PSEUDO-ANOSOVS OPTIMIZING THE RATIO OF TEICHMÜLLER TO CURVE GRAPH TRANSLATION LENGTH

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Abstract. Given ϕ a pseudo-Anosov map, let $\ell_{\mathcal{T}}(\phi)$ denote the translation length of ϕ in the Teichmüller space, and let $\ell_{\mathcal{C}}(\phi)$ denote the stable translation length of ϕ in the curve graph. Gadre–Hironaka–Kent–Leininger showed that, as a function of Euler characteristic $\chi(S)$, the minimal possible ratio $\tau(\phi) = \frac{\ell_{\mathcal{T}}(\phi)}{\ell_{\mathcal{C}}(\phi)}$ is $\log(|\chi(S)|)$, up to uniform additive and multiplicative constants. In this short note, we introduce a new construction of such *ratio optimizers* and demonstrate their abundance in the mapping class group. Further, we show that ratio optimizers can be found arbitrarily deep into the Johnson filtration as well as in the point pushing subgroup.

1. Introduction

Let $S = S_{g,p}$ denote the orientable surface of genus g with p punctures, and let $\omega(g,p) = \omega(S) = 3g + p - 4$ be its *complexity*. Let $\mathrm{Mod}(S)$ denote the mapping class group of S, $\mathrm{Teich}(S)$ the Teichmüller space equipped with the Teichmüller metric, and $\mathcal{C}(S)$ the curve graph of S.

Consider the coarsely-defined map $\pi_{g,p} : \mathrm{Teich}(S) \to \mathcal{C}^0(S)$, which sends a marked hyperbolic surface to the simple closed curve(s) of shortest length. The map $\pi_{g,p}$ was originally studied by Masur-Minsky, who, as part of the proof of the δ-hyperbolicity of $\mathcal{C}(S)$, demonstrated the existence of a constant $K = K(g,p)$ such that $\pi_{g,p}$ is coarsely K– Lipschitz [MM99]. Recall that a map $f : X \to Y$ between metric spaces is *coarsely K–Lipschitz* if there is an $L \geq 0$ such that $d_Y(f(a), f(b)) \leq K \cdot d_X(a,b) + L$ for all $a, b \in X$.

Let $K(g,p)$ denote the optimal possible value of the Lipschitz constant for $\pi_{g,p}$ as a function of $S_{g,p}$; that is,

$$K(g,p) = \inf\{c \in \mathbb{R} : \pi_{g,p} \text{ is coarsely } c\text{-Lipschitz}\}.$$

Gadre–Hironaka–Leininger–Kent showed that $K(S_{g,0}) \sim \frac{1}{\log(g)}$ [GHKL13]. Thus, not only is $\pi_{g,p}$ coarsely Lipschitz, but it is coarsely contracting, and the optimal contraction factor approaches 0 as $g \to \infty$. Following this work, Valdivia showed that for $r \in \mathbb{Q}$ a fixed rational number, the optimal Lipschitz constant for a sequence of surfaces S_{g_i,p_i} with $g_i/p_i = r$, also decays logarithmically in complexity (relative to constants that a priori depend on r) [Val14].

In one direction, [GHKL13] follow Masur and Minsky's original proof, while controlling the portions of the argument that a priori grow with the complexity of S. Conversely, to show that a Lipschitz constant on the order of $1/\log(g)$ is optimal, they construct, for each g, a pseudo-Anosov map $\psi \in \mathrm{Mod}(S_g)$ such that the ratio $\tau(\psi) = \ell_{\mathcal{T}}(\psi)/\ell_{\mathcal{C}}(\psi)$ of its translation length in $\mathrm{Teich}(S)$, denoted $\ell_{\mathcal{T}}(\psi)$, to its stable translation length in $\mathcal{C}(S)$, denoted $\ell_{\mathcal{C}}(\psi)$, is on the order of $\log(g)$.

Date: October 4, 2015.

2000 *Mathematics Subject Classification.* 46L55.

Key words and phrases. curve complex, mapping class group.

The purpose of this short note is to give a new construction of pseudo-Anosov maps for which $\tau(\psi)$ is optimal, i.e. on the order of $\log(\omega(S))$. We call such pseudo-Anosovs *ratio optimizers* and our construction shows their abundance in the mapping class group:

Theorem 1.1. *There exists a function $f(\omega) = O(\log(\omega))$, and a Teichmüller disk $\mathcal{D} \subset$ Teich$(S_{g,p})$ such that there are infinitely many conjugacy classes of primitive pseudo-Anosovs ψ with $\tau(\psi) = \frac{\ell_{\mathcal{T}}(\psi)}{\ell_{\mathcal{C}}(\psi)} < f(\omega(g,p)))$, and the invariant axis of ψ is contained in $\mathcal{D}$.*

We will see in Corollary 3.5 that the function $f(\omega)$ can be taken to be $\log(2B \cdot \omega)$ where $B \geq 1$ is a constant not depending on ω.

In addition to establishing the abundance of ratio optimizers, our methods show that ratio optimizers can be constructed in subgroups of mapping class groups which are well-known not to contain pseudo-Anosov mapping classes that minimize Teichmüller space translation length alone. In particular, we build ratio optimizers arbitrarily deep into the Johnson filtration as well as in the point pushing subgroup for a mapping class group of a surface with a single puncture.

