precisely, let G act by homeomorphisms on a compact, perfect, metrizable space M. Then a point $\zeta \in M$ is called *conical* if there is a sequence (g_n) and distinct points $\alpha, \beta \in M$ such that $g_n \zeta \to \alpha$ and $g_n \eta \to \beta$ for all $\eta \in M \setminus \{\zeta\}$. A point $\zeta \in M$ is called *bounded parabolic* if the stabilizer of ζ in G is infinite, and acts cocompactly and properly discontinuously on $M \setminus \{\zeta\}$. We say that the action of G on M is a *convergence action* if G acts properly discontinuously on triples of elements of M, and the action is *geometrically finite* if it is a convergence action and every point of M is either a conical limit point or a bounded parabolic point. Note that there are only countably many parabolic points. Finally, the *maximal parabolic subgroups* are the stabilizers of bounded parabolic points. An action of a group G on a proper δ-hyperbolic space X is *geometrically finite* if the induced action on the limit set is geometrically finite. We refer the reader to [Far98, Bow12, Osi06, GM08] for the relevant background material.

Fix a relatively hyperbolic group $(G, \mathcal{P})$, a generating set S of G, and let d_G denote distance in G with respect to S. Let $\hat{G}$ be the vertices of $\mathrm{Cay}(G, S \cup \mathcal{P})$ with the induced metric, which we denote by $\hat{d}$ or $d_{\hat{G}}$. Here $\mathrm{Cay}(G, S \cup \mathcal{P})$ is the corresponding electrified Cayley graph, that is the Cayley graph of G with respect to the generating set $S \cup \bigcup \mathcal{P}$. We remind the reader that $\mathrm{Cay}(G, S \cup \mathcal{P})$ is hyperbolic and that $\partial \mathrm{Cay}(G, S \cup \mathcal{P})$ naturally includes as a subspace into ∂G, the complement of which is the collection of parabolic fixed points.

Let $\mathfrak{h}$ be the exponent of convergence for $\mathrm{Cay}(G, S)$. That is,

$$\mathfrak{h} := \lim_{n \to \infty} \frac{1}{n} \log |B_n|,$$

where B_n denotes the ball of radius n in G with respect to d_G. For $\zeta \in \partial G$ and $g, h \in G$ let

$$\beta_\zeta(g, h) = \limsup_{z \to \zeta} (d(g, z) - d(h, z))$$

be the Busemann function for the word metric on G.

Following [Yan13], the Bowditch boundary ∂G is equipped with a nonatomic measure ν, which is given by the Patterson-Sullivan construction by taking average on balls *for the word metric* on G. Such a measure ν is supported on conical points and is $\mathfrak{h}$-*quasiconformal*, in the sense that there is a $D > 0$ such that:

$$D^{-1} e^{-\mathfrak{h}\beta_\zeta(g,e)} \leq \frac{dg_\star \nu}{d\nu}(\zeta) \leq D e^{-\mathfrak{h}\beta_\zeta(g,e)} \tag{11}$$

for all $g \in G$ and $\zeta \in \partial G$.

Definition 9.1. We define a relatively hyperbolic group G to be *doubly ergodic* if its action on $\partial G \times \partial G$ with the measure $\nu \times \nu$ is ergodic.

We will also see (Proposition 9.17) that a relatively hyperbolic group is doubly ergodic if it admits a geometrically finitely action on a $CAT(-1)$ proper metric space. For instance, geometrically finite Kleinian groups satisfy this hypothesis. Note that, once G admits such an action, Theorem 9.2 works for isometric actions of G on *any* hyperbolic, metric space X.

In this section, we will prove the following result.

Theorem 9.2. *Let G be a doubly ergodic, relatively hyperbolic group, and let (G, Γ) be a geodesic combing. Then for any nonelementary action of G on a hyperbolic metric space X, the graph structure (G, Γ) is nonelementary.*

Combining this result with Theorem 1.6, the discussion in Section 2.3, and the fact that relatively hyperbolic groups have pure exponential growth for any generating set ([Yan13], Theorem 1.9) Theorem 9.2 establishes Theorem 2.3 in the introduction.

In fact, using very recent work of W. Yang [Yan16], the theorem may be extended to all nontrivial relatively hyperbolic groups, as relatively hyperbolic groups contain strongly contracting elements by [ACT15] and [Yan16] (see Corollary 2.6 in the introduction). However, we give a self-contained argument here.

9.1. **Fellow traveling in the Cayley graph and coned-off space.** We first remind the reader that a K–quasigeodesic γ in a metric space X is a map $\gamma \colon I \to X$ from a subinterval $I \subset \mathbb{R}$ such that for all $s, t \in I$

$$\frac{1}{K} \cdot |t - s| - K \leq d(\gamma(s), \gamma(t)) \leq K \cdot |t - s| + K.$$

We will need the following proposition, which is certainly known to experts. We provide a proof for completeness.

Proposition 9.3. *For $K, C \geq 0$, there are $D, L \geq 0$ such that the following holds. Suppose that $\gamma = [a, b]$ is a geodesic in $\mathrm{Cay}(G, S)$ with length at least L which projects to a K–quasigeodesic in $\mathrm{Cay}(G, S \cup \mathcal{P})$. Let γ' be any other geodesic in $\mathrm{Cay}(G, S)$ whose endpoints have distance no more than C from a, b in $\mathrm{Cay}(G, S \cup \mathcal{P})$. Then there are $a', b' \in \gamma'$ such that*

$$d_G(a, a') \leq D \quad \text{and} \quad d_G(b, b') \leq D.$$

We will use the following theorem of Osin:

Theorem 9.4 ([Osi06], Theorem 3.26)**.** *There is an $\nu \geq 0$ such that if p, q, r are sides of a geodesic triangle in $\mathrm{Cay}(G, S \cup \mathcal{P})$, then for any vertex v on p there exists a vertex u on either q or r such that*

$$d_G(v, u) \leq \nu.$$

Proof of Proposition 9.3. Suppose that $d_G(a, b) > K(2C + 4\nu + K)$ so that $d_{\hat{G}}(a, b) > 2C + 4\nu$, for ν as in Theorem 9.4.

