EFFECTIVE MASUR–MINSKY DISTANCE FORMULAS AND APPLICATIONS TO HYPERBOLIC 3-MANIFOLDS

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Abstract. The Masur–Minsky distance formula estimates word length in the mapping class group of a surface S up to additive and multiplicative errors bounded by some constant $C(S)$ which depends on the surface S. We obtain effective versions of this formula by estimating the constants $C(S)$ as a function of the absolute value of the Euler characteristic of S. In particular we prove lower and upper bounds for C which depend exponentially and factorially on $|\chi(S)|$, respectively.

Our main estimates have several applications to the geometry of surfaces and hyperbolic 3-manifolds. Among other results, we give effective versions of the following theorems: (1) Brock's theorem relating the volume of a closed hyperbolic 3–manifold fibering over the circle to the translation length of its monodromy acting on the pants graph; (2) the upper bound on the word length of a smallest conjugator for pseudo-Anosovs g, h in terms of the input word lengths $|g|$ and $|h|$; (3) that covering maps induce quasi-isometric embeddings of pants graphs of surfaces.

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1. Introduction

Let $S = S_{g,p}$ denote the orientable surface of genus g with p punctures, and define the *complexity* of S to be $\omega(S) = 3g+p-3$; henceforth we assume that $\omega(S) > 1$. Let $\text{Mod}(S_{g,p})$ denote the corresponding mapping class group, the group of orientation preserving homeomorphisms of S up to isotopy. The *curve complex* of S, denoted by

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$\mathcal{C}(S)$ is the simplicial complex whose vertices are isotopy classes of essential simple closed curves on S, and whose k-simplices correspond to $(k+1)$-component multicurves on S.

The curve complex $\mathcal{C}(S)$ is a locally infinite, infinite diameter δ-hyperbolic metric space whose coarse geometry is intimately related to the algebra of $\mathrm{Mod}(S)$ and to the geometry and topology of hyperbolic 3-manifolds ([31], [32], [34], [35]). Indeed, for all but finitely many surfaces, the automorphism group of $\mathcal{C}(S)$ is isomorphic to $\mathrm{Mod}^{\pm}(S)$—the so-called *extended mapping class group* consisting of orientation reversing and preserving isotopy classes of self-homeomorphisms ([26], [27], [28]), but due to the local infiniteness of $\mathcal{C}(S)$, this action of $\mathrm{Mod}^{\pm}(S)$ is not proper.

One of the most useful and important tools for circumventing this difficulty is the Masur–Minsky *distance formula*, which relates the distance between two mapping classes g, h in the word metric with respect to some fixed finite generating set, to the geometry of $\mathcal{C}(S)$ via *hierarchies of tight geodesics* [32]:

Theorem 1.1 ([32])**.** *Fix S. There exist constants C and K such that the following holds: Let μ_1, μ_2 be two complete clean markings on S. Then*

$$d_{\mathcal{M}}(\mu_1, \mu_2) \asymp_C |H(\mu_1, \mu_2)| \asymp_C \sum_{Y \subseteq S} [[d_Y(\mu_1, \mu_2)]]_K,$$

where $d_{\mathcal{M}}(\cdot, \cdot)$ denotes distance in the marking complex $\mathcal{M} = \mathcal{M}(S)$, $H(\mu_1, \mu_2)$ is any complete hierarchy with initial marking μ_1 and terminal marking μ_2, and the sum is taken over all essential, properly embedded subsurfaces $Y \subseteq S$.

Furthermore, if $d_Y(\mu_1, \mu_2) > K$, then Y is a domain of H, and the geodesic g of H supported on Y has length $|g|$ satisfying

$$|d_Y(\mu_1, \mu_2) - |g|| < K.$$

In Theorem 1.1, the symbol $\asymp_C$ denotes equality up to additive and multiplicative error bounded above and below by C:

$$Z \asymp_C W \Leftrightarrow \frac{1}{C} W - C \leq Z \leq C \cdot W + C.$$

Furthermore, $[[x]]_K := x$ if $x \geq K$ and 0 otherwise.

The main focus of this paper is to estimate C and K in Theorem 1.1 as functions of $\omega = \omega(S)$, and in so doing, to obtain effective versions of these coarse equalities with respect to the surface S. With these estimates at hand, we provide effective and explicit connections between the topology and geometry of surfaces and 3-manifolds. Such applications are our primary motivation and are discussed extensively below.

Our first result proves that C grows at least exponentially in ω and at most like ω^{ω}, and K grows linearly in ω. In what follows, an affine function of a variable x is simply a function of the form $x \mapsto C \cdot x + b$ for some constants C and b.

Theorem 1.2. *For each ω, let $C(\omega)$ and $K(\omega)$ be the minimal constants such that Theorem 1.1 holds for any surface S with $\omega(S) = \omega$. Then there exist non-constant affine functions $f, h : \mathbb{N} \to \mathbb{N}$ such that*

$$f(\omega) \leq K(\omega) \leq h(\omega),$$

and

$$2^{f(\omega)} \leq C(\omega) \leq h(\omega)^{h(\omega)}.$$

Note that Theorem 1.1 is a statement about 6 inequalities so C is the maximum over all multiplicative constants, their reciprocals, and any additive constants appearing in any of these inequalities.

The exponential growth of C is not necessary in all 6 of these inequalities; more effective control can be obtained by treating these inequalities as independent statements, summarized as follows.

Theorem 1.3. *There is an $R \geq 0$ so that if $\mu_1, \mu_2, H(\mu_1, \mu_2)$ are as in the statement of Theorem 1.1 and $h(\omega)$ is as in the statement of Theorem 1.2, then the following inequalities hold:*

(1) $d_{\mathcal{M}}(\mu_1, \mu_2) \leq 5 \cdot |H(\mu_1, \mu_2)|$ *(Lemma 3.1).*
(2) $|H(\mu_1, \mu_2)| \leq h(\omega)^{h(\omega)} \cdot d_{\mathcal{M}}(\mu_1, \mu_2) + h(\omega)^{h(\omega)}$ *(Propositions 3.5 and 4.1).*
(3) $\sum_{Y \subseteq S} [[d_Y(\mu_1, \mu_2)]]_{f(\omega)} \leq h(\omega)^{h(\omega)} \cdot |H(\mu_1, \mu_2)| + h(\omega)^{h(\omega)}$ *(Proposition 3.5).*
(4) $|H(\mu_1, \mu_2)| \leq h(\omega)^{h(\omega)} \cdot \sum_{Y \subseteq S} [[d_Y(\mu_1, \mu_2)]]_R + h(\omega)^{h(\omega)}$ *(Proposition 3.5).*
(5) $d_{\mathcal{M}}(\mu_1, \mu_2) \leq h(\omega)^{h(\omega)} \cdot \sum_{Y \subseteq S} [[d_Y(\mu_1, \mu_2)]]_{f(\omega)} + h(\omega)^{h(\omega)}$ *(Lemma 3.1, Proposition 3.5).*
(6) $\sum_{Y \subseteq S} [[d_Y(\mu_1, \mu_2)]]_R \leq f(\omega) \cdot d_{\mathcal{M}}(\mu_1, \mu_2)$ *(Proposition 4.1).*

For the pants graph $\mathcal{P}(S)$ there is an analogous version of Theorem 1.1, which relates the distance in the pants graph $d_{\mathcal{P}}(p_1, p_2)$ between two pants decompositions, the length of a *hierarchy without annuli* interpolating between p_1 and p_2, and the sum of all sufficiently large subsurface projections taken over all non-annular subsurfaces [32]. A version of Theorem 1.3 holds in this context, which we summarize below:

Theorem 1.4. *Let p_1, p_2 be pants decompositions on S and $H(p_1, p_2)$ a hierarchy without annuli with initial marking p_1 and terminal marking p_2. Then inequalities $(2) - (5)$ of Theorem 1.3 hold with $H(\mu_1, \mu_2)$ replaced with $H(p_1, p_2)$ and $d_{\mathcal{M}}(\cdot, \cdot)$ replaced with $d_{\mathcal{P}}(\cdot, \cdot)$. Furthermore, the following hold:*

$$(1') \;\; d_{\mathcal{P}}(p_1, p_2) \leq |H(p_1, p_2)|;$$

and

$$(6') \quad \sum_{Y \subseteq S} [[d_Y(\mu_1, \mu_2)]]_{3184} \leq 600 \cdot d_{\mathcal{P}}(p_1, p_2).$$

1.1. **The conjugacy problem for pseudo-Anosovs.** Masur–Minsky use the distance formula to prove the following important result [32, Theorem 7.2]:

Theorem 1.5. [32] *Fix S and a finite generating set $\mathcal{S}$ of $\mathrm{Mod}(S)$, there exists a constant $q = q(\mathcal{S})$ satisfying the following: Let $\phi, \theta \in \mathrm{Mod}(S)$ be conjugate pseudo-Anosovs. Then there exists a mapping class r conjugating ϕ to θ whose word length $|r|$ satisfies*

$$|r| < q \cdot (|\phi| + |\theta|).$$

As q necessarily depends on the choice of generating set, one can not hope to directly estimate q as a function of $\omega(S)$ without first having specified a choice of generating set for each mapping class group. After making such choices, we obtain the following effective and explicit estimate for q:

Theorem 1.6. *There is an affine function h and there are finite generating sets $\mathcal{S}(g,p)$ of $\mathrm{Mod}(S_{g,p})$ such that*

$$q(\mathcal{S}(g,p)) \leq h(\omega)^{h(\omega)}.$$

1.2. **Brock's convex core and mapping torus theorems.** Let $X, Y \in \mathcal{T}(S_{g,p})$, the Teichmüller space of $S_{g,p}$. Bers' simultaneous uniformization theorem [5] demonstrates that there exists a unique quasi-fuchsian 3-manifold $Q(X,Y)$ such that the two marked Riemann surfaces at infinity are X and Y. Recall that the *Bers' constant* $B(g,p)$ is a number such that any hyperbolic surface topologically equivalent to $S_{g,p}$ admits at least one pants decomposition with total length at most B.