For a group Γ, let $\Gamma^{(k)}$ denote the kth term of its lower central series. That is $\Gamma^{(1)} = [\Gamma, \Gamma]$ is the commutator subgroup and $\Gamma^{(k+1)} = [\Gamma^{(k)}, \Gamma]$. For any $k \geq 0$ there is a surjective homomorphism

$$\mathrm{Mod}(S) \to \mathrm{Out}(\pi_1(S)/\pi_1(S)^{(k)}),$$

whose kernel, denoted J_k, is the *kth term of the Johnson filtration.* These subgroups were introduced by Johnson in [Joh83]. Note that J_1 is the Torelli subgroup of $\mathrm{Mod}(S)$ and J_2 is the so-called Johnson kernel.

Theorem 1.2. *There exists a uniform constant $C_J \geq 0$ satisfying the following. Let $S = S_{g,p}$, with $g \geq 2$ and $p = 0$ or $p = 1$, and denote by $J_k(S)$ the kth term of the Johnson filtration of* $\mathrm{Mod}(S)$*. Then there exists $\phi_k \in J_k(S)$ with*

$$\tau(\phi_k) = \frac{\ell_{\mathcal{T}}(\phi)}{\ell_{\mathcal{C}}(\phi)} \leq C_J \log \omega(S).$$

That is, there are ratio optimizers arbitrarily deep into the Johnson filtration.

We remark that Theorem 1.2 is entirely different from the situation of minimizing $\ell_{\mathcal{T}}(\phi)$ alone. In fact, Farb–Leininger–Margalit [FLM08] have shown that the minimal Teichmüller space translation length among pseudo-Anosov mapping class in J_1 is uniformly bounded from above and below, independent of genus. This is in contrast to work of Penner who shows that among all pseudo-Anosov homeomorphisms this quantity is on the order of $1/g$ [Pen91].

Finally, for a surface $S_{g,1}$ with $g \geq 2$, we denote the kernel of the natural map

$$\mathrm{Mod}(S_{g,1}) \to \mathrm{Mod}(S_{g,0})$$

by PP_g. This is the point pushing subgroup of $\mathrm{Mod}(S_{g,1})$; it consists of mapping classes which are isotopic to the identity after ignoring the puncture. Similar to the situation discussed above, it is known that pseudo-Anosov mapping classes in PP_g cannot minimize Teichmüller space translation length [Dow11]. However, this is not an issue for ratio optimizers:

Theorem 1.3. *There exists a uniform constant $C_P \geq 0$ satisfying the following. Let $S = S_{g,1}$ with $g \geq 2$ and let $PP_g \leq$ $\mathrm{Mod}(S)$ be the point pushing subgroup of its mapping* *class group. Then there is* $\phi \in PP_g$ *with*

$$\tau(\phi) = \frac{\ell_{\mathcal{T}}(\phi)}{\ell_{\mathcal{C}}(\phi)} \leq C_P \log \omega(S).$$

Acknowledgments: Both authors are partially supported by the National Science Foundation. The first by NSF grant DMS- 1502623 and the second by NSF grant DMS-1400498.

2. Background

2.1. **Curves, filling pairs and projections.** Let $S_{g,p}$ denote the genus g surface with $p \geq 0$ punctures. The *complexity* of S is defined as $\omega(S) = \omega(g,p) = 3g + p - 4$. For all surfaces in this paper we assume $\omega(S) > 0$. A simple closed curve c on S is *essential* if it is not homotopically trivial and if it is not homotopic into a neighborhood of a puncture. Given two essential simple closed curves α, β their *geometric intersection number*, denoted $i(\alpha, \beta)$, is defined as

$$i(\alpha, \beta) := \min_{x \sim \alpha} |x \cap \beta|,$$

where $\sim$ denotes homotopy. If $|\alpha \cap \beta| = i(\alpha, \beta)$, we say α and β are in *minimal position.* Note that any collection of pairwise non-homotopic essential curves can be placed in pairwise minimal position on S. Indeed, when S is equipped with any complete hyperbolic metric, any pair of closed geodesics is in minimal position and there exists a unique geodesic in each free homotopy class of essential curve.

A pair of essential simple closed curves α, β are in minimal position on a closed surface S_g if and only if no complementary component of $\alpha \cup \beta$ is a *bigon*, a disk whose boundary is comprised of one arc of α and one of β [FM11].

A collection of curves $\Gamma = \{\gamma_1, ..., \gamma_n\}$ in pairwise minimal position is said to *fill* a surface S if the complement of their union consists of a disjoint union of topological disks and once-punctured disks. Equivalently, Γ fills S so long as every essential simple closed curve α has positive geometric intersection number with at least one curve in Γ. Let $i_{g,p}$ denote the minimum possible geometric intersection number for a filling pair α, β on $S_{g,p}$. A simple Euler characteristic argument shows that $i_{g,p}$ must grow linearly in $\omega(g,p)$. In [AH15] and [AT14] the quantities $i_{g,p}$ were determined.

Lemma 2.1 (Minimally intersecting filling pairs)**.** *Minimally intersecting filling pairs intersect as follows:*

1. *If* $g \neq 2, 0$ *and* $p = 0$, $i_{g,p} = 2g - 1$.
2. *If* $g \neq 2, 0$ *and* $p \geq 1$, $i_{g,p} = 2g + p - 2$.
3. *If* $g = 0$ *and* $p \geq 6$ *is even, then* $i_{g,p} = p - 2$. *On the other hand if* p *is odd,* $i_{g,p} = p - 1$.
4. *If* $g = 2$ *and* $p \leq 2$, $i_{g,p} = 4$.
5. *If* $g = 2$ *and* $p \geq 2$ *is even, then* $i_{g,p} = 2g+p-2$*; and if* $p \geq 3$ *is odd,* $i_{g,p} \leq 2g+p-1$.