Let c, c' be geodesics in $\mathrm{Cay}(G, S \cup \mathcal{P})$ joining the endpoints of γ, γ' respectively. Note that by assumption the initial and terminal endpoints of these geodesics are at $\hat{d}$–distance less than C from one another. Pick vertices c_a, c_b on c (ordered a, c_a, c_b, b) so that $d_{\hat{G}}(a, c_a) = d_{\hat{G}}(b, c_b) = C + 2\nu$. (This is possible since $d_{\hat{G}}(a, b) > 2C + 4\nu$.) Consider a geodesic quadrilateral with opposite sides c, c'. Applying Theorem 9.4 twice, we may find vertices $c'_a, c'_b \in c'$ such that $d_G(c_a, c'_a) \leq 2\nu$ and $d_G(c_b, c'_b) \leq 2\nu$.

Now using, for example, ([Hru10], Lemma 8.8), we can find vertices $\gamma_a, \gamma_b \in \gamma$ and $\gamma'_a, \gamma'_b \in \gamma'$ which have d_G–distance at most L from c_a, c_b, c'_a, c'_b, respectively, where $L \geq 0$ depends only on $(G, \mathcal{P})$ and S. Note that $d_G(\gamma_a, \gamma'_a) \leq 2(L + \nu)$ and $d_G(\gamma_b, \gamma'_b) \leq 2(L + \nu)$. Moreover,

$$d_{\hat{G}}(a, \gamma_a) \leq d_{\hat{G}}(a, c_a) + d_{\hat{G}}(c_a, \gamma_a) \leq C + 2\nu + L,$$

and so $d_G(a, \gamma_a) \leq K(C + 2\nu + L + 1)$ since both a and γ_a occur along γ. Similarly, $d_G(b, \gamma_b) \leq K(C + 2\nu + L + 1)$.

Putting everything together, after setting $a' = \gamma'_a$ and $b' = \gamma'_b$, we see that each of $d_G(a, a')$ and $d_G(b, b')$ are less than $2(L + \nu) + K(C + 2\nu + L + 1)$ and this completes the proof. ☐

9.2. **Patterson-Sullivan measures and sphere averages.** Continuing with the notation from the previous section, let $\mathfrak{h}$ be the exponent of convergence for $\mathrm{Cay}(G, S)$.

Definition 9.5. For $g \in G$, we define the *large shadow* $\Pi_r(g)$ at g to be the set of $\zeta \in \partial G$ such that there exists *some* geodesic in $\mathrm{Cay}(G, S)$ from 1 converging to ζ intersecting $B_r(g)$. Similarly, the *small shadow* $\pi_r(g)$ is the set of $\zeta \in \partial G$ such that *every* geodesic in $\mathrm{Cay}(G, S)$ from 1 converging to ζ intersects $B_r(g)$.

In Theorem 1.7 and Proposition A4 of [Yan13] Yang constructs an $\mathfrak{h}$-quasiconformal ergodic density ν without atoms for the word metric on the Bowditch boundary ∂G. In Lemma 4.3 of [Yan13] he shows that this satisfies the shadow lemma: for large enough r:

$$\nu(\pi_r(g)) \simeq \nu(\Pi_r(g)) \simeq e^{-\mathfrak{h} \cdot d_G(1, g)} \tag{12}$$

(up to a uniform multiplicative constant). In particular, ν has full support on ∂G. In what follows, S_n denotes the set of elements $g \in G$ with $d_S(1, g) = n$.

Lemma 9.6. *There is a $C > 0$ such that for any Borel set $A \subset G \cup \partial G$ one has*

$$\limsup_{n \to \infty} \frac{|A \cap S_n|}{|S_n|} \leq C\nu(\overline{A}),$$

where $\overline{A}$ denote the closure of A in $G \cup \partial G$.

Proof. Let $A \subset G \cup \partial G$ be a Borel set. Since the number of elements in a ball of radius r in $\mathrm{Cay}(G, S)$ is universally bounded, a point of ∂G lies in at most D small shadows $\pi_r(g), g \in S_n$ where D depends only on r. Thus,

$$\sum_{g \in S_n \cap A} \nu\left(\pi_r(g)\right) \leq D\nu\left(\bigcup_{g \in S_n \cap A} \pi_r(g)\right)$$

Moreover, if we denote $A_n := A \setminus B_{n-1}$ then $A_{n+1} \subseteq A_n$ and

$$\bigcap_{n \in \mathbb{N}} \bigcup_{g \in A_n} \pi_r(g) \subset \overline{A}.$$

Indeed, if $\zeta \in \bigcap_{n \in \mathbb{N}} \bigcup_{g \in A_n} \pi_r(g)$, then there are $g_n \in A$ with $|g_n| \geq n$ such that some (any) geodesic from the identity to ζ meets $B_r(g_n)$. Hence, $g_n \to \zeta$ and so $\zeta \in \overline{A}$.

Thus, since $S_n \cap A \subseteq A_n$ we have for large enough n

$$\nu\left(\bigcup_{g \in S_n \cap A} \pi_r(g)\right) \leq \nu\left(\bigcup_{g \in A_n} \pi_r(g)\right) \leq 2\nu(\overline{A})$$

so by exponential growth and the shadow lemma (12)

$$\frac{|S_n \cap A|}{|S_n|} \simeq e^{-\mathfrak{h}n}|S_n \cap A| \simeq \sum_{g\in S_n\cap A} \nu(\pi_r(g)) \lesssim 2D\nu(\overline{A}).$$

□

9.3. **Growth quasitightness for relatively hyperbolic groups.** We will now establish a form of relative growth quasitightness for a relatively hyperbolic group G.