Brock's convex core theorem [11] relates the volume of the convex core $\mathrm{core}(Q)$ of $Q(X,Y)$, to the distance in the pants graph $\mathcal{P}$ between a pair of pants decompositions P_X, P_Y, one of which is short on X, and the other short on Y:

Theorem 1.7. [11] *There exists a constant $\eta = \eta(\omega(g,p))$ such that the following holds: Let $X, Y \in \mathcal{T}(S_{g,p})$ and let P_X, P_Y be pants decompositions such that P_X (resp. P_Y) has length at most $B(g,p)$ on X (resp. on Y). Then*

$$d_{\mathcal{P}}(P_X, P_Y) \asymp_\eta \mathrm{vol}(\mathrm{core}(Q(X,Y))).$$

Using Theorem 1.4, we obtain an effective version of Theorem 1.7:

Theorem 1.8. *Let X, Y, P_X, P_Y be as in the statement of Theorem 1.7. Then*

$$\frac{1}{h(\omega)^{h(\omega)}} d_{\mathcal{P}}(P_X, P_Y) - h(\omega)^{h(\omega)} \leq \mathrm{vol}(\mathrm{core}(Q(X,Y))).$$

Furthermore, when S is closed,

$$\mathrm{vol}(\mathrm{core}(Q(X,Y))) \leq h(\omega)^3 \cdot (d_{\mathcal{P}}(P_X, P_Y) + 1).$$

Remark 1.9. Futer–Purcell–Schleimer have also worked out an effective version of Theorem 1.7 using different methods.

Brock's mapping torus theorem [12] relates the volume of a hyperbolic mapping torus to the translation length of its monodromy on $\mathcal{P}(S)$. Define the *stable translation length* of $\psi \in \mathrm{Mod}(S)$ on $\mathcal{P}(S)$ to be

$$\overline{\tau}_{\mathcal{P}}(\psi) := \lim_{n\to\infty} \frac{d_{\mathcal{P}}(P, \psi^n(P))}{n},$$

for any pants decomposition $P \in \mathcal{P}$. Using Theorem 1.8, we obtain the following effective version of Brock's result:

Theorem 1.10. *Let $\psi \in \mathrm{Mod}(S)$ be pseudo-Anosov, and let M_ψ denote the mapping torus, $M_\psi := S \times [0,1]/((x,1) \sim (\psi(x), 0))$, equipped with its unique finite volume complete hyperbolic metric. Then*

$$\frac{1}{h(\omega)^{h(\omega)}}\overline{\tau}_{\mathcal{P}}(\psi) - h(\omega)^{h(\omega)} \leq \mathrm{vol}(M_\psi).$$

Remark 1.11. The inequality which bounds volume from above by translation length has already been made effective by Agol [1], who showed that the multiplicative and additive error can be taken to be uniform in ω.

1.3. **Covering maps and Pants graphs.** Let $p : S \to S'$ be a covering map between closed orientable surfaces. The covering induces a coarse map $p^* : \mathcal{C}(S') \to \mathcal{C}(S)$ between curve complexes, by sending a curve γ on S' to a component of its pre-image under p. We recall the following theorem of Rafi and Schleimer [37]:

Theorem 1.12. [37] *There exists a constant $C = C(S, S')$ such that p^* is a C-quasi-isometry.*

Tang later gave a short proof of Theorem 1.12 using hyperbolic geometry [40], and a result relating the geometry of a quasi-fuchsian manifold to the curve complex ([9], [13]). Building on Tang's ideas and using Theorem 1.8, we give an effective proof of an analogous theorem for the pants graph.

Given a cover $p : S \to S'$, fix a hyperbolic metric σ on S'; then σ lifts to a metric $\tilde{\sigma}$ on S. A pants decomposition on S' will lift to a multi-curve on S and we complete this multicurve to a shortest possible pants decomposition with respect to $\tilde{\sigma}$ to obtain a map $p^* : \mathcal{P}(S') \to \mathcal{P}(S)$. We prove:

Theorem 1.13. *Let $p : S \to S'$ be a covering map between closed orientable surfaces. Then there exists an affine function $t(\omega)$ such that p^* is an $\omega^{t(\omega)}$-quasi-isometry.*

1.4. **Connections to other work.** We conclude the introduction by highlighting applications related to the Weil–Petersson metric on Teichmüller space, and to algorithms for determining the Nielsen–Thurston type of braids.

Recall the following result of Schlenker [39]:

Theorem 1.14. [39] *Let S be a closed orientable surface of genus at least 2, and let $x, y \in \mathcal{T}(S)$. Then there is a constant $K_S > 0$ such that*

$$\mathrm{vol}(\mathrm{core}(Q(X,Y))) \leq 3\sqrt{\pi(g-1)}\, d_{WP}(X,Y) + K_S,$$

where d_{WP} denote distance in the Weil-Petersson metric.

Combining Theorem 1.14 with Theorem 1.8, we obtain an effective version of Brock's quasi-isometry [11] between the pants graph and $\mathcal{T}(S)$ equipped with the Weil-Petersson metric:

Theorem 1.15. *Let S be a closed surface of genus at least 2, $X, Y \in \mathcal{T}(S)$, and let $P_X, P_Y \in \mathcal{P}(S)$ be shortest pants decompositions on X and Y, respectively. Then there exists an affine function $r(\omega)$ such that*

$$d_{\mathcal{P}}(P_X, P_Y) \asymp_{r(\omega)^{r(\omega)}} d_{WP}(X, Y).$$

That is, the map $\mathcal{Q} : \mathcal{T}(S) \to \mathcal{P}(S)$ sending a marked Riemann surface to its shortest pants decomposition is a quasi-isometry, with constants bounded above by $(r(\omega))!$.

Proof. The inequality $d_{\mathcal{P}}(P_X, P_Y) \prec_{r(\omega)^{r(\omega)}} d_{WP}(X, Y)$ is obtained by simply combining Theorems 1.8 with 1.14. That is, we use the volume of $\operatorname{core}(Q(X, Y))$ as an intermediary. We also use a bound on the constant K_S which follows from Schlenker's proof [39] and from work of Bridgeman (see Proposition 2 of [10]); K_S is bounded above by some affine function of $\omega(S)$. As for the opposite inequality, Brock's original argument [11] yields a bound on the Lipschitz constants for $\mathcal{Q}$ which grows at most quadratically in ω. $\square$

Finally, consider the following existence theorem of Calvez [16]:

Theorem 1.16. [16] *Let $B(n)$ denote the Braid group on n strands. Given $b \in B(n)$, there exists an algorithm which determines the Nielsen–Thurston type of b, which terminates in quadratic time (with respect to word length $|b|$).*

Calvez's algorithm uses the Garside structure of the Braid group, and we note that the above theorem is strictly an existence result: the algorithm is technically not well-defined, since it relies on Theorem 1.5 of Masur-Minsky [32] and as a result, its run-time is a function of the multiplicative constant q. Thus, Theorem 1.6 yields the well-definedness of Calvez's algorithm.

1.5. **Techniques and tools.** The upper bound on marking distance in terms of subsurface projections is proved by analyzing Masur and Minsky's original argument ([32]), keeping care of the various constants and analyzing their dependencies on the complexity $\omega(g, p)$. We are able to obtain more control on these constants using the following effective results:

1. Uniform hyperbolicity of curve graphs, as shown independently by the first author [2], Bowditch [7], Clay-Rafi-Schleimer [19], and the third author with Hensel and Przytycki [25];
2. The combinatorial proof of the bounded geodesic image theorem obtained by the third author [43], which when coupled with (1), yields uniform control on the bounded geodesic image theorem.

In addition to this, several completely new arguments are needed, especially for the lower bound on marking distance. For instance, the techniques used to prove inequality (6′) in Theorem 1.4—that pants distance can be bounded below by subsurface projections with a uniform cut-off and uniform multiplicative error—do not appear anywhere in the literature.

The lower bound of Theorem 1.2 is obtained by constructing a long hierarchy, all of whose geodesics are relatively short. This produces examples where the constants involved are particularly bad and demonstrates the complexity dependence inherent in the Masur–Minsky hierarchy machinery.