In our application to pseudo-Anosov mapping classes in the Johnson filtration, we also require information about filling pairs of separating curves. Let $i_{g,p}^{sep}$ denote the minimum geometric intersection number taken over all filling pairs α, β where both α, β are separating curves. Then we have the following:

Lemma 2.2 (Separating filling pairs)**.** *There exists a constant* $C \geq 0$ *such that if* $g \geq 2$ *and* $p = 0$ *or* 1*, there is a filling pair* (α, β) *on* $S_{g,p}$ *with both* α, β *separating curves, satisfying* $i(\alpha, \beta) \leq C \cdot \omega(g,p)$.

Proof. First suppose $p = 0$, and define α_2, β_2 to be any pair of separating curves which fill S_2; similarly let α_3, β_3 be any pair of separating curves filling S_3. These will be the seeds of an inductive construction.

Now let ρ, γ be a pair of simple separating arcs on $S_{2,1}$ (which we interpret as the genus 2 surface with one boundary component, as opposed to one puncture) having the property that any essential arc in $S_{2,1}$ intersects either ρ or γ. Then given α_g, β_g, we form $\alpha_{g+2}, \beta_{g+2}$ as follows: excise a small open disk centered at one of the points in $\alpha_g \cap \beta_g$. After excising, α_g and β_g have become arcs which we denote by $\tilde{\alpha}_g, \tilde{\beta}_g$. We then glue on a copy of $S_{2,1}$, matching the endpoints of $\tilde{\alpha}_g$ to those of γ, and similarly matching the endpoints of $\tilde{\beta}_g$ to those of ρ. We obtain a pair of simple closed curves $\alpha_{g+2}, \beta_{g+2}$ on S_{g+2}, and we claim that these curves are both separating and that they fill.

Note first that $\alpha_{g+2}, \beta_{g+2}$ are in minimal position since no complementary region is a bigon, and therefore it suffices to prove that if κ is any essential simple closed curve on S_{g+2}, κ is not disjoint from $\alpha_{g+2} \cup \beta_{g+2}$. If κ can be isotoped into the original copy of S_g, it must intersect either $\tilde{\alpha}_g$ or $\tilde{\beta}_g$ since α_g, β_g fill on S_g. Therefore, we can assume that κ projects non-trivially to the copy of $S_{2,1}$; that is, that κ intersects this copy of $S_{2,1}$ in at least one arc which is not boundary parallel. This arc must intersect either ρ or γ since any arc does so by construction. Therefore κ intersects $\alpha_{g+2} \cup \beta_{g+2}$ and we conclude that the new pair fills S_{g+2}.

That $\alpha_{g+2}, \beta_{g+2}$ are both separating is immediate since both are obtained by concatenating a pair of separating arcs in disjoint subsurfaces. Finally, $i(\alpha_{g+2}, \beta_{g+2}) \leq i(\alpha_g, \beta_g) + i(\gamma, \rho)$.

If $p = 1$, then by puncturing one complementary region of $S_{g,0} \setminus (\alpha_g \cup \beta_g)$, (α_g, β_g) is a filling pair on $S_{g,1}$ with the desired properties. □

2.2. **Annular projections and the bounded geodesic image theorem.** For an annulus $Y \subset S$ whose core curve α is essential, let $\tilde{Y}$ be the cover of S associated to the conjugacy class of the cyclic subgroup of $\pi_1(S)$ represented by α. Let $\overline{Y}$ be the compactification of $\tilde{Y}$ obtained by choosing a hyperbolic metric on S and lifting it to $\tilde{Y}$. The curve graph $\mathcal{C}(Y)$ of the annulus Y is the graph whose vertices are homotopy classes of properly embedded, simple arcs of $\overline{Y}$ whose endpoints lie on distinct boundary components. Two vertices x and y of $\mathcal{C}(Y)$ are joined by an edge of $\mathcal{C}(Y)$ if and only if x and y can be represented by arcs in $\overline{Y}$ with disjoint interiors. There is a projection π_Y from the vertices of the curve graph of S to arcs of $\mathcal{C}(Y)$, known as *subsurface projection*. Given $\beta \in \mathcal{C}^0(S)$ realize α and β with minimal intersection in S. If β is disjoint from α then define $\pi_Y(\beta) = \emptyset$. Otherwise, the preimage of β in the cover $\tilde{Y}$ contains simple, properly embedded arcs with well-defined endpoints on distinct components of $\partial\overline{Y}$ and we define $\pi_Y(\beta) \subset \mathcal{C}^0(Y)$ to be this collection of arcs in $\overline{Y}$.

If $\alpha \in \mathcal{C}^0(S)$ is a curve which is the core of an annulus Y then we also use the notation $\mathcal{C}(\alpha)$ for the curve complex $\mathcal{C}(Y)$ and we denote its path metric by d_α. Let $\pi_\alpha : \mathcal{C}(S) \setminus N_1(\alpha) \to \mathcal{C}(\alpha)$ be the corresponding subsurface projection, where $N_1(\cdot)$ is the closed 1-neighborhood in $\mathcal{C}_1(S)$. Also, write $d_\alpha(\beta, \gamma)$ for $\mathrm{diam}_{\mathcal{C}(\alpha)}(\pi_\alpha(\beta) \bigcup \pi_\alpha(\gamma))$.