Let w be an element of G. A *w-path* is an infinite path of the form $l_w = \bigcup_{i\in\mathbb{Z}} w^i\gamma_w$ in the Cayley graph $\mathrm{Cay}(G,S)$, where $\gamma_w = [1,w]$ is a geodesic segment joining the identity and w. Of course there may be finitely many choices of l_w for each w.

Definition 9.7. The element w is called K**–bounded** if some w-path l_w (with the arc length parameterization) in the Cayley graph $\mathrm{Cay}(G,S)$ projects to a K-quasigeodesic in the electrified graph $\mathrm{Cay}(G, S\cup\mathcal{P})$.

The following lemma is well-known. See for example [DMS10, ADT].

Lemma 9.8. *For each K, there is a function $f\colon \mathbb{N}\to\mathbb{N}$ such that if w is K–bounded, then every w-path l_w is an f-stable quasigeodesic in the Cayley graph* $\mathrm{Cay}(G,S)$*.*

Recall that l_w being f–stable means that any K–quasigeodesic with endpoints on l_w has Hausdorff distance at most $f(K)$ from the subpath of l_w its endpoints span.

Given $w\in G$ and $c\geq 0$, we say that a (finite or infinite) geodesic γ *c-almost contains w* if there exists $g\in G$ such that $d_G(g,\gamma)\leq c$ and $d_G(gw,\gamma)\leq c$. Let $X_{w,c}$ be the set of $h\in G$ such that there exists a geodesic γ from identity to h which does not c-almost contain w. That is, for every $g\in N_c(\gamma)$, γ does NOT pass within distance c of gw.

Proposition 9.9. *For each $K\geq 1$, there is $c\geq 0$ such that for every K–bounded $w\in G$ we have*

$$\frac{|S_n\cap X_{w,c}|}{e^{\mathfrak{h}n}} \to 0$$

as $n\to\infty$.

We remark that, for fixed c, it suffices to prove the proposition for sufficiently long w, that is, where $|w|_S$ is sufficiently large. We will prove this proposition by using the ergodicity of the double boundary (Proposition 9.17). To do this, we will apply Proposition 9.3 several times, for K the boundedness constant. Hence, we fix K once and for all, and consider the constant D produced by that proposition as a function of C alone and write $D = D(C)$.

Let $Z_{w,c}$ be the set of pairs (α,β) in $\partial G\times\partial G$ such that for every bi-infinite geodesic γ in $\mathrm{Cay}(G,S)$ joining α and β there exist infinitely many $x\in N_c(\gamma)$ such that γ passes within c of xw. Let $Z^n_{w,c}$ be the set of pairs (α,β) in $\partial G\times\partial G$ such that for every bi-infinite geodesic γ joining α and β there are at least n elements $x\in N_c(\gamma)$ such that γ passes within c of xw. By definition, $Z_{w,c} = \bigcap_{n\in\mathbb{N}} Z^n_{w,c}$. Moreover, for each n,w,c, the sets $Z_{w,c}$ and $Z^n_{w,c}$ are G-invariant subsets of $\partial G\times\partial G$.

Furthermore we have

Lemma 9.10. *For each $K\geq 0$ there is a constant $c_0 = c_0(K)$ such that for all $c\geq c_0$, $Z_{w,c}$ contains a pair of conical points for every K–bounded $w\in G$.*

Proof. Let f be the function given by Lemma 9.8. By definition, the w-path l_w projects to a K-quasigeodesic in $\mathrm{Cay}(G,S\cup\mathcal{P})$, hence it has two distinct limit points $(w^{-\infty}, w^{\infty})$ in the Bowditch boundary ∂G. Then, by connecting further and further points on l_w by a geodesic in the $\mathrm{Cay}(G,S)$, using f-stability and the Ascoli-Arzelá theorem, one constructs a geodesic in $\mathrm{Cay}(G,S)$ which connects w^{∞} to $w^{-\infty}$ and c-fellow travels l_w, once $c\geq 2f(1)$. Hence, $(w^{-\infty},w^{\infty})\in Z_{w,c}$. □

Lemma 9.11. *For each K, there is a $c_1 = c_1(K)$ and $L_1 = L_1(K)$ such that for $c \geq c_1$ and for any K–bounded $w \in G$ with $|w| \geq L_1$, the set $Z^n_{w,c}$ has nonempty interior. More precisely, the interior of $Z^n_{w,c}$ contains every pair of conical points in Z^n_{w,c_0} where c_0 is as in Lemma 9.10.*

Proof. Suppose that $(\alpha, \beta) \in Z^n_{w,c_0}$ is a pair of conical points, and pick a geodesic γ joining α and β. Then by definition there are segments $[x_j, x'_j] \subseteq [\alpha, \beta]$ for $1 \leq j \leq n$ and points z_j such that $d_G(x_j, z_j), d_G(x'_j, z_j w) \leq c_0$ (for $1 \leq j \leq n$).

Now, let $\alpha_i, \beta_i \in \partial G$ such that $\alpha_i \to \alpha$ and $\beta_i \to \beta$. In particular, for some uniform R, the projections of the geodesics $[\alpha_i, \beta_i]$ to $\mathrm{Cay}(G, S \cup \mathcal{P})$ R–fellow travel the projection of $[\alpha, \beta]$ for longer and longer intervals. Hence, there is an $N \geq 0$ so that for $i \geq N$, each $[\alpha_i, \beta_i]$ passes with $\hat{d}$–distance R from $[x_j, x'_j] \subset [\alpha, \beta]$.

Since w is K–bounded, the geodesic segment $z_j \cdot [1, w] = [z_j, z_j w]$ projects to a K–quasigeodesic, so we may apply Proposition 9.3 (with $C = R + c_0$) to find constants D, L such that if $|w| \geq L$ there exist points $y_j, y'_j \in [\alpha_i, \beta_i]$ such that $d_G(y_j, z_j), d_G(y'_j, z_j w) \leq D$.