2. Preliminaries

2.1. **Coarse geometry.** Let X and Y be metric spaces. A map $\phi : X \to 2^Y$ is called *coarsely well-defined* if there exists a constant $K \geq 0$ so that for each $x \in X$ the diameter of $\phi(x)$ in Y is bounded above by K. In this case, we think of ϕ as a map from X to Y; this can be formalized by replacing ϕ with a map ϕ' that for each $x \in X$, chooses one point arbitrarily in $\phi(x)$. Henceforth, when we refer to ϕ as a map between metric spaces, we allow for the possibility that ϕ is a coarsely well-defined map from one metric space to the power set of the other.

Given $\lambda \geq 1, k \geq 0$, a map $\phi : X \to Y$ is a (λ, k)-*quasi-isometric embedding* if for all $a, b \in X$,

$$\frac{1}{\lambda} d_Y(\phi(a), \phi(b)) - k \leq d_X(a, b) \leq \lambda d_Y(\phi(a), \phi(b)) + k.$$

The above inequalities can be more succinctly expressed by the notation

$$d_X(a, b) \asymp_{\lambda,k} d_Y(\phi(a), \phi(b)).$$

That is, to express the inequality $A \leq \lambda \cdot B + k$, we write $A \leq_{\lambda,k} B$, and similarly for $A \geq_{\lambda,k} B$.

A *quasi-isometry* between metric spaces X and Y is a quasi-isometric embedding $\phi : X \to Y$ satisfying the additional property that there exists a constant $R \geq 0$ so that for each $y \in Y$, there exists $x \in X$ with $d_Y(\phi(x), y) \leq R$. Given $a, b \in \mathbb{R} \cup \{\pm\infty\}$, a (λ, k)–*parameterized quasi geodesic* in X is a (λ, k) quasi-isometric embedding of an interval $[a, b] \subseteq \mathbb{R}$ (equipped with the standard metric) into X. A *geodesic* in X is by definition a $(1, 0)$-quasi-geodesic, i.e. an isometric embedding of $[a, b] \subset \mathbb{R}$ into X.

Finally, recall that the metric space X is *geodesic* if any points $x, y \in X$ can be joined by a geodesic in X. The metric space X is called δ-*hyperbolic* (or Gromov hyperbolic, or just hyperbolic) if whenever $T_{x,y,z}$ is a geodesic triangle with vertices x, y, z in X, then any edge of $T_{x,y,z}$ is contained in the δ-neighborhood of the union of the other two edges [23, 24].

2.2. **Simple closed curves.** A *simple closed curve* on a surface S is the image of an embedding of S^1 into S. A simple closed curve is *essential* if it is not homotopically trivial and not homotopic into a regular neighborhood of a puncture. A *multi-curve* is a disjoint union of pairwise non-homotopic essential simple closed curves. A *pants decomposition* is a multi-curve on S with the maximum possible number of components. Note that $\omega(g, p) = 3g+p-3 > 1$ exactly means that $S_{g,p}$ admits a multi-curve with at least two components. As two simple closed curves are homotopic if and only if they are isotopic, we will sometimes blur the distinction between isotopy and homotopy.

Given a pair of simple closed curves α, β the *geometric intersection number* $i(\alpha, \beta)$ is the minimal number of intersection points between any simple closed curve homotopic to α and any simple closed curve homotopic to β:

$$i(\alpha, \beta) := \min_{x \sim \alpha, y \sim \beta} |x \cap y|,$$

where $\sim$ denotes homotopy. Curves α and β are said to be in *minimal position* if the number of their points of intersection is equal to their geometric intersection number. A pair of simple closed curves α, β are in minimal position if and only if no complementary region of their union is a *bigon* ([20, Proposition 1.7]), a simply-connected region bounded by one arc of α and one arc of β.

2.3. The mapping class group Mod(S) and the curve and pants complexes.

The *mapping class group* $\mathrm{Mod}(S)$ of a surface S is the group of isotopy classes of orientation preserving homeomorphisms of S:

$$\mathrm{Mod}(S) \cong \mathrm{Homeo}^+(S)/\sim,$$

where $\sim$ denotes isotopy between homeomorphisms. Any mapping class sends punctures to punctures, but is allowed to permute the set of punctures.

Curve complexes. Suppose $\omega(g, p) = 3g + p - 3 > 1$. Then the *curve complex* of $S_{g,p}$, denoted by $\mathcal{C}(S_{g,p})$, is the simplicial complex whose vertices correspond to isotopy classes of essential simple closed curves on S, and $k + 1$ such vertices span a k-simplex precisely when there is a multi-curve in S whose components represent these vertices. The *curve graph*, denoted by $\mathcal{C}_1(S)$, is the 1-skeleton of $\mathcal{C}(S)$: vertices are isotopy classes of essential simple closed curves, and two such vertices span an edge if and only if the corresponding curves have geometric intersection number 0. The curve complex (and curve graph) is made into a metric space by identifying each k-simplex with the standard Euclidean k-simplex with unit length edges.

One plus the dimension of $\mathcal{C}(S)$ is equal to the number of components in any pants decomposition of S. Moreover, $\mathcal{C}(S)$ is a *flag complex*, meaning that $k + 1$ vertices span a k-simplex if and only if there exists the 1-skeleton of a k-simplex spanned by those vertices in the curve graph.

The mapping class group $\mathrm{Mod}(S)$ acts by isometries on $\mathcal{C}(S)$, by extending the natural action on the vertex set simplicially to the higher dimensional simplices. For all but finitely many surfaces, the automorphism group of $\mathcal{C}(S)$ is isomorphic to the *extended mapping class group* $\mathrm{Mod}^{\pm}(S)$, the super-group of $\mathrm{Mod}(S)$ consisting of isotopy classes of homeomorphisms of S (orientation reversing or preserving) ([26], [27], [28]).

The *arc and curve graph* of S, denoted by $\mathcal{AC}(S)$, is the graph where each vertex either corresponds to the isotopy class of an essential simple closed curve, or the isotopy class of a properly embedded arc on S that is not homotopic into a puncture. Again, edges correspond to pairs of vertices that can be realized disjointly on S.

We will also require a version of the curve complex for several surfaces S with $\omega(S) \leq 1$. If S is the once-punctured torus, we define $\mathcal{C}(S)$ to be the graph whose vertices correspond to isotopy classes of essential simple closed curves, and two such vertices span an edge precisely when the corresponding curves have geometric intersection number 1. When S is the 4-punctured sphere $S_{0,4}$, we define $\mathcal{C}(S)$ to be the graph where adjacency corresponds to pairs of isotopy classes with geometric intersection number 2. When $\omega(S) = 1$, we have that $\mathcal{C}(S)$ is isomorphic to the Farey graph, and we call adjacent vertices *Farey neighbors*. Finally, when S is an annulus, consider a hyperbolic isometry $\phi \in \mathrm{Isom}^+\mathbb{H}^2$ and the hyperbolic cylinder C obtained by quotienting $\mathbb{H}^2$ by the action of the infinite cyclic subgroup $\langle\phi\rangle$. The cylinder C is not compact, but it admits a Gromov compactification $\bar{C}$. Then we define the *annular curve graph* of S to be the graph whose vertices are geodesic simple arcs in $\bar{C}$ with an endpoint on each boundary component, and edges correspond to pairs of arcs with disjoint interiors. Letting η denote the core curve of S, we will sometimes refer to $\mathcal{C}(S)$ as $\mathcal{C}(\eta)$.

We record for later use Equation 2.3 of [32], which implies that distance in the annular complex is coarsely the same as intersection number (counting only interior intersections):

$$d_A(\alpha, \beta) = 1 + i(\alpha, \beta) \tag{2.1}$$

Masur and Minsky showed [31] that for each S, there exists $\delta = \delta(S)$ so that the curve complex is $\delta(S)$-hyperbolic. Bowditch later reproved this result ([8]), and his proof gave logarithmic upper bounds on the growth of $\delta(S_{g,p})$ as a function of g and p. Recently, the first author ([2]), Bowditch ([7]), Clay–Rafi–Schleimer ([19]), and the third author together with Hensel and Przytycki ([25]), all independently proved that the curve graph $\mathcal{C}_1(S)$ is *uniformly hyperbolic*, meaning that there exists a single δ so that all curve graphs are δ-hyperbolic. We record this in the following theorem:

Theorem 2.1 ([2], [7], [19], [25])**.** *Curve graphs are uniformly hyperbolic. Furthermore, Hensel–Przytycki–Webb prove that given a geodesic triangle T in any curve graph, there is a point which is a distance of at most* 17 *from from all points on T.*

The pants graph. Let $\omega(S) > 1$. The *pants graph*, denoted by $\mathcal{P}(S)$, of a surface S is the graph whose vertices correspond to (isotopy classes of) pants decompositions of S, and whose edges correspond to pairs of pants decompositions (P, P') so that P is obtained from P' via a *pants move*: delete one curve c in P', and replace it with a curve that intersects c minimally over all possible choices of replacement. The mapping class group $\mathrm{Mod}(S)$ also acts by isometries on $\mathcal{P}(S)$, and Margalit has shown that the full automorphism group of $\mathcal{P}(S)$ is $\mathrm{Mod}^{\pm}(S)$ ([30]) for almost all surfaces.