From [MM99], we recall the following:

Lemma 2.3 (Masur–Minsky)**.** *Let S be a surface with $\omega(S) > 1$. For $\alpha \in \mathcal{C}^0(S)$ and any path $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_n$ of curves in $\mathcal{C}(S)$ each intersecting α essentially, we have:*

(1) $\mathrm{diam}_{\mathcal{C}(\alpha)} \pi_\alpha(\gamma) \leq 1$

(2) $d_\alpha(\gamma_0, \gamma_n) \leq n+1$
(3) *If* T_α *is the Dehn twist about* α*, then* $d_\alpha(\gamma, T^N(\gamma)) \geq N-2$*.*

Finally, we recall the bounded geodesic image theorem of Masur–Minsky [MM00]. The version we state here is due to Webb and gives a uniform, computable constant [Web13]. It is stated below for arbitrary subsurfaces $Y \subset S$, but we will use it only for annuli.

Theorem 2.4 (Bounded geodesic image theorem)**.** *There exists* $M \geq 0$ *so that for any surface* S *and any geodesic* g *in* $\mathcal{C}(S)$*, if each vertex of* g *has nontrivial projection to the subsurface* Y *then* $\operatorname{diam}(\pi_Y(g)) \leq M$*.*

2.3. **Mangahas' Lemma.** Let $C_A = N_1(\alpha)$ and $C_B = N_1(\beta)$ be 1-neighborhoods of the curves α and β in $\mathcal{C}(S)$ and let M be as in the bounded geodesic image theorem (Theorem 2.4). The following is a special case of a combination of Lemma 5.3 of [Man13] along with the claim used in its proof. Recall that for a word w in the free group $F(a,b)$, the *syllable length* of w, denote $|w|_s$, is the number of powers of a or b that occur in the reduced form for w.

Lemma 2.5 (Mangahas)**.** *Let* a, b *be powers of Dehn twists about curves* α, β*, respectively, such that* $d_S(\alpha, \beta) \geq 3$*, i.e.* α *and* β *fill* S*. Suppose that for all* $k \neq 0$

$$d_\alpha(C_B, a^k \cdot C_B) > 2M+4 \quad \text{and} \quad d_\beta(C_A, b^k \cdot C_A) > 2M+4.$$

Then for any word w *in* $\langle a, b\rangle$*, either*

$$d_S(w\cdot\alpha, \alpha) \geq |w|_s \quad \text{or} \quad d_S(w\cdot\beta, \beta) \geq |w|_s.$$

We remark that if $a = T_\alpha^{l_1}$ and $b = T_\beta^{l_2}$ then by Lemma 2.3 the hypotheses of Lemma 2.5 are satisfied so long as $|l_1|, |l_2| \geq 2M+7$.

2.4. **Pseudo-Anosovs and Teichmüller disks.** For curves α and β which jointly fill the surface S and have intersection number $i(\alpha,\beta) = n$, there is a representation $\Psi : \langle T_\alpha, T_\beta\rangle \to PSL_2(\mathbb{R})$ given by

$$\Psi(T_\alpha) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad \Psi(T_\beta) = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}. \tag{2.1}$$

Thurston showed that the pseudo-Anosov mapping classes of the subgroup $\langle T_\alpha, T_\beta\rangle \leq \operatorname{Mod}(S)$ are exactly the ones mapping to hyperbolic matrices in $PSL_2(\mathbb{R})$ (i.e. matrices with 2 distinct eigenvalues). Further, he showed that the dilatation of such a pseudo-Anosov is equal to the largest eigenvalue of its representative matrix [Thu88]. Since the Teichmüller space translation length of a pseudo-Anosov mapping class ϕ is equal to the logarithm of its dilatation, this allows a direct computation of $\ell_{\mathcal{T}}(\phi)$ for $\phi \in \langle T_\alpha, T_\beta\rangle$. For proofs of these facts see [FM11].

The representations $\Psi : \langle T_\alpha, T_\beta\rangle \to PSL_2(\mathbb{R})$ in (2.1) comes from the singular flat structure $S(\alpha,\beta)$ on S associated to the pair (α,β). This is the structure induced by the quadratic differential q on S whose vertical foliation is equal to α and whose horizontal foliation is equal to β as measured foliations. Alternatively, one can consider the dual square complex to the graph $\alpha \cup \beta$, which induces a complex structure on S along with the quadratic differential q obtained by taking dz^2 in the interior of each square. The Dehn twists T_α and T_β can each be realized by an affine map with respect to the singular flat structure, and the "derivative" map induces the representation to $PSL_2(\mathbb{R})$. See [FM11] for details.

The image in $\mathcal{T}(S)$ of the $SL_2(\mathbb{R})$ orbit of a quadric differential q is known as a *Teichmüller disk*. Given a filling pair α, β, by $\mathbb{D}(\alpha,\beta)$ we mean the Teichmüller disk corresponding to the quadratic differential described above, determined by the dual square complex to $\alpha\cup\beta$. The subgroup of $\mathrm{Mod}(S)$ preserving $\mathbb{D}(\alpha,\beta)$ is known as the Veech group $V(\alpha,\beta)$ and equals the image in $\mathrm{Mod}(S)$ of the affine homeomorphisms of the singular flat surface $S(\alpha,\beta)$. In particular, we note that $\langle T_\alpha, T_\beta\rangle \le V(\alpha,\beta)$.