Setting $c_1 = D$, $L_1 = L$ we see that for sufficiently large $i \geq 0$, $(\alpha_i, \beta_i) \in Z^n_{w,c_1}$ and this completes the proof. □

Since G is doubly ergodic, the action of G on $\partial G \times \partial G$ is ergodic, hence Lemma 9.11 implies

Lemma 9.12. *For $c \geq c_1(K)$ and for any K–bounded $w \in G$ with $|w| \geq L_1(K)$, the set $Z^n_{w,c}$ has full $\nu \times \nu$ measure. Hence, under the same hypotheses the set $Z_{w,c}$ has full $\nu \times \nu$ measure.*

Proof. For each n, the set $Z^n_{w,c}$ is G-invariant, hence by ergodicity its measure is either 0 or 1. Since it has nonempty interior and the measure $\nu \times \nu$ has full support, then it must have full measure. The second claim follows since $Z_{w,c} = \cap_n Z^n_{w,c}$. □

Let $\Lambda_{w,c} \subset \partial G$ be the set of conical points $\alpha \in \partial G$ such that for every geodesic ray γ from the identity converging to α there are infinitely many points $g \in N_c(\gamma)$ such that γ passes within distance c of gw. Using Proposition 9.3 just as in Lemma 9.11 we have:

Lemma 9.13. *For each $K \geq 0$ there is a $c_2 = c_2(K)$ and $L_2 = L_2(K)$ such that for $c \geq c_2$, and for any K–bounded $w \in G$ with $|w| \geq L_2$, if $(\alpha, \beta) \in Z_{w,c_1}$, then either α or β is in $\Lambda_{w,c}$.*

This, together with the fact that conical limit points have full ν-measure, implies

Corollary 9.14. *For each $c \geq c_2$ and for any K-bounded $w \in G$ with $|w| \geq L_2$, the set $\Lambda_{w,c}$ has full ν measure.*

Lemma 9.15. *For each $K \geq 0$ there is a $c_3 = c_3(K)$ and $L_3 = L_3(K)$ such that for each $c \geq c_3$ and for any K–bounded $w \in G$ with $|w| \geq L_3$, the closure of $X_{w,c}$ is contained in $\partial G \setminus \Lambda_{w,c_2}$.*

Proof. If this were false, then for all large $c \geq 0$ there would be a sequence $(y_i) \subseteq X_{w,c}$ converging to $\eta \in \Lambda_{w,c_2}$. Since η is not a parabolic fixed point, then one can view η as belonging to the boundary of $\mathrm{Cay}(G, S \cup \mathcal{P})$. Then the projections to $\mathrm{Cay}(G, S \cup \mathcal{P})$ of any geodesics $\gamma_i = [1, y_i]$ must R–fellow travel the projection to $\mathrm{Cay}(G, S \cup \mathcal{P})$ of $[1, \eta]$ for longer and longer intervals, where R is independent of w. If η were in Λ_{w,c_2}, then just as in the proof of Lemma 9.11, we would obtain by applying Proposition 9.3 (with $C = c_2 + R$) two constants L_3, c_3 such that for $|w| \geq L$ and for large i, the geodesic γ_i c_3–almost contains w. Hence, for $c \geq c_3$ we obtain a contradiction to $y_i \in X_{w,c}$ for all i. This completes the proof. □

We are now in position to prove Proposition 9.9.

Proof of Proposition 9.9. By Lemma 9.15 and Corollary 9.14, for $c \geq c_3$

$$\nu(\overline{X_{w,c}}) \leq \nu(\partial G \setminus \Lambda_{w,c_2}) = 0$$

Hence, applying Lemma 9.6, for large enough $c > 0$ we have

$$\limsup_n e^{-\mathfrak{h}n} |S_n \cap X_{w,c}| \leq C\nu(\overline{X_{w,c}}) = 0.$$

□

9.4. **The loop semigroup is nonelementary.** We will now assume that $(G, \mathcal{P})$ is a doubly ergodic relatively hyperbolic group which admits a geodesic combing for the generating set S.

Recall that a finitely generated group G is *geodesically completable* if any finite generating set S of G can be extended to a finite generating set $S' \supseteq S$ for which there exists a geodesic biautomatic structure. Moreover, by work of Antolín-Ciobanu [AC16], if the parabolic subgroups are geodesically completable, then every generating set for G can be extended to a generating set for which G has a geodesic combing. From here on, we will use such a generating set S.

Then for each $w \in G$ and constant c, let us recall that $Y_{w,c}$ is the set of paths γ in the directing graph from the initial vertex which do not c-almost contain w, i.e. such that one cannot write $\mathrm{ev}(\gamma) = a_1 w a_2$ in G, with $|a_i| \leq c$ for $i = 1, 2$. By identifying paths from the identity with group elements, it is immediate from the definition that $Y_{w,c} \subseteq X_{w,c}$. Hence, by Proposition 9.9, also $Y_{w,c}$ has zero density if w is K–bounded.

Proposition 9.16. *Let (G, Γ) be a geodesic combing for a doubly ergodic, relatively hyperbolic group. Then (G, Γ) is nonelementary.*

Proof. We are going to prove that the graph structure is thick relative to a nonelementary, free subgroup $F < G$, which yields the claim by Proposition 6.3. Let v be a vertex of maximal growth and w be any K–bounded word. Let $d = \mathrm{diam}\, \Gamma$. Let c be the constant from Proposition 9.9. Let h_1 be a group element representing a path from the initial vertex to v, and consider the set

$$\Sigma = \{h_1 h_2 \ : \ h_2 \in \Gamma_v\}$$

Since v has maximal growth and $Y_{w,c}$ has zero density, the set Σ contains a path h which does not belong to $Y_{w,c}$. Then there is a path $h = h_1 h_2$ such that h_1 has length $\leq d$, h_2 is entirely contained in the component C_v containing v, and h_2 contains a subpath of the form $w' = awb$ where a and b have length less than $c + d$. Let s be a path from v to the start of w' and t be a path from the end of w' to v, each of length at most D. Then $sw't$ is in Γ_v and $w = as^{-1}(swt)t^{-1}b \subset B\Gamma_v B$ where B is a finite set.