2.4. **Subsurface projections and the marking complex.** A *non-annular subsurface* of S is a component of the complement of a regular neighborhood of a multicurve in S. An *annular subsurface* is a closed regular neighborhood of an essential simple closed curve. An *essential subsurface* Y is one such that each boundary component is essential.

Subsurface projection. Let $Y \subseteq S$ be a non-annular essential proper subsurface whose interior is not homeomorphic to a 3-times punctured sphere. Note that since Y is essential the homomorphism induced by inclusion $\pi_1(Y) \to \pi_1(S)$ is injective. Let S^Y denote the covering space of S associated to the induced subgroup $\pi_1(Y) \leq \pi_1(S)$. The cover S^Y is not compact, but it admits a compactification $\overline{S^Y}$ (take the Gromov compactification minus the punctures of the homeomorphic lift of Y in S^Y) which is homeomorphic to Y. Then we define $\mathcal{AC}(Y)$, the arc and curve graph of the subsurface Y, to be equal to $\mathcal{AC}(\overline{S^Y})$. When Y is annular we define $\mathcal{AC}(Y) = \mathcal{C}(Y)$.

There is a map $\pi_Y \colon \mathcal{AC}(S) \to 2^{\mathcal{AC}(Y)}$ defined as follows: $\pi_Y(\alpha)$ is the set of essential curves and arcs in the preimage of α under the covering map $S^Y \to S$. This is a coarsely well-defined map because when $\pi_Y(\alpha)$ is non-empty it consists of disjoint arcs and/or curves, so its diameter in $\mathcal{AC}(Y)$ is at most 1. We say that α *cuts* Y if any simple closed curve representing α intersects Y. Note that $\pi_Y(\alpha)$ is non-empty if and only if α cuts Y.

Suppose that α cuts Y. In the case that Y is not an annulus, there is a simple closed curve $\psi_Y(\alpha)$ closely related to $\pi_Y(\alpha)$. We define $\psi_Y(\alpha)$ by first defining $\sigma_Y \colon 2^{\mathcal{AC}(Y)} \to 2^{\mathcal{C}(Y)}$. If a set A of arcs and/or curves contains a curve, we set $\sigma_Y(A)$ to be that curve. Otherwise, A consists only of arcs. Choose one such arc $\lambda \in A$, and let $\mathcal{N}(\lambda)$ denote a thickening of λ together with the at most two boundary components of Y that λ intersects. Then we define $\sigma_Y(A)$ to be a boundary component of $\mathcal{N}(\lambda)$ that is essential in Y. We define $\psi_Y := \sigma_Y \circ \pi_Y$. When Y is annular, we define $\psi_Y := \pi_Y$ since $\mathcal{AC}(Y) = \mathcal{C}(Y)$. Note that ψ_Y is again a coarsely well-defined assignment called the *subsurface projection.*

The marking graph. Let $P = \{p_1, p_2, ..., p_n\}$ be a multicurve on S. A *marking with base* P is a set $\mu = \{\beta_1, ..., \beta_n\}$ such that for each i, β_i is either equal to p_i, or equal to an ordered pair (p_i, t_i) for t_i a diameter-1 subset of the annular curve graph $\mathcal{C}(p_i)$. Such a t_i is called a *transversal* for p_i. A marking μ is called *complete* if P is a pants decomposition of S, and for each i we have $\beta_i = (p_i, t_i)$ i.e. each component of P has an associated transversal. A complete marking μ is called *clean* if each transversal t_i is equal to $\psi_{p_i}(w_i)$, where the only base curve that w_i intersects is p_i, and w_i and p_i are Farey neighbors in the curve graph of the appropriate component of $S - (P \backslash \{p_i\})$.

The *marking graph* $\mathcal{M}(S)$ of S, introduced by Masur and Minsky in [32] is the graph whose vertices are clean complete markings μ of S, and whose edges correspond to pairs (μ, μ') where μ' is obtained from μ by one of the following two *elementary moves* (see [32, Section 2] for details):

(1) twist move: μ and μ' have the exact same base, and for exactly one i, the transverse curve w'_i is obtained from w_i by a Dehn twist or a half-twist once about the base curve p_i (a half-twist is possible when p_i and w_i intersect twice);
(2) flip move: μ' is obtained from μ by exchanging one transverse curve with the corresponding base curve. That is, for one i, p_i becomes the transversal and w_i becomes the base curve. Since μ is clean, w_i will not intersect any of the other components of P and thus after this exchange the base is still a pants decomposition. However, it is possible that w_i intersects other transversals,

and that therefore the resulting complete marking is not clean. If this occurs, we *clean* the marking by replacing each of the other transversals with new ones so that the resulting marking is clean, and so that the replacement transversals minimize distance (amongst all choices of replacements that yield a clean marking) in the respective annular curve graphs to the original transversals. We say that the resulting clean marking μ' is *compatible* with the complete marking obtained by exchanging w_i and p_i.

There is a bounded number of choices for a clean marking that is compatible with a complete marking, and there is always at least one such clean marking (see section 2 of [32]). Since there are only finitely many complete clean markings up to the action of $\mathrm{Mod}(S)$, there is a uniform bound on the distance within all annular curve graphs of the replacement transversals in any cleaning. Therefore, $\mathcal{M}(S)$ is a connected and locally finite graph, admitting a properly discontinuous, co-finite simplicial action of $\mathrm{Mod}(S)$. It follows that $\mathcal{M}(S)$ is quasi-isometric to $\mathrm{Mod}(S)$ equipped with the word metric associated to a finite generating set. See [32] for details.

We remark that subsurface projection to an essential subsurface Y can be extended over a complete marking μ as follows. If Y is an annulus whose core curve is in the base of μ, then we define $\psi_Y(\mu) \in \mathcal{C}(Y)$ to be the transversal associated to that base curve. In all other instances, we define $\psi_Y(\mu)$ to be the standard subsurface projection of the base of μ to Y. Then we have the following formula, due to Masur–Minsky, which expresses distance in $\mathcal{M}(S)$, coarsely in terms of subsurface projections ([32]):

Theorem 2.2 ([32])**.** *There exists a constant $D = D(S)$ so that for all $T > D$, there is $N = N(T)$ satisfying the following. If $\mu_1, \mu_2 \in \mathcal{M}(S)$ are clean complete markings, then*

$$d_{\mathcal{M}}(\mu_1, \mu_2) \asymp_N \sum_{Y \subseteq S} [[d_Y(\mu_1, \mu_2)]]_T,$$

where $d_{\mathcal{M}}(\cdot,\cdot)$ denotes distance in $\mathcal{M}(S)$, the sum is taken over all essential subsurfaces Y of S, and $[[x]]_T = x$ for all $x \geq T$ and 0 otherwise.

As stated in the introduction, the main goal is this paper is to determine how the constants of Theorem 2.2 depend on the surface S.

2.5. **Hierarchies.** A *hierarchy* between two clean complete markings μ, μ' is a path in $\mathcal{M}(S)$ starting at μ and ending at μ', which can be built up as a collection of certain specially chosen geodesics in curve graphs of subsurfaces. We begin by defining these geodesic paths – so-called *tight geodesics*– which are the fundamental building blocks of a hierarchy. For $Y \subseteq S$ a subsurface of a surface S potentially with boundary, the *relative boundary* of Y consists of all boundary curves of Y which are not peripheral in S.

Definition 2.3. For any S with $\omega(S) > 1$, a *tight sequence* in $\mathcal{C}(S)$ is a sequence $\{\sigma_0, ..., \sigma_n\}$ of simplices in $\mathcal{C}(S)$ (perhaps of different dimension) so that:

(1) σ_0 and σ_n are vertices of $\mathcal{C}(S)$.

(2) For all $i \neq j$ and any vertices $v_i \in \sigma_i$ and $v_j \in \sigma_j$,
$$d_{\mathcal{C}(S)}(v_i, v_j) = |i - j|.$$
(3) For each $i \neq 0, n$, the simplex σ_i corresponds to the relative boundary of the subsurface $F(v_i, v_j)$, obtained by taking a regular neighborhood of $\sigma_i \cup \sigma_j$ after filling in each simply-connected component of $S \setminus (\sigma_i \cup \sigma_j)$ with a disk.

If S is a once-punctured torus or 4-holed sphere ($\omega(S) = 1$), then a tight geodesic is simply the vertex sequence of a curve graph geodesic. Finally, if S is an annulus, then a tight geodesic is again the vertex sequence of an annular curve graph geodesic, however we also require that if $x \in \partial\overline{S}$ is an endpoint of some arc in a tight geodesic $\{v_0, ..., v_n\}$, then x is an endpoint of either v_0 or v_n.