3. Ratio optimizers via QI trees

Let $S = S_{g,p}$. Choose simple closed curves $\alpha,\beta\in\mathcal{C}^0(S)$ which fill S, that is, for which $d_S(\alpha,\beta)\ge 3$. For notational convenience, set $i = i_{\alpha,\beta} = i(\alpha,\beta)$. We will use later that α and β can be chosen so that $i(\alpha,\beta)\le i_{g,p}$ where $i_{g,p}$ is as in Section 2.1 and depends linearly on the complexity $\omega(S)=\omega(g,p)$ of S. Let M be the bounded geodesic image constant of Theorem 2.4. Recall that M is independent of the complexity of S. Let $B = 2M+7$ and set $a = T_\alpha^B$ and $b = T_\beta^B$.

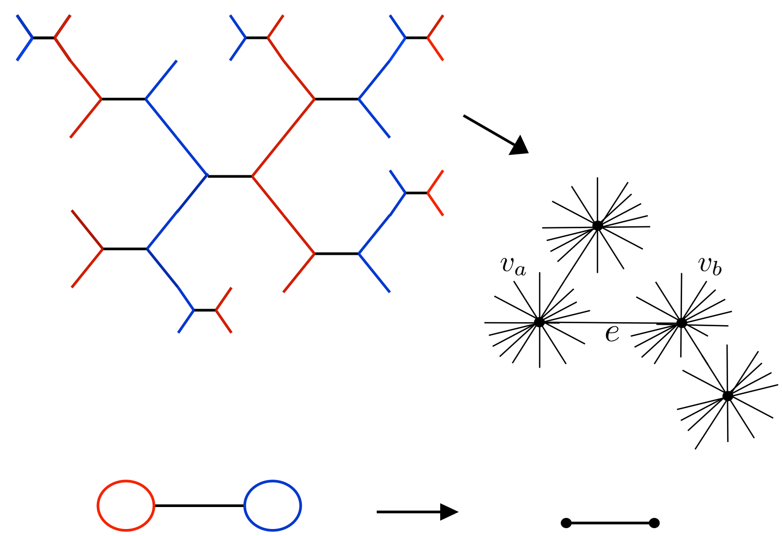

Figure 1. The tree $T_{\alpha,\beta}$ as a collapse of the trivalent tree.

It is well known that the subgroup of $\mathrm{Mod}(S)$ generated by a,b is isomorphic to the free group of rank 2 (see, for example, [Lei04, Theorem 6.1] [Man10, Proposition 3.3]) and we make the identification $F_2 = \langle a,b\rangle \le \mathrm{Mod}(S)$. Let $T = T_{\alpha,\beta}$ be the Bass–Serre tree for the splitting $\langle a\rangle * \langle b\rangle$. In details, T is the F_2–tree obtained by taking the universal cover of the "barbell" graph whose loops are labeled by $\{a,b\}$ and collapsing the lift of each a–edge and each b–edge. See Figure 1. Denote the image of the axis for a by v_a and note that this is the unique vertex of T which is fixed by a. Similarly, denote the image of the axis for b by v_b. Note that these vertices are joined by an edge, which we call e, and e is a fundamental domain for the action $F_2 \curvearrowright T$.

We now define an equivariant map $\mathcal{O} : T \to \mathcal{C}(S)$. The vertex v_a is mapped to α, v_b is mapped to β and e is mapped to a fixed geodesic from α to β in $\mathcal{C}(S)$. Using the identification $F_2 \to \langle a,b\rangle \le \mathrm{Mod}(S)$, extend the map to all of T by equivariance. This is well-defined since a fixes α and b fixes β. The main result of this section comes from the following proposition:

Proposition 3.1 (The tree quasi-isometrically embeds)**.** *With notation as above, the $\langle a, b\rangle$–equivariant map $\mathcal{O} : T_{\alpha,\beta} \to \mathcal{C}(S)$ is a $(3, 7)$–quasi-isometric embedding.*

We remark that by examining the proof of Lemma 2.5, the constants in Proposition 3.1 can be improved upon. This, however, will not be necessary for our application.

Before turning to the proof of Proposition 3.1, we first make a few remarks on distance in the (infinite valence) tree T. For a reduced word w in $\langle a, b\rangle$, a *syllable* of w is a maximal subword of the form a^k or b^k. The *syllable length* of w, denoted $|w|_s$, is the number of syllables in the word w. For example, for the reduced word $w = a^{k_1} b^{k_2} \dots a^{k_l}$, $|w|_s = l$. The syllable spelling of w is exactly the normal form associated to the tree T and, hence, can be used to compute distance in T. In particular, let x and y be two vertices of T; there are four cases depending on whether x and y are in the orbit of v_a or v_b. For example, suppose that $x = v_a$ and that $y \in F \cdot v_a \setminus \{v_a\}$. Then there is $w \in F$ with $y = w \cdot x$ and we can write $w = a^{l_1} w' a^{l_2}$ as a reduced syllable decomposition. Now it is easily seen that $d_T(x, y) = d_T(v_a, w'v_a) = |w'|_s + 1$. If $y \in F \cdot v_b$ then write $y = w \cdot v_b$ with $w = a^{l_1} w' b^{l_2}$ as a reduced syllable decomposition. Again we see that $d_T(x, y) = d_T(v_a, w'v_b) = |w'|_s + 1$. These elementary observations will be used in the proof of Proposition 3.1.