To complete the proof, it suffices to show that $B\Gamma_v B$ contains a nonelementary subgroup (Proposition 6.3). Using a standard ping-pong argument, construct a free subgroup $H = \langle f, g \rangle \leq G$ which K–quasi-isometrically embeds in $\mathrm{Cay}(G, S \cup \mathcal{P})$ and which K–quasi-isometrically embeds into X for some $K \geq 0$. (Indeed, by [TT15], a random 2-generator subgroup of G will have this property.) For this K, let B be the finite subset produced above enlarged to contain $f^{\pm}, g^{\pm}$. Then for any $w \in H$, at least one of w, wf, or wg is cyclically reduced in H and hence K–bounded in G. Hence $w \in B\Gamma_v B$ and so $H \leq B\Gamma_v B$, as required. □

9.5. **Double Ergodicity.** We conclude this section by proving that a group G which admits a geometrically finite action on a $\mathrm{CAT}(-1)$ space is doubly ergodic.

Let us assume that G acts geometrically finitely on a $\mathrm{CAT}(-1)$ space Y. Recall that an orbit map $G \to Y$ induces an embedding $\partial G \to \partial Y$ [Bow12, Theorem 9.4]. We continue to denote the pushforward of the measure ν to ∂Y by ν.

Proposition 9.17. *Suppose G acts geometrically finitely on a $\mathrm{CAT}(-1)$ space Y. Then the action of G on $\partial Y \times \partial Y$ is ergodic with respect to $\nu \times \nu$.*

We remind the reader that ν is quasiconformal with respect to the word metric rather than the metric on Y.

Proof. Assume G acts geometrically finitely on a CAT(-1) space Y, with elements of $\mathcal{P}$ being the parabolic subgroups. As above, the Bowditch boundary ∂G is identified with a closed subspace of the Gromov boundary of Y. Let $d = d_G$ still denote the word metric on G. For $\zeta \in \partial G$ and $g, h \in G$ let $\beta_\zeta(g,h)$ be the Busemann function for the word metric on G. By W. Yang's Lemma 2.20 in [Yan13] there is a $C > 0$ such that for every conical $\zeta \in \partial G$ we have

$$|\limsup_{z\to\zeta}[d(g,z) - d(h,z)] - \liminf_{z\to\zeta}[d(g,z) - d(h,z)]| < C \tag{13}$$

Recall that the Patterson-Sullivan measure ν (for the word metric) gives full measure to conical points and is $\mathfrak{h}$-quasiconformal, i.e. there is a $D > 0$ such that:

$$D^{-1}e^{-\mathfrak{h}\beta_\zeta(g,e)} \leq \frac{dg_\star\nu}{d\nu}(\zeta) \leq De^{-\mathfrak{h}\beta_\zeta(g,e)} \tag{14}$$

for all $g \in G$ and $\zeta \in \partial G$. We claim there is a G-invariant measure in the measure class of $\nu \times \nu$. Indeed, let

$$\rho_e(\zeta,\xi) = \lim \sup_{z\to\zeta, y\to\xi} \left(\frac{d(e,y) + d(e,z) - d(y,z)}{2}\right).$$

Define a locally finite measure m' on $(\partial G \times \partial G) \setminus Diag$ by

$$dm'(\zeta,\xi) = e^{2\mathfrak{h}\rho_e(\zeta,\xi)}\ d\nu(\zeta)\ d\nu(\xi)$$

The measure m' is G quasi-invariant with a uniformly bounded derivative. Indeed, we can compute

$$2\rho_e(g^{-1}\zeta, g^{-1}\xi) - 2\rho_e(\zeta,\xi) =$$

$$= \limsup_{z\to\zeta, y\to\xi}\left[d(e,g^{-1}y) + d(e,g^{-1}z) - d(g^{-1}y, g^{-1}z)\right] - \limsup_{z\to\zeta, y\to\xi}\left[d(e,y) + d(e,z) - d(y,z)\right] =$$

$$= \limsup_{y\to\xi}\left[d(g,y) - d(e,y)\right] + \limsup_{z\to\zeta}\left[d(g,z) - d(e,z)\right] + O(1) = \beta_\xi(g,e) + \beta_\zeta(g,e) + O(1)$$

(where we could distribute the limsup since the limsup and liminf are within bounded difference as in (13)). Hence, combining this with (14) one gets that the Radon-Nykodym cocycle is uniformly bounded, i.e.

$$\frac{dg_\star m'}{dm'}(\zeta,\xi) = e^{2\mathfrak{h}\rho_e(g^{-1}\zeta, g^{-1}\xi) - 2\mathfrak{h}\rho_e(\zeta,\xi)}\ \frac{dg_\star\nu}{d\nu}(\zeta)\ \frac{dg_\star\nu}{d\nu}(\xi) \cong 1$$

Hence, by a general fact in ergodic theory the Radon-Nykodym cocycle is also a coboundary (see [Fur02], Proposition 1). Thus, there exists a G-invariant measure m on $(\partial G \times \partial G) \setminus Diag$ in the same measure class as m', hence also in the same measure class as $\nu \times \nu$. By [Yan13], the Patterson-Sullivan measure is supported on conical limit points. Thus, m is also supported on pairs of conical limit points. By Theorem 2.6 of [Kai94], any quasi-product G-invariant Radon measure on the double boundary of a CAT(-1) space which gives full measure to pairs of conical limit points of G is ergodic. Thus, $\nu \times \nu$ is ergodic. □

10. RAAGs, RACGs and graph products

Let Λ be a finite simplicial (undirected) graph. Recall that the corresponding right-angled Artin group (RAAG) $A(\Lambda)$ is the group given by the presentation

$$A(\Lambda) := \langle v \in V(\Lambda) : [v,w] = 1 \iff (v,w) \in E(\Lambda)\rangle.$$

The corresponding right-angled Coxeter group (RACG) $C(\Lambda)$ is the group obtained from $A(\Lambda)$ by adding the relators $v^2 = 1$ for each $v \in V(\Lambda)$. In each case, $S = \{v^{\pm 1} : v \in V(\Lambda)\}$ is called the set of *standard* (or *vertex*) generators of the group.