Definition 2.4. Given $Y \subseteq S$ any subsurface, a *tight geodesic* in $\mathcal{C}(Y)$ is a tight sequence $g = \{\sigma_0, ..., \sigma_n\}$ in $\mathcal{C}(Y)$, together with a pair of markings $\mathbf{I}(g), \mathbf{T}(g)$, called the *initial and terminal markings of* g, such that σ_0 (resp. σ_n) is contained in the base of $\mathbf{I}(g)$ (resp. $\mathbf{T}(g)$). The surface Y is called the *domain of* g, which we denote by $D(g) = Y$.

We now define the relation of *subordinacy* between tight geodesics, which dictates how tight geodesics in different subsurfaces can be amalgamated into the hierarchy. For $Y \subseteq S$, $\omega(Y) \geq 1$, and μ a marking on S, the *restriction of* μ *to* Y, denoted by $\mu|_Y$, is the set of all base curves of μ (together with their transversals if they exist) that intersect Y essentially. If Y is an annulus, we define $\mu|_Y$ to be equal to $\psi_Y(\mu)$. Note that the restriction of a marking to a domain can be equal to the empty set.

Definition 2.5. Given g a tight geodesic in $\mathcal{C}(Y)$, $Y \subseteq S$, a domain Z is called a *component domain* of g if there exists some simplex σ_j in g such that either:
(1) Z is a connected component of $Y \setminus \sigma_j$, or
(2) Z is an annulus whose core curve is a vertex in σ_j.

If Z satisfies this definition with respect to the vertex σ_j, we say that Z is a component domain of (Y, σ_j) or of $(D(g), \sigma_j)$.

If Z is a component domain of $(D(g), \sigma_j)$ we define the *initial and terminal markings of* g *relative to* Z, denoted by $\mathbf{I}(Z, g)$ and $\mathbf{T}(Z, g)$ respectively, as follows. If σ_j is not the first vertex of g (i.e. if $j \neq 0$), then define $\mathbf{I}(Z, g) := \sigma_{j-1}|_Z$; otherwise, define $\mathbf{I}(Z, g) := \mathbf{I}(g)|_Z$. Similarly, if σ_j is not the last vertex of g, define $\mathbf{T}(Z, g) := \sigma_{j+1}|_Z$, and otherwise define $\mathbf{T}(Z, g) := \mathbf{T}(g)|_Z$.

Definition 2.6. If Z is a component domain of g and $\mathbf{T}(Z, g) \neq \emptyset$, we say that Z is *directly forward subordinate* to g, or *d.f.s.* to g for short, and we write $Z \overset{d}{\searrow} g$. Analogously, if Z is a component domain of g and $\mathbf{I}(Z, g) \neq \emptyset$, we say that Z is *directly backwards subordinate* to g, or *d.f.b.* to g, and we write $g \overset{d}{\swarrow} Z$. If g and h are two tight geodesics, then we say that h is *directly forward subordinate* (resp. *directly backwards subordinate*) to g if $D(h)$ is d.f.s. (resp. d.f.b.) to g and $\mathbf{T}(h) = \mathbf{T}(D(h), g)$ (resp. if $\mathbf{I}(h) = \mathbf{I}(D(h), g)$). We define *forward subordinacy* (resp. *backwards subordinacy*) for domains and for tight geodesics to be the transitive closure of direct forward subordinacy (resp. direct backwards subordinacy).

We are now ready to define hierarchies.

Definition 2.7. A *hierarchy* H between markings μ, μ' on S is a collection of tight geodesics satisfying:

(1) There exists a unique geodesic $g_H \in H$, called the *main geodesic*, whose domain is the surface S. The initial and terminal markings of g_H are equal to μ, μ' respectively, and are denoted by $\mathbf{I}(H), \mathbf{T}(H)$.
(2) If g, h are tight geodesics in H and Z is a domain with $g \overset{d}{\swarrow} Z$ and $Z \overset{d}{\searrow} h$, then there is a unique tight geodesic $k \in H$ with $D(k) = Z$, and $g \overset{d}{\swarrow} k$ and $k \overset{d}{\searrow} h$
(3) Except for the main geodesic g_H, every geodesic $h \in H$ has the property that there exist tight geodesics $g, k \in H$ so that h is d.b.s. to g and h d.f.s. to k.

The *length* of a hierarchy H, denoted by $|H|$, is equal to the sum of all of the lengths of the tight sequences of tight geodesics contained in H.

Masur–Minsky prove that between any two markings μ and μ', there exists at least one and at most finitely many hierarchies. Furthermore, they provide a way of "resolving" a hierarchy H into a sequence of adjacent markings in $\mathcal{M}(S)$, coarsely of length $|H|$, between μ and μ'. For these facts, the reader is referred to [32]. Finally, in conjunction with Theorem 2.2, they prove that hierarchy paths constitute parameterized quasi-geodesics in the marking graph:

Theorem 2.8 ([32])**.** *There exist constants $K = K(S), R = R(S)$ so that if H is a hierarchy path between complete clean markings $\mu, \mu' \in \mathcal{M}(S)$, then* (1) *$H$ constitutes an $(R(S), R(S))$-quasigeodesic in the marking graph, and* (2) *for any $Y \subseteq S$, if $d_Y(\mu, \mu') > K$, then Y is a domain of some tight geodesic in H.*

3. Upper bounds on marking distance: Making effective the Masur–Minsky argument

In this section, we prove the upper bounds of Theorem 1.2. To keep the exposition succinct, we rely on the original arguments of Masur and Minsky [32] when possible and indicate where new arguments are necessary.

3.1. **Hierarchy bounds.** We begin by showing that one can turn a hierarchy into a sequence of (clean) markings compatible with a resolution of the hierarchy, where the distance between consecutive markings is independent of complexity. As opposed to the other statements in this subsection, Lemma 3.1 requires a bit of care. This is because the processing of transforming a hierarchy resolution into a path of markings requires careful control over the transversal curves for each annulus. Our proof does so by keeping track of what we call *twist* and *insurance opportunities* for each annulus; these record the number of elementary moves between markings that potentially alter the transversal for a given annulus.

Lemma 3.1 (Effectivizing Lemma 5.5)**.** *If H is a complete finite hierarchy such that $\boldsymbol{I}(H)$ and $\boldsymbol{T}(H)$ are clean, then there is a sequence of clean markings $(\mu_j)_{j=0}^{M}$,* *successive ones differing by elementary moves, so that* $\mu_0 = \boldsymbol{I}(H), \mu_M = \boldsymbol{T}(H)$, *and* $M \leq 5|H|$.

Proof. In a pants decomposition P we say that a pair of curves of P are adjacent if there is an arc connecting them whose interior is disjoint from P. Take any resolution $(\tau_j)_{j=0}^N$. We have $N \leq |H|$ by [32, Proposition 5.4]. Also recall that to each τ_j there is associated a non-clean marking μ_{τ_j}.

Whenever a forward elementary move τ_j, τ_{j+1} occurs in an annular domain Y then we call this a *twist opportunity* for Y.

Suppose τ_j, τ_{j+1} is a forward elementary move determined by (h, v) and (h, v') where $D(h)$ is homeomorphic to $S_{0,4}$ or $S_{1,1}$. Furthermore suppose that c is an adjacent curve of v in base(μ_{τ_j}). Then we call this an *insurance opportunity* for Y where Y is the annulus whose core is c.

There are at most $4N \leq 4|H|$ twist and insurance opportunities in total, summed over all annuli.

Now we inductively construct a path of clean, complete markings $(\mu_i)_i$ from $\mathbf{I}(H)$ to $\mathbf{T}(H)$: Set $i = 0$, $j = 0$ and $\mu_i = \mathbf{I}(H)$.

Inductive Step: Note that base(μ_{τ_j}) = base(μ_i). We consider the forward elementary move τ_j, τ_{j+1}. If the move happens in a domain which is not annular, $S_{1,1}$ or $S_{0,4}$ then do nothing, set $j = j + 1$ then repeat the Inductive Step. If the move is in an annular domain Y then we note the twist opportunity for Y, set $j = j + 1$ then repeat the Inductive Step. Finally, if the move is determined by (h, v) and (h, v') with $D(h) = S_{1,1}$ or $S_{0,4}$ then we note the insurance opportunities for the adjacent annuli of v (at most 4 of them) and then we extend our path of clean, complete markings $(\mu_k)_k$ as follows:

We have already defined μ_i. Write t_v for the clean transverse curve of v in μ_i. Suppose $t_v = v'$ then we set μ_{i+1} to be the result after flipping v in μ_i to v' (and replacing other transversals so that they are clean), then set $i = i + 1$, $j = j + 1$ and repeat the Inductive Step.

On the other hand, suppose that $t_v \neq v'$. Set t to be the noted number of twist opportunities of v and s to be the noted number of insurance opportunities of v.

We claim that in at most $s + t$ full- and half-twists about v, we can twist t_v in μ_i to v'. Assuming this claim, we then define μ_{i+1}, μ_{i+2},...,μ_{i+r}, where $r \leq s + t$ when we perform these twists, define μ_{i+r+1} to be the result after performing the Flip move that interchanges v and v'. Then we update the value of i to $i + r + 1$ and update the value of j to $j + 1$ then repeat the Inductive Step.