Proof of Proposition 3.1. Recall the definition of C_A, C_B from the statement of Lemma 2.5. Let $x, y \in T$ and set $\gamma = \mathcal{O}(x)$ and $\delta = \mathcal{O}(y)$. Using equivariance and the fact that $d_S(\alpha, \beta) = 3$ we easily see that, $d_S(\gamma, \delta) \le 3 \cdot d_T(x, y)$.

For the other inequality, we may assume (by equivariance) that x equals either v_a or v_b; since the proofs in each case are identical we assume that $x = v_a$. First, suppose that y is in the orbit of v_a, i.e. that $y = w \cdot v_a$ for $w \in F$. By the definition of $\mathcal{O}$ and the triangle inequality,

$$\begin{aligned} d_S(\gamma, \delta) &= d_S(\alpha, w \cdot \alpha) \\ &\ge d_S(\beta, w \cdot \beta) - 6, \end{aligned}$$

and thus by Lemma 2.5, $d_S(\gamma, \delta) \ge |w|_s - 6 \ge d_T(x, y) - 7$.

If on the other hand y is in the obit of v_b, then we choose $w \in F$ so that $y = w \cdot v_b$. By the triangle inequality $d_S(\alpha, w \cdot \beta) \ge d_S(\alpha, w \cdot \alpha) - 3$ and $d_S(\alpha, w \cdot \beta) \ge d_S(\beta, w \cdot \beta) - 3$. Hence, we may apply Lemma 2.5 to conclude

$$\begin{aligned} d_S(\gamma, \delta) &= d_S(\alpha, w \cdot \beta) \\ &\ge |w|_s - 3 \\ &\ge d_T(x, y) - 4. \end{aligned}$$

□

We will say that $w \in \langle a, b\rangle$ is *cyclically reduced* if it has the smallest syllable length among any of its conjugates. The following is an immediate corollary of Proposition 3.1.

Corollary 3.2. *Let $w \in \langle a, b\rangle \le \mathrm{Mod}(S)$ which is cyclically reduced. Then*

$$|w|_s \le \ell_{\mathcal{C}}(w) \le 3|w|_s.$$

The next lemma is elementary and is used to bound the stretch factors of pseudo-Anosovs obtained by iterating composition.

Lemma 3.3. *For any $i \ge 2$, let*

$$a = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}$$

and set $w = a^{\epsilon_1} b^{\delta_1} \dots a^{\epsilon_k} b^{\delta_k}$, *where* $\epsilon_i, \delta_i \in \{\pm 1\}$. *Then* $\mathrm{trace}(w) \le (2i)^{|w|}$.

Proof. For a 2-by-2 matrix A, let $|A|_1$ denote its l^1-norm and $|A|_2$ its l^2-norm. For any such A, $|A|_1 \le 2|A|_2$. Moreover, for any matrices A and B, $|AB|_2 \le |A|_2 |B|_2$. Hence,

$$\begin{aligned}\mathrm{trace}(w) &\le |w|_1 \\ &\le 2|w|_2 \\ &\le 2\prod_i |a^{\epsilon_i} b^{\delta_i}|_2 \\ &\le 2(i^2+1)^k < (2i)^{2k}.\end{aligned}$$

□

Theorem 3.4 (Ratio bounds)**.** *Let* α, β *be a filling pair of simple closed curves on* S, *and set* $a = T_\alpha^B, b = T_\beta^B$. *Let* w *be a cyclically reduced word in* a, b *satisfying* $|w| = |w|_s$. *Then*

$$\tau(w) = \frac{l_{\mathcal{T}}(w)}{l_{\mathcal{C}}(w)} \le \log(2B \cdot i(\alpha, \beta)).$$

Proof. Recall, as noted in Section 2.4, $\ell_{\mathcal{T}}(w)$ is equal to the logarithm of largest eigenvalue of the matrix corresponding to w. Applying Lemma 3.3 and Corollary 3.2, we compute

$$\begin{aligned}\tau(w) &\le \frac{\log((2i)^{|w|})}{|w|_s} \\ &= \frac{|w|}{|w|_s}\log(2i).\end{aligned}$$

Since $|w| = |w|_s$ and $i = B \cdot i(\alpha, \beta)$, this completes the proof. □

The following corollary of Theorem 3.4 gives our construction of ratio optimizers.

Corollary 3.5 (Ratio optimizers)**.** *Let* α, β *be a filling pair of simple closed curves satisfying* $i(\alpha, \beta) = \eta_S$. *Then for* w *a cyclically reduced word as in Theorem 3.4,*

$$\begin{aligned}\tau(w) = \frac{l_{\mathcal{T}}(w)}{l_{\mathcal{C}}(w)} &\le \log(B \cdot \eta_S) \\ &\le \log(2B \cdot \omega(S)).\end{aligned}$$

4. Counting ratio optimizers in a Teichmüller Disk

In this section, we show that our construction yields infinitely many ratio optimizers whose maximal cyclic subgroups in $\mathrm{Mod}(S)$ are pairwise non-conjugate. That is, we will exhibit infinitely many ratio optimizers $\phi_1, \phi_2, ...$ such that for each $i \ne j$, no power of ϕ_i is conjugate to a power of ϕ_j. Since each of our ratio optimizers is contained in the group generated by T_α, T_β for α, β a minimally intersecting filling pair, it will follow that the Teichmüller disk $\mathbb{D}(\alpha, \beta)$ is stabilized by infinitely many primitive, pairwise non-conjugate ratio optimizers. This will complete the proof of Theorem 1.1.