In greater generality, let Λ be a finite simplicial graph, and for each vertex v of Λ let us pick a finitely generated group G_v, which we call *vertex group*. Then we define the *graph product* from the relative presentation

$$G(\Lambda) := \langle g \in G_v \ : \ [g,h] = 1 \iff g \in G_v, h \in G_w \text{ and } (v,w) \in E(\Lambda) \rangle$$

as the group generated by the vertex groups G_v with the relation that two vertex groups commute if and only if the corresponding vertices are joined by an edge. Clearly, RAAGs are special cases of graph products when $G_x = \mathbb{Z}$ for all x, and RACGs are graph products with $G_x = \mathbb{Z}/2\mathbb{Z}$. Graph products were first introduced by Green [Gre90] and have received much attention, see for example [BHP93, Chi94, Gre90, Hag08, HM95, HW+99, Mei96, Mei95, Rad03].

In this section, we are going to apply our counting techniques to graph products.

10.1. **Geodesic combing for graph products.** Let us call a group *admissible* if it has a geodesic combing with respect to some finite generating set (i.e., in the language of the previous sections, if it has an admissible generating set). Recall that a recurrent component of a directed graph Γ is nontrivial if it contains at least one closed path. A component is terminal if there is no path exiting it. Finally, a graph structure (G, Γ) is *recurrent* if every vertex admits a directed path to every vertex other than the initial one.

Recall that, given a graph Λ, the *opposite graph* is the graph Λ^{op} with the same vertex set as Λ and such that $(v,w) \in E(\Lambda^{op})$ if and only if $(v,w) \notin E(\Lambda)$. We will assume that Λ is *anticonnected*, i.e. that the opposite graph Λ^{op} is connected. This implies that $G(\Lambda)$ is not a direct product of graph products associated to subgraphs of Λ. As an example, if Λ is a square, then the opposite graph is the disjoint union of two segments, hence Λ is not anticonnected, while if Λ is a pentagon, then its opposite graph is also a pentagon, hence Λ is anticonnected.

Proposition 10.1. *Let Λ be anticonnected, and choose for each vertex x a group G_x with a geodesic combing (G_x, Γ_x) for the generating set S_x. Then the graph product $G(\Lambda)$ with the generating set $S = \cup_x S_x$ admits a geodesic combing which is recurrent.*

We call the generating set S in Proposition 10.1, the *standard* generating set for $G(\Lambda)$. Note that this agrees with the standard vertex generators for the special case of right-angled Artin and Coxeter groups.

The proof of Proposition 10.1 will be an explicit construction of a recurrent graph structure for $G(\Lambda)$ with the standard generators. First, Hermiller-Meier [HM95] produce a geodesic combing for $G(\Lambda)$ which is not recurrent. In the next few lemmas, we will show that if Λ is anticonnected we can modify their construction in order to produce a new graph structure which is reccurent. Of course it is necessary to assume that Λ is anticonnected, as the counting theorems fail for RAAGs which decompose as direct products (Example 1).

We begin by reviewing the construction of [HM95]. First introduce a total ordering on the vertices of Λ such that the first two vertices in the ordering are not adjacent in Λ. Each vertex of Λ will be labeled by a capital letter $A, B, \dots$.

Then for each pair of vertices (I, J) such that I and J are not adjacent in Λ and with $I > J$ one constructs the (I,J)-*admissible tree* in the following way. An (I,J)-*admissible word* is a finite sequence $IJK_1K_2 \dots K_r$ (with $r \geq 0$) such that:

(1)
$$J < K_1 < K_2 < \dots < K_r$$
and

(2) if $K_i \leq I$ for some $i \leq r$, then K_i is not adjacent to at least one vertex among $I, J, K_1, \dots, K_{i-1}$.

Given (I,J), the (I,J)-admissible tree is the finite directed tree, whose vertices are labeled by letters and whose paths spell exactly the (I,J)-admissible words. In particular, such a tree will have I as a root and there is only one edge coming out of this vertex, with endpoint J. Here, and in what follows, a directed edge always has the same label as its terminal vertex.

Moreover, [HM95] define the *header graph* (the terminology is ours) as the graph with one vertex for each letter, and an edge $A \to B$ if and only if $A < B$.

Finally, they construct the graph structure for $C(\Lambda)$ (the corresponding RACG) as follows. Consider the union of an initial vertex v_0, the header graph and all (I, J)-admissible trees. First, one identifies the vertex I of the header graph with the root of the (I, J)-admissible tree for each possible J. Then, one adds one edge from v_0 to each vertex of the header graph, and if $K > L$ and K, L are not adjacent in Λ, one joins by a directed edge each vertex labeled K in the union of the (I, J)-admissible trees with the L vertex in the (K, L)-admissible tree. As shown by Hermiller-Meier, this graph $\mathfrak{G}$ gives a bijective, geodesic graph structure for $C(\Lambda)$ [HM95, Section 5 and Proposition 3.3]. In fact, they show that $\mathfrak{G}$ recognizes the geodesic language of normal forms with respect to the ordering of the vertices of Λ, but we will not need this stronger fact.

Let C be the subgraph of $\mathfrak{G}$ obtained by removing all vertices in the header graph and the initial vertex. That is, C is the subgraph induced on the vertices on all (I, J)–admissible trees, excluding the initial vertex of the tree (which is labeled I). The next lemma is key to our modification.

Lemma 10.2. *If Λ is anticonnected, then the graph C is irreducible, i.e. there is one directed path from each vertex to any other vertex. Hence, $\mathfrak{G}$ has a unique nontrivial recurrent component C and this component is terminal.*

Proof. We will show that C is indeed irreducible. This will suffice since the header graph has no directed loops (it only has directed edges which increase in the ordering) and there are no edges leaving C by construction.