To prove the claim, suppose that we require at least $s+t+1$ such twists. First note that t_v and v' only differ by twists and half-twists as they are each Farey neighbors of v in curve graph of $D(h)$. Let q be the geodesic in H with domain v. Then the minimum distance between $\mathbf{I}(q)$ and $\mathbf{T}(q)$ is at most t. (Note that $(q, u) \in \tau_i$ where u is the last vertex of q.) Factoring into account the insurance opportunities of v, we have that the minium distance between $\pi_v(t_v)$ and $\pi_v(v')$ is at most $s + t$. This is because of the triangle inequality (the Replacement procedure during a Flip move provides a path in the annular complex).

On the other hand, plugging in $|n| \geq s+t+1$ into [32, Equation (2.6)] shows that the distance between $\pi_v(t_v)$ and $\pi_v(v')$ is exactly $s+t+3$, a contradiction in the case of $S_{1,1}$.

In the case of $S_{0,4}$ the argument is slightly more complicated by the fact that there are several lifts of t_v and v'. Nonetheless, if they differ by more than n half-twists then one can show that the minimum distance is at least $2+\lfloor (|n|-1)/2 \rfloor$. This is similar to [32, Equation (2.7)]. Plugging in $|n| \geq 2s+2t+1$ shows that the minimum distance between $\pi_v(t_v)$ and $\pi_v(v')$ is at least $2+s+t$, a contradiction. The claim is proved.

Now we repeat the Inductive Step as previously mentioned.

Finally, the last defined μ_i only differs from $\mathbf{T}(H)$ by full- and half-twists along its base curves. By the same claim as above, the number of such twists is bounded by the number of twist and insurance opportunities. This completes the proof. □

We now move on to making effective Section 6 of [32]. Our main tool here is the uniform bounds on the bounded geodesic image theorem, due to the third author. We state this here for easy reference:

Theorem 3.2 (Uniform bounded geodesic image theorem [43].). *There is a constant $M \geq 0$ such that for any surface S with $\omega(S) \geq 1$ and any subsurface $Y \subset S$, if g is a geodesic in $\mathcal{C}(S)$ all of whose vertices cut Y, then* $\mathrm{diam}_Y(g) \leq M$.

Using arguments similar to those found there and also in [25], it can be shown that the bounded geodesic image theorem constant M in Theorem 3.2 is at most 100.

We begin by showing that the constant in the Sigma projection lemma can be chosen to grow linearly in $\omega(S) - \omega(Y)$. In Section 8, we show that this cannot be improved.

Lemma 3.3 (Effectivizing Lemma 6.1: Sigma Projection). *For any hierarchy H and any domain Y in S,*

$$\mathrm{diam}_Y(\pi_Y(\sigma^+(Y,H))) \leq (\xi(S) - \xi(Y))M,$$

where M is the bounded geodesic image theorem constant of Theorem 3.2. Further, if Y is properly contained in the top domain of $\Sigma(Y)$, then

$$\mathrm{diam}_Y(\pi_Y(\sigma(Y,H))) \leq 2(\xi(S) - \xi(Y))M + M.$$

Lemma 3.4 (Effectivizing Lemma 6.2: Large link). *If Y is any domain in S and*

$$\mathrm{diam}_Y(\pi_Y(\sigma(Y,H))) > 2(\xi(S) - \xi(Y))M + M,$$

then Y is the support of a geodesic h in H. Further, for any geodesic $h \in H$ with $Y = D(h)$,

$$||h| - d_Y(\mathbf{I}(H), \mathbf{T}(H))| \leq 2(\xi(S) - \xi(Y))M.$$

As these lemmas follow immediately from Masur-Minsky's original set-up, we refer the reader to [32] for proofs.

3.2. **Counting in hierarchies.** The large link lemma (Lemma 3.4) is one of the main ingredients for proving the coarse equality between the sum of subsurface projections and the total length of a hierarchy. Indeed, the coarse equality between $|H|$ and sums of projections follows from it combined with a lemma (Lemma 9.6 of [33]) stating the following: summing together quantities that are each individually coarsely equal to the lengths of the geodesics in the hierarchy yields a result that is coarsely equal to $|H|$.

In this subsection, we give an effective version of this lemma. Concretely, let $f : \mathbb{N} \to \mathbb{N}$ be arbitrary. Let g be a tight geodesic whose domain $D(g) \subset S$ appears in a complete hierarchy path, and let $\xi(g)$ be equal to the complexity $\omega(D(g))$ of $D(g)$.

Proposition 3.5. *Let ϕ be an $\mathbb{N}$-valued function on the set of geodesics of H satisfying*

$$|g| - f(\xi(g)) \leq \phi(g) \leq |g| + f(\xi(g)),$$

for every geodesic g of a complete hierarchy H. Then

$$\sum_{h \in H} |h| \leq_{A,B} \sum_{h \in H} \phi(h),$$

for

$$A, B \leq \sum_{i=1}^{\omega(S)} \prod_{j=i}^{\omega(S)} [f(j) + 4].$$

Proof. We follow the outline of Minsky's original argument, the main strategy being induction on $\omega(S)$; when possible we will use the same notation as in the original argument. The base case $\omega(S) = 0$ is immediate.

When $\omega(S) = m > 0$, given $g \in H$ we define

$$\beta(g) = \sum_{\text{h f.s. to g or h= g}} |h|;$$

$$\beta'(g) = \sum_{\text{h f.s. to g or h= g}} \phi(h).$$

By the "structure of sigma" theorem of Masur-Minsky [32], if f is f.s. to g, there is a unique geodesic h such that f is f.s. to h and such that h is d.f.s. to g. Therefore,

$$\beta(g) = |g| + \sum_{\text{h d.f.s. to g}} \beta(h),$$

and similarly

$$\beta'(g) = \phi(g) + \sum_{\text{h d.f.s. to g}} \beta'(h).$$

Remark 3.6. Note that by equation (9.19) of [33], the above summations each have at most $|g| + 4$ summands.

Then by the induction hypothesis, there exists $\psi_A(m-1), \psi_B(m-1)$ satisfying the inequality of Proposition 3.5 such that

$$\beta(g) \leq |g| + \sum_{\text{h d.f.s. to g}} \psi_A(m-1)\beta'(h) + \psi_B(m-1).$$

Using Remark 3.6, this is in turn at most

$$(1+\psi_B(m-1))|g| + 4\psi_B(m-1) + \psi_A(m-1)\left[\sum_{\text{h d.f.s. to g}} \beta'(h)\right]$$

Note also that $\phi(g)$ satisfies

$$|g| - f(\xi(g)) \leq \phi(g) \leq |g| + f(\xi(g)), \tag{3.1}$$

and therefore the above is at most

$$\leq [f(\xi(g)) + \phi(g)] \cdot [1+\psi_B(m-1)] + 4\psi_B(m-1) + \psi_A(m-1)\left[\sum_{\text{h d.f.s. to g}} \beta'(h)\right]$$

$$\leq (1+\psi_B(m-1))\phi(g) + \psi_A(m-1)\left[\sum_{\text{h d.f.s. to g}} \beta'(h)\right] + f(\xi(g))(1+\psi_B(m-1)) + 4\psi_B(m-1)$$

$$\leq \max[1+\psi_B(m-1), \psi_A(m-1)] \cdot \beta'(g) + (f(\xi(g))+4)(1+\psi_B(m-1)).$$

Therefore, we set

$$\psi_A(m) := \max[1+\psi_B(m-1), \psi_A(m-1)], \psi_B(m) := (f(m)+4)(1+\psi_B(m-1)).$$

Since $\psi_A(1) = 1$,

$$\psi_A(m) = 1 + \psi_B(m-1) < \psi_B(m).$$

By the induction hypothesis,

$$\psi_B(m) \leq (f(m)+4)\left(1 + \sum_{i=1}^{m-1}\prod_{j=i}^{m-1}[f(j)+4]\right)$$

$$= f(m)+4+[(f(1)+4)\cdot(f(2)+4)\cdot \ldots \cdot(f(m-1)+4)\cdot(f(m)+4)]+\ldots+[f(m)+3)(f(m)+4)]$$

$$= \sum_{i=1}^{m}\prod_{j=i}^{m}[f(j)+4].$$

This completes the proof of the inequality

$$\sum_{h\in H}|h| <_{A,B} \sum_{h\in H}\phi(h). \qquad \square$$

In particular, define $f(n) := 2M(\omega(S) - n + 1)$, where $1 \leq M \leq 100$ is as in Theorem 3.2. Then we have shown (Lemma 3.4) that if $H(\mu_1, \mu_2)$ is a complete hierarchy between markings μ_1, μ_2, and g is a tight geodesic of H,

$$|d_{D(g)}(\mu_1, \mu_2) - |g|| \leq f(\xi(g)).$$

That is, the length of a tight geodesic is within $f(\xi(g))$ of the distance between the projections of the initial and terminal markings to the domain of that tight geodesic. Furthemore, if $Y \subset S$ is such that the distance between the projections to Y of the initial and terminal markings is at least $f(\omega(Y))$, then Y must appear in H. Thus Proposition 3.5 can be used to bound the multiplicative and additive errors A, B in the coarse equality

$$|H(\mu_1, \mu_2)| \leq_{A,B} \sum_{Y \subseteq S} [[d_Y(\mu_1, \mu_2)]]_{f(\omega(Y))}. \tag{3.2}$$

Indeed,

$$A, B \leq \sum_{i=1}^{\omega(S)} \left[\prod_{j=i}^{\omega(S)} (2M(\omega(S) - j + 1)) + 4 \right] < (4M\omega(S))! \tag{3.3}$$

Note that the cut-off $f(\omega(Y))$ is large when the complexity of Y is small. Indeed, f grows linearly in the *co-complexity* of Y, which we define to be $2M(\omega(S) - \omega(Y) + 1)$.