To begin, let $w_1, w_2, ...$ be an infinite collection of distinct cyclically reduced words in a, b, satisfying:

(1) $w_i \ne a^{\pm}, b^{\pm}$
(2) $|w_i| = |w_i|_s$.

Since each w_i is cyclically reduced and all words in the collection are distinct, the words are pairwise non-conjugate. Furthermore, by property (1), each word w_i has nonzero translation length on the tree $T_{\alpha,\beta}$, and therefore by Corollary 3.2, each corresponds to a pseudo-Anosov mapping class under the map sending the free group generated by a, b to the subgroup generated by the Dehn twists T_α^B, T_β^B.

We will refer to the pseudo-Anosov image of w_i by $\mathcal{O}(w_i)$. By Proposition 3.1, $\mathcal{O}(w_j)$ admits a uniformly quasigeodesic axis A_j. By property (2) above, we may pass to a subsequence such that for each $k \geq 0$ the initial subword of w_i of length k is eventually constant as $i \to \infty$. Translating this fact to the tree T and possibly passing to a further subsequence, we have the following property: the axis of w_i in T shares a segment centered around the origin of length at least i with the axis for w_{i-1}. Thus, there exists a bi-infinite quasigeodesic $\mathcal{R}$ in $\mathcal{C}(S)$ and a point $x \in \mathcal{R}$ so that $\mathcal{O}(w_i)$ admits an axis that shares a segment of length i with $\mathcal{R}$, centered about x. Furthermore, as a consequence of Corollary 3.2 these axes do not fellow travel in $\mathcal{C}(S)$.

Now, let $\Gamma_1 \subset \{\mathcal{O}(w_2), \mathcal{O}(w_3), ...\}$ denote the set of words whose maximal cyclic subgroups are conjugate in $\mathrm{Mod}(S)$ to the maximal cyclic subgroup determined by $\mathcal{O}(w_1)$. By hyperbolicity of $\mathcal{C}(S)$, there is a constant $K > 0$ so that for $\mathcal{O}(w_i) \in \Gamma_1$, there is a conjugator $c_i \in \mathrm{Mod}(S)$ such that the quasigeodesic $c_i \cdot A_i$ K-fellow travels with A_1. Let l denote the stable translation length of $\mathcal{O}(w_1)$.

It follows that there exists a uniform constant r depending only on l, the hyperbolicity constant for $\mathcal{C}(S)$, and the quasigeodesic constants determined in Proposition 3.1, so that for any two points t, s on $c_i \cdot A_i$, there exists a power of $\mathcal{O}(w_1)$ sending t within r of s. Hence the same is true for any two points on A_i, after replacing $\mathcal{O}(w_1)$ with its conjugate by c_i^{-1}.

We first show that $|\Gamma_1| < \infty$. Assume by contradiction that Γ_1 is infinite. Then there exists $\mathcal{O}(w_i) \in \Gamma_1$ with i arbitrarily large. Choose such an $i \gg 1$, and let y, z denote the endpoints of the segment of $\mathcal{R}$ that A_i shares. Note that by construction $y, z \in A_j$ for all $j \geq i$.

Then for any $j > i$, $c_i^{-1}c_j$ sends A_j to A_i, and post-composing this with some power e_j of $c_i^{-1}\mathcal{O}(w_1)c_i$ sends each of y and z within r of themselves. By acylindricity of the action of $\mathrm{Mod}(S)$ on $\mathcal{C}(S)$ [Bow08], there are at most finitely many mapping classes with this property. Hence,

$$\left\{(c_i^{-1}\mathcal{O}(w_1)^{e_j}c_i)c_i^{-1}c_j\right\}_{j>i} := \left\{c_j'\right\}_{j>i}$$

is a finite collection of mapping classes. We note that $c_i^{-1}c_j \neq c_i^{-1}c_k$ for any $j \neq k$, as the axes A_j and A_k do not fellow travel, and the inverse of $c_i^{-1}c_j$ sends the endpoints at infinity of A_i to those of A_j. Moreover, c_j' is obtained from $c_i^{-1}c_j$ by post-composition with a map that fixes the endpoints of A_i, and therefore $c_j' \neq c_k'$ for $k \neq j$. Thus $|\Gamma_1| < \infty$.

It follows that we may pass to a subsequence so that no corresponding pseudo-Anosov determines the same maximal cyclic subgroup up to conjugacy as $\mathcal{O}(w_1)$. Now we simply iterate this argument. By the exact same logic, the set Γ_i of all maps in our collection which determine the same (up to conjugacy) maximal cyclic subgroup as $\mathcal{O}(w_i)$ is finite, and thus we can pass to a further subsequence all of whose terms determine pseudo-Anosov mapping classes which are distinct, up to conjugacy and powers, from those already obtained.