Since the (unique) type J vertex in the (I, J)–admissible tree has a directed path to each of its vertices and each vertex is in some (I, J)–admissible tree, it suffices to show that from any vertex of C we can reach the type J vertex of any (I, J)–admissible tree. Hence, fix some vertex v of $\mathfrak{G}$ and I, J, which are vertices of Λ.

Here is a main point: Any type I vertex of any admissible tree is joined to the type J vertex of the (I, J)–admissible tree. Hence, it suffices to get from v to any type I vertex of any admissible tree. To do this let X be the type of v.

Fix a path $X = X_0, X_1, \ldots X_n = I$ in the complement graph of Λ. (Here we use that Λ is anticonnected.) That is, X_i and X_{i+1} are not adjacent in Λ. We have to get from v to any type I vertex. We do this inductively as follows: We have either $X < X_1$ or $X > X_1$. In the first case, there is a type X_1 vertex v_1 in the admissible tree containing v along with a directed path $v \to v_1$. (Since no consecutive pair in our fixed path are adjacent in Λ, condition (2) above holds automatically.) In the second case, there is an edge from v to the unique X_1 vertex of the (X, X_1)–admissible tree, call this vertex v_1. In either case we get a directed path $v \to v_1$, were v_1 is a type X_1 vertex.

We now repeat this argument to produce a path from $v_1 \to v_2$, where v_2 is a type X_2 vertex. Continuing in this manner, we produce $v \to v_1 \to \ldots \to v_n$, were v_n has type I. Since v_n then has a directed edge to the type J vertex of the (I, J)–admissible tree (as discussed above), this completes the proof, □

We now know that the union of the admissible trees (excluding the initial vertices) is an irreducible graph. However, the header graph by construction is not irreducible. However, in the following lemma we observe that all words we can spell in the header graph can also be spelled in one of the admissible trees. Hence, we can modify $\mathfrak{G}$ (essentially, by removing the header graph) in order to get a recurrent graph $\mathfrak{G}^r$ which recognizes the same language as $\mathfrak{G}$.

Lemma 10.3. *If Λ is anticonnected, there exists a recurrent graph $\mathfrak{G}^r$ which recognizes the same language as $\mathfrak{G}$.*

Proof. Assume that Λ is anticonnected and that its vertices are ordered so that the first two vertices A, B do not commute (i.e. they are not adjacent). We modify $\mathfrak{G}$ so that the resulting graph $\mathfrak{G}^r$ still recognizes the same language as $\mathfrak{G}$, and it is recurrent.

The modification is simple and requires only one observation: we note that any strictly increasing sequence $X_1 \dots X_r$ can be spelled in the (B, A)–admissible tree, starting from some vertex. In fact, if $X_1 = A$ then $BX_1 \dots X_r$ is (B, A)-admissible, since the only required condition is that whenever $X_i \leq B$ the vertex X_i is not adjacent to some X_l with $l < i$. However, the only two letters not greater than B are A, B, and A and B are not adjacent by construction. Similarly, if $X_1 \neq A$ then $BAX_1 \dots X_r$ is (B, A)-admissible.

Thus, the new graph $\mathfrak{G}^r$ is given by removing the header graph and joining the initial vertex v_0 to each vertex of the (B, A)-admissible tree. Any word which is recognized by $\mathfrak{G}$ is made of an increasing word followed by a word spelled in the union of the admissible trees. In $\mathfrak{G}^r$, such a word is spelled by spelling the increasing sequence in the (B, A)-admissible tree, and the second part as before. This proves the claim. $\square$

The graph $\mathfrak{G}^r$ is a recurrent graph which by Lemma 10.3 gives a bijective, geodesic graph structure for the right-angled Coxeter group $C(\Lambda)$. We now modify the construction to produce a geodesic combing for each graph product $G(\Lambda)$.

Let Γ_I be the graph structure of the vertex group G_I, let $v_{0,I}$ be the initial vertex of G_I, let $s_{1,I}, \dots, s_{k,I}$ be the labels of the edges going out of $v_{0,I}$, and let $v_{1,I}, \dots, v_{I,k}$ be the targets of these edges, respectively. Moreover, let Γ'_I be the subgraph of Γ_I given by removing the initial vertex.

To construct the graph structure for $G(\Lambda)$, let us consider the disjoint union of a vertex $\widetilde{v}_0$, which will serve as initial vertex, and a copy of Γ'_I for each vertex v of type I in $\mathfrak{G}^r$. Moreover, for any edge in $\mathfrak{G}^r$ of type $I \to J$ let us connect each vertex of the corresponding Γ'_I with the vertices $v_{1,J}, \dots, v_{k,J}$ of the corresponding Γ'_J with edges labeled, respectively, $s_{1,J}, \dots, s_{k,J}$. Finally, for each edge from v_0 in $\mathfrak{G}^r$ to some other vertex of type I, let us connect the new initial vertex $\widetilde{v}_0$ with vertices $v_{1,J}, \dots, v_{k,J}$ of Γ'_J with edges labeled, respectively, $s_{1,J}, \dots, s_{k,J}$.

This new graph Γ_G gives a bijective, geodesic structure for $G(\Lambda)$ with respect to the standard generators. This follows since, by construction, Γ_G parameterizes the same language of geodesic normal forms for $G(\Lambda)$ given in [HM95]. Moreover, since Γ_G is modeled on the recurrent graph $\mathfrak{G}^r$ one easily sees that Γ_G is itself recurrent. This completes the proof of Proposition 10.1.

Corollary 10.4. *Let $G(\Lambda)$ be a graph product of admissible groups, which does not decompose as a direct product. Then there exists a thick graph structure for its standard generating set.*

Proof. From Proposition 6.2, the graph structure given by the above proposition is thick since Γ_G is recurrent. $\square$

As a consequence of thickness, we are ready to establish the following counting result for loxodromics.