We also note that the inequality

$$|H(\mu_1, \mu_2)| \geq_{A',B'} \sum_{Y \subseteq S} [[d_Y(\mu_1, \mu_2)]]_{f(\omega(Y))}$$

holds for A', B' bounded above by a linear function of $\omega(S)$. This follows from the fact that the length of a hierarchy bounds the distance in the marking graph between μ_1 and μ_2 from above, up to a uniform multiplicative error by Lemma 3.1, and from the fact that marking graph distance is bounded from below by sums of projections, up to a multiplicative error that grows at most linearly in the complexity of S (see Proposition 4.1 below).

3.3. **Upper bounds on marking distance.** Lemma 3.1 proves that marking graph distance $d_M(\mu_1, \mu_2)$ is bounded above by twice the length of any hierarchy H with $\mathbf{I}(H) = \mu_1$ and $\mathbf{T}(H) = \mu_2$. Furthermore, (3.2) and (3.3) imply that the length of such a hierarchy is bounded above by sums of projections (with cut-offs that grow linearly in the co-complexity of the subsurface), up to errors bounded by $(4M\omega(S))!$. Thus, marking graph distance is bounded above by sums of projections, up to errors bounded by $2 \cdot (4M\omega(S))!$.

4. Lower bounds on marking distance

4.1. **Linear lower bounds.** We give an effective proof of the lower bound in the Masur-Minsky distance formula. Although the proof is new, it is largely inspired by [32].

To simplify the exposition, set $L = 4$ and $B = 10$. Recall that for overlapping subsurfaces X and Y of S we have the following two facts:

(1) $\pi_Y : \mathcal{M}(S) \to \mathcal{C}(Y)$ is coarsely 4- Lipschitz ([32]).
(2) For any $\mu \in \mathcal{M}(S)$,

$$\min\{d_X(\mu, \partial Y), d_Y(\mu, \partial X)\} \leq B. \tag{4.1}$$

Equation 4.1 is known as the Behrstock inequality [4]. That one can take $B = 10$ is a consequence of Leininger's effective argument, found in work of Mangahas [29].

For $\mu, \nu \in \mathcal{M}(S)$, let $\Omega(\mu, \nu, K)$ be the set of subsurfaces Y with $d_Y(\mu, \nu) \geq K$.

Proposition 4.1. *For $K = 5B + 3L$ and $s = 2\omega(S)$,*

$$\sum_Y [d_Y(\mu, \nu)]_K \leq 5sL \cdot d_M(\mu, \nu).$$

Proof. Take $K = 5B+3L$ and fix $\mu, \nu \in \mathcal{M}(S)$ along with a geodesic $\mu = \mu_0, \mu_1, \ldots, \mu_N = \nu$. For each $Y \in \Omega(\mu, \nu, K)$ choose $i_Y, t_Y \in \{0, \ldots, N\}$ as follows: i_Y is the largest index k with $d_Y(\mu_0, \mu_k) \leq 2B + L$ and t_Y is the smallest index k greater than i_Y with $d_Y(\mu_k, \mu_N) \leq 2B + L$. Write $I_Y = [i_Y, t_Y] \subset \{0, 1, \ldots, N\}$ and note that this subinterval is well-defined and that since the projection of adjacent vertices in the geodesic have d_Y less than or equal to L, $d_Y(\mu_0, \mu_k), d_Y(\mu_k, \mu_N) \geq 2B + 1$ for all $k \in I_Y$ and $d_Y(\mu_{i_Y}, \mu_{t_Y}) \geq B + L$.

Claim 1. If $Y, Z \in \Omega(\mu, \nu, K)$ with $Y \pitchfork Z$ then $I_Y \cap I_Z = \emptyset$.

Proof. Toward a contradiction, take $k \in I_Y \cap I_Z$. Since Y and Z overlap we have either $d_Y(Z, \mu_0) \leq B$ or $d_Z(Y, \mu_0) \leq B$. Assume the former; the latter case is proven by exchanging the occurrences of Y and Z in the proof. By the triangle inequality,

$$\begin{aligned} d_Y(Z, \mu_k) &\geq d_Y(\mu_0, \mu_k) - d_Y(Z, \mu_0) \\ &\geq 2B + 1 - B \geq B + 1. \end{aligned}$$

So by condition B, we have $d_Z(Y, \mu_k) \leq B$ and

$$\begin{aligned} d_Z(Y, \mu_N) &\geq d_Z(\mu_k, \mu_N) - d_Z(Y, \mu_k) \\ &\geq 2B + 1 - B \geq B + 1. \end{aligned}$$

and we conclude, again by condition B, that $d_Y(Z, \mu_N) \leq B$. This, together with our initial assumption, implies

$$d_Y(\mu_0, \mu_N) \leq d_Y(\mu_0, Z) + d_Y(Z, \mu_N) \leq 2B < K$$

contradicting that $Y \in \Omega(\mu, \nu, K)$. □

Returning to the proof of the proposition, we have a covering $\{I_Y : Y \in \Omega(\mu, \nu, K)\}$ of $\{0, 1, \ldots, N\}$. By the claim above and our assumption on the number of pair-wise non-overlapping codomains, each $k \in \{0, 1, \ldots, N\}$ is contained in at most s intervals of the covering. Hence,

$$\sum_{Y \in \Omega(\mu, \nu, K)} |t_Y - i_Y| \leq s \cdot d_M(\mu, \nu).$$

Finally, using the Lipschitz condition on the projections,

$$\begin{aligned} d_Y(\mu,\nu) &\leq d_Y(\mu_{i_Y},\mu_{t_Y}) + 4B + 2L \\ &\leq L|t_Y - i_Y| + 4B + 2L \end{aligned}$$

Since, for each $Y \in \Omega(\mu,\nu,K)$, $d_Y(\mu,\nu) \geq 5B + 3L$ we have $\frac{1}{5L} \cdot d_Y(\mu,\nu) \leq |t_Y - i_Y|$ and so putting this with the inequality above

$$\sum_{Y \in \Omega(\mu,\nu,K)} d_Y(\mu,\nu) \leq 5sL \cdot d_M(\mu,\nu)$$

as required. □

In summary, we have shown that for $\mu,\nu \in \mathcal{M}(S)$

$$\sum_Y [d_Y(\mu,\nu)]_{62} \leq 40\omega(S) \cdot d_M(\mu,\nu).$$

4.2. **Uniform lower bounds for the pants graph.** In this section, given $Y \subseteq S$, let σ_Y denote the "surgery map" which takes in a simplex in $\mathcal{AC}(Y)$ and outputs a vertex of $\mathcal{C}(Y)$ by surgering arcs along ∂Y, as described in the preliminaries. Thus, $\psi_Y = \sigma_Y \circ \pi_Y$. Let δ be the slim triangles uniform constant for curve graphs (Theorem 2.1). In this subsection, we denote distance in $\mathcal{C}(Y)$ by $d_{\mathcal{C}(Y)}$ and distance in $\mathcal{AC}(Y)$ by $d_{\mathcal{AC}(Y)}$.

The main result of this section is that distance in the pants graph can be bounded below in terms of subsurface projections with a uniform multiplicative constant, completely independent of $\omega(S)$. We note that in the argument given to prove Proposition 4.1 significant progress is made simultaneously in a number of subsurfaces that nest in one another. This number can be some linear function in $\omega(S)$, and this yields multiplicative constants that a priori grow linearly in ω. To overcome this problem, a more careful tactic is employed here, which seems to be new. It involves an analysis of where and when progress is being made by the pants path in chains of nested subsurfaces of S.

We assume throughout this section that all subsurfaces are non-annular.

The strategy for proving Theorem 4.8 is as follows. We are given a path $P(0),\dots,P(n)$ in the pants graph of a surface $S_{g,p}$ with $3g+p-3>1$. We wish to bound n from below. To each non-annular subsurface Y we associate a geodesic G_Y in $\mathcal{C}(Y)$ between the images of $P(0)$ and $P(n)$ under ψ_Y. We consider the $P(i)$ under the map ψ_Y followed by nearest point projection to the geodesic G_Y—the composition of these two maps is called Π_Y.

The projection Π_Y has three key properties. The first key property is that these projections are coarsely Lipschitz with some uniform constant. The second key property is that if $\Pi_Y P(i)$ is far from both $\Pi_Y P(0)$ and $\Pi_Y P(n)$ and Z contains the subsurface Y, then $\Pi_Z P(i)$ is close to ∂Y in $\mathcal{C}(Z)$. In other words, while progress is being made by the coarse path $\Pi_Y P(i)$ in $\mathcal{C}(Y)$, $\Pi_Z P(i)$ cannot move in $\mathcal{C}(Z)$ as they are "stuck" next to ∂Y; this is the main aspect of the argument that addresses the issue of progress being made simultaneously in nested subsurfaces. The third key property is similar to the Behrstock inequality (4.1), and states that if $\Pi_Z P(i)$ is far from both $\Pi_Z P(0)$ and $\Pi_Z P(n)$ whenever $Z \in \{Y, Y'\}$, then Y and Y' cannot overlap. This allows us to form chains of nested subsurfaces and understand when and where progress is being made.

To each edge $P(i), P(i+1)$ in the pants path we associate at most two non-annular subsurfaces—our way of doing this is described in Section 4.2.3. Lemma 4.7 states that each non-annular subsurface Y has sufficiently many edges associated to it in terms of $d_{\mathcal{C}(Y)}(\psi_Y P(0), \psi_Y P(n))$, and from this we immediately deduce a lower bound on n and we get Theorem 4.8. We now give the details.

For each Y, pick a geodesic G_Y in $\mathcal{C}(Y)$ from a vertex of $\psi_Y P(0)$ to a vertex of $\psi_Y P(n)$ such that G_Y is shortest possible over all choices of vertices of $\psi_Y P(0)$ and $\psi_Y P(n)$.

Fix near_{G_Y} to be a function that accepts a finite subset $F \subset \mathcal{C}_0(Y)$ and returns a vertex of G_Y of minimum distance from F in $\mathcal{C}(Y)$, i.e. near_{G_Y} is a nearest point projection onto G_Y.

Set $\Pi_Y P(i) = \mathrm{near}_{G_Y} \psi_Y P(i)$. Since G_Y is shortest possible we can set $\Pi_Y P(0)$ and $\Pi_Y P(n)$ equal to the endpoints of G_Y.

4.2.1. *Statements of the three key lemmas.* In this subsection we state three key lemmas for the structure of the projections Π_Y for non-annular subsurfaces Y.

Lemma 4.2 (bounded jumps)**.** *There exists a uniform constant J such that for any non-annular subsurface Y*

$$d_{\mathcal{C}(Y)}(\Pi_Y P(i), \Pi_Y P(i+1)) \leq J.$$

We think of J as the "jump" constant.

We write

$$d_Y(i) = d_{\mathcal{C}(Y)}(\Pi_Y P(i), \{\Pi_Y P(0), \Pi_Y P(n)\}).$$

Lemma 4.3 (awake downstairs implies stationary upstairs)**.** *There exists a uniform constant A such that whenever Y and Z are non-annular subsurfaces with Y nested in Z and*

$$d_Y(i) \geq A$$

then

$$\mathrm{diam}_{\mathcal{C}(Z)}(\Pi_Z P(i) \cup \partial Y) \leq 3.$$

We say that $P(i)$ is *awake in* Y if $d_Y(i) \geq A$. We call A the "awake" constant. To avoid too many names for constants, we choose A large enough so that the following lemma also holds.

Lemma 4.4. *There exists a uniform constant A such that if*

$$d_Z(i) \geq A$$

whenever $Z \in \{Y, Y'\}$ where Y and Y' are non-annular, then Y and Y' do not overlap i.e. either they miss or one is nested in the other.

4.2.2. *Proofs of the three key lemmas.* In this subsection we prove Lemmas 4.2, 4.3 and 4.4. The proofs are standard.

The proof of Lemma 4.2 (bounded jumps) follows readily from the fact that ψ_Y and near_{G_Y} are coarsely Lipschitz with uniform constants. This uses uniform hyperbolicity of the curve graphs.

Proof of Lemma 4.2. Suppose that the pants move between $P(i)$ and $P(i+1)$ occurs in a subsurface X where $\omega(X) = 1$. Consider the image of $P(i)$ and $P(i+1)$ under π_Y. Now $P(i)$ and $P(i+1)$ coincide outside X, so we may assume that X and Y cut and that $X \neq Y$. Furthermore, Y is non-annular so Y is not nested in X, and so X and Y overlap. By considering the essential and non-peripheral curves of ∂X we have that

$$\text{diam}_{\mathcal{AC}(Y)}(\pi_Y P(i) \cup \pi_Y P(i+1)) \leq 2.$$

Therefore

$$\text{diam}_{\mathcal{C}(Y)}(\psi_Y P(i) \cup \psi_Y P(i+1)) \leq 6.$$

In a δ-hyperbolic, geodesic metric space (X, d), if two vertices p, q satisfy $d(p,q) \leq k$, then their nearest point projections p', q' to some geodesic Q satisfy $d(p', q') \leq k + 8\delta + 2$ (if p' and q' are at least $6\delta + 2$ far apart then consider two points on Q of distance $2\delta + 1$ from either p' or q', then these are 2δ away from a geodesic between p and q).

Thus, it follows that we may take $J = 8\delta + 8$, where $\delta \leq 17$. □

The proof of Lemma 4.3 makes use of the Bounded Geodesic Image Theorem (Theorem 3.2). We argue towards a contradiction: if $\Pi_Z P(i)$ is far from ∂Y then we can construct a concatenation of two geodesics that connect $\psi_Z P(i)$ with either $\psi_Z P(0)$ or $\psi_Z P(n)$, such that each vertex of the concatenation cuts Y. By the Bounded Geodesic Image Theorem, this proves that $\psi_Y P(i)$ is close to either $\psi_Y P(0)$ or $\psi_Y P(n)$. By the definition of near_{G_Y}, we have that $\Pi_Y P(i)$ is close to either $\Pi_Y P(0)$ or $\Pi_Y P(n)$, a contradiction. The details are as follows.

Proof of Lemma 4.3. First, we claim that for any pants decomposition P we have that

$$\text{diam}_{\mathcal{C}(Y)}(\psi_Y \psi_Z P \cup \psi_Y P) \leq 2. \tag{4.2}$$

Indeed, $\pi_Y \sigma_Z \pi_Z P$ misses $\pi_Y \pi_Z P = \pi_Y P$ and hence any curve of $\sigma_Y \pi_Y \sigma_Z \pi_Z P$ intersects any curve of $\sigma_Y \pi_Y P$ at most twice and we get the desired upper bound.

If $A \geq M + 5$ then $\text{length}(G_Y) \geq M + 5$ or equivalently $d_{\mathcal{C}(Y)}(\psi_Y P(0), \psi_Y P(n)) \geq M + 5$. By inequality (4.2), $d_{\mathcal{C}(Y)}(\psi_Y \psi_Z P(0), \psi_Y \psi_Z P(n)) \geq M + 1$. By the Bounded Geodesic Image Theorem (Theorem 3.2), if each vertex of G_Z cuts Y then $\text{diam}_{\mathcal{C}(Y)}(\psi_Y \Pi_Z P(0) \cup \psi_Y \Pi_Z P(n)) \leq M$ and it immediately follows that

$$d_{\mathcal{C}(Y)}(\psi_Y \psi_Z P(0), \psi_Y \psi_Z P(n)) \leq M.$$

This is a contradiction and we conclude that some vertex v of G_Z misses Y.

By considering ∂Y we see that the set of vertices of G_Z that miss Y has diameter at most 2 and contains the vertex v. Hence, if we assume that $\text{diam}_{\mathcal{C}(Z)}(\Pi_Z P(i) \cup \partial Y) \geq 3$ then there exists a geodesic g that connects $\Pi_Z P(i)$ to $\Pi_Z P(0)$ (or to $\Pi_Z P(n)$—from now on without loss of generality it connects to $\Pi_Z P(0)$) with the property that each vertex of g cuts Y.

Write g' for a shortest geodesic that connects $\Pi_Z P(i)$ to some vertex of $\psi_Z P(i)$. We consider the concatenation of g and g'. We may use the Bounded Geodesic Image Theorem, unless g' contains some vertex that misses Y. So suppose that some vertex v' of g' misses Y. We have that $d_{\mathcal{C}(Z)}(v, v') \leq 2$, so by definition of near_{G_Z} we have that $d_{\mathcal{C}(Z)}(v', \Pi_Z P(i)) \leq 2$, and this immediately gives $\text{diam}_{\mathcal{C}(Z)}(\Pi_Z P(i) \cup \partial Y) \leq 3$.

Therefore if we assume that $\text{diam}_{\mathcal{C}(Z)}(\Pi_Z P(i) \cup \partial Y) \geq 4$ and that $d_Y(i) \geq M + 5$ then we may use the Bounded Geodesic Image Theorem with g and g' as defined above to deduce that $d_{\mathcal{C}(Y)}(\psi_Y \psi_Z P(i), \psi_Y \psi_Z P(0)) \leq 2M$. By inequality (4.2), we have that $d_{\mathcal{C}(Y)