5. Ratio optimizers in the Johnson filtration and point pushing subgroups

Fix $S = S_{g,p}$ with $g \geq 2$ and $p \in \{0,1\}$ and let α and β be separating curves of S which fill and intersect minimally, i.e. $i(\alpha,\beta) = i^{sep}_{g,p}$. Recall that by Lemma 2.2, there is a constant $C \geq 0$, independent of S, such that $i(\alpha_g,\beta_g) \leq C \cdot \omega(g,p)$. Let $a = T^B(\alpha)$ and $b = T^B(\beta)$.

Theorem 5.1. *There is a constant $C_J \geq 0$ satisfying the following. Let $S = S_{g,0}$ or $S_{g,1}$ with $g \geq 2$ and denote by $J_k(S)$ the kth term of the Johnson filtration of* $\mathrm{Mod}(S)$*. Then there exist $f_k \in J_k(S)$ with*

$$\tau(f_k) = \frac{\ell_{\mathcal{T}}(f_k)}{\ell_{\mathcal{C}}(f_k)} \leq C_J \log \omega(S).$$

In other words, there are ratio optimizers arbitrarily deep into the Johnson filtration.

Proof. Set $w_1 = aba$ and $w_2 = bab$ for a and b as defined at the beginning of this section. Set

$$f_k = [\ldots[[w_1,w_2],w_1]\ldots w_*]$$

which is k iterated commutators alternating between w_1 and w_2. Note that by construction $|f_k| = |f_k|_s$, i.e. each syllable of f_k has length 1. Since by definition $a, b \in J_1$ the same is true for w_1, w_2. Further, as the Johnson filtration $\{J_k\}$ is a central series, see [BL94] and [Mor91], we have $f_k \in J_k$.

By Corollary 3.2, $\ell_{\mathcal{C}}(f_k) \geq \frac{1}{3}|f_k|_s = \frac{1}{3}|f_k|$. Moreover, using Lemma 3.3 we can directly compute an upper bound for the dilatation, which (up to a uniform constant) is a product of $|f_k|$ with $\log(\omega(S))$. This completes the proof. □

We now construct ratio optimizers in the point-pushing subgroup $PP_g < \mathrm{Mod}(S_{g,1})$ of the mapping class group of a once-punctured surface. To achieve this, it suffices to construct a pair (α,β) of curves on $S_{g,1}$ which (1) fill the surface, (2) have geometric intersection number at most some fixed polynomial function of g, and (3) such that α and β are isotopic after forgetting the puncture. Assuming the existence of such a pair, note that the pseudo-Anosov $T^B_\alpha T^{-B}_\beta$ lies in PP_g, and by Theorem 3.4, it will be a ratio optimizer. From this, we will obtain:

Theorem 5.2. *There exists a uniform constant $C_P \geq 0$ satisfying the following. Let $S = S_{g,1}$ with $g \geq 2$ and let $PP_g \leq$* $\mathrm{Mod}(S)$ *be the points pushing subgroup of its mapping class group. Then there is $\phi \in PP_g$ with*

$$\tau(\phi) = \frac{\ell_{\mathcal{T}}(\phi)}{\ell_{\mathcal{C}}(\phi)} \leq C_P \log \omega(S).$$

To construct the desired filling pair, begin with a filling pair (ρ,δ) of non-separating curves on a closed surface $S_{g,0}$ with $i(\rho,\delta)$ bounded above by some fixed linear function of g. For example, (ρ,δ) could be a minimally intersecting filling pair on $S_{g,0}$.

Let δ_1, δ_2 be two parallel copies of δ, and puncture the surface $S_{g,0}$ on the interior of the annulus bounded by δ_1 and δ_2 to form the surface $S = S_{g,1}$. Note that $f_\delta := T^3_{\delta_1} \circ T^{-3}_{\delta_2}$ is a point-pushing map in $\mathrm{Mod}(S_{g,1})$. We claim that ρ fills with $f_\delta(\rho)$, and that $i(\rho, f_\delta(\rho))$ is bounded above by a quadratic function of g.

We first show that these two curves jointly fill $S_{g,1}$; that is we must show that if γ is any essential simple closed curve, γ must intersect either ρ or $f_\delta(\rho)$. We use the following inequality as seen in [Iva92]:

Lemma 5.3. *Let $c_1, ..., c_m$ be a collection of pairwise disjoint, pairwise non-homotopic simple closed curves on a surface S with negative Euler characteristic, let $\boldsymbol{S} := (s_1, ...s_m) \in \mathbb{Z}^m$, and let $T^{\mathbb{S}}$ denote the composition of Dehn twists $c_1^{s_1} \circ ... \circ c_m^{s_m}$. Then for any simple closed curves γ, ρ,*

$$\sum_{i=1}^{m}(|s_i - 2|)i(\rho, c_i)i(c_i, \gamma) - i(\rho, \gamma) \leq i(T^{\boldsymbol{S}}(\rho), \gamma)$$

$$\leq \sum_{i=1}^{m}|s_i|i(\rho, c_i)i(c_i, \gamma) + i(\rho, \gamma).$$

Now suppose $i(\gamma, \rho) = 0$. Then since ρ fills with δ on the original closed surface, it follows that $i(\delta_j, \gamma) \neq 0$ for $j = 1, 2$. Thus the left hand side of the inequality of Lemma 5.3 is non-zero, so γ must intersect $f_\delta(\rho)$.

The quadratic bound on $i(\rho, f_\delta(\rho))$ follows from the linear bound on $i(\rho, \delta)$, and another application of Lemma 5.3. This completes the proof of Theorem 5.2.

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