Theorem 10.5. *Let G be a graph product of admissible groups which is not a nontrivial direct product, and let S be its set of standard (vertex) generators. Then for any nonelementary action $G \curvearrowright X$ on a hyperbolic space X, the set of loxodromics for the action is generic with respect to S, i.e.*

$$\frac{\#\{g \in G : |g|_S \leq n \text{ and } g \text{ is } X-\text{loxodromic}\}}{\#\{g \in G : |g|_S \leq n\}} \longrightarrow 1,$$

as $n \to \infty$.

Proof. The pair (G, S) admits a thick graph structure by Corollary 10.4, hence the claim follows from Theorem 6.4. $\square$

11. Exact exponential growth for RAAGs and RACGs

We conclude by proving a fine estimate on the number of elements in a ball for RAAGs and RACGs.

Theorem 11.1. *Let G be a right-angled Artin or Coxeter group which is not virtually cyclic and does not decompose as a nontrivial direct product, and let us consider S its standard generating set. Then there exists constants $\lambda > 1$, $c > 0$ such that the following limit exists:*

$$\lim_{n\to\infty} \frac{\#\{g \in G \ : \ |g|_S = n\}}{\lambda^n} = c. \tag{15}$$

We say that a group which satisfies (15) has *exact exponential growth*. Let us remark that such a property is *not* invariant with respect to quasi-isometries of the metric, and hence it depends very carefully on the generating set. Moreover, the proof shows that λ is a Perron number, which is one of properties conjectured by [KP11] for cocompact Coxeter groups acting on $\mathbb{H}^n$.

In fact, Theorem 11.1 will follow immediately from the following theorem for general graph products.

Theorem 11.2. *Let $G(\Lambda)$ be a graph product of admissible groups, and assume that Λ is anticonnected (so that the group does not split trivially as a product) and has at least 3 vertices. Then $G(\Lambda)$ has exact exponential growth.*

Note that it makes sense to assume that the number of vertices is at least 3. In fact, if $n = 1$ then $G(\Lambda)$ can be any group with a geodesic combing, while if $n = 2$ then $G(\Lambda)$ can be the free product of any two admissible groups. In particular, if it is a RAAG then it must be the free group on 2 generators, which has exact exponential growth, and if it is a RACG it must be $\mathbb{Z}/2\mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$, which is virtually cyclic.

Let us remark that the growth function for graph products has been worked out by Chiswell [Chi94] (see also [AP14]); however, it does not seem obvious how to prove exact exponential growth by this method.

Let us consider the recurrent graph $\mathfrak{G}^r$ defined in the previous section, and denote as $\mathfrak{G}^r_0 = \mathfrak{G}^r \setminus \{v_0\}$. By the previous section, we know that $\mathfrak{G}^r_0$ is irreducible. The final step in the proof of Theorem 11.2 is the following lemma.

Lemma 11.3. *If Λ is anticonnected and has at least 3 vertices, then the graph $\mathfrak{G}^r_0$ is aperiodic.*

Proof. Let us assume, consistently with the previous section, that the vertices of Λ are ordered. Let us call A, B the two smallest vertices, with $A < B$, and assume that A, B are not adjacent. Moreover, let C be the largest vertex in the ordering. Then let us observe that the sequences BAC and $BABC$ are (B, A) admissible, hence in the (B, A)-admissible tree there is a Y-shaped subtree with five vertices: one labeled A, two labeled B (let us denote them B_1, B_2) and two labeled C (let us denote them C_1, C_2) so that the paths in this subtree are $B_1 \to A \to C_1$ and $B_1 \to A \to B_2 \to C_2$. Now, since the graph is irreducible, there exists a path from C_1 to A; let us denote its vertices as $C_1 \to v_1 \to v_2 \cdots \to v_k \to A$. Then by definition, the type of v_1 is smaller than C, and is not adjacent to C. Thus, by construction, there is also an edge from C_2 to v_1; hence, in the graph there are two loops: one loop is given by $A \to C_1 \to v_1 \to \cdots \to v_k \to A$ and the other is $A \to B \to C_2 \to v_1 \to \cdots \to v_k \to A$. Since the lengths of these two closed paths differ by one, the greatest common divisor of the lengths of all paths is 1, hence $\mathfrak{G}^r_0$ is aperiodic. □

Note that the statement is false if the number of vertices is 2: indeed, then there is only one loop of length 2, hence the period is 2.

Now, let us consider a general graph product $G(\Lambda)$. By the previous section, by replacing vertices of $\mathfrak{G}^r$ with graphs which recognize the geodesic combings of vertex group, we get a new graph Γ_G which gives a geodesic combing for $G(\Lambda)$. By the previous lemma we get:

Corollary 11.4. *If Λ is anticonnected and has at least three vertices, then the graph $\Gamma'_G = \Gamma_G \setminus \{v_0\}$ is irreducible and aperiodic.*

Proof of Theorem 11.2. Since the graph Γ'_G is irreducible and aperiodic, then by the Perron-Frobenius theorem its adjacency matrix A has a unique eigenvalue $\lambda > 1$ of maximum modulus, and that eigenvalue is real, positive, and simple. Moreover, the coordinates of the corresponding eigenvector are all positive. Finally, the sequence $\frac{A^n}{\lambda^n}$ converges to the projection to the eigenspace. In particular, none of the basis vectors is orthogonal to the eigenvector, hence for any i, j there exists $c_{ij} > 0$ such that

$$\lim_{n\to\infty} \frac{(A^n)_{ij}}{\lambda^n} = c_{ij}.$$

Now, each path of length n from the initial vertex starts with an edge to the irreducible graph, hence

$$\frac{\#S_n}{\lambda^n} = \sum_{v_0 \to v_i} \frac{\#S_{n-1}(v_i)}{\lambda^n} \to \sum_{v_0 \to v_i} \sum_j \frac{(A^{n-1})_{ij}}{\lambda^n} \to \sum_{v_0 \to v_i} \sum_j \frac{c_{ij}}{\lambda} = c > 0$$

which establishes exact exponential growth. □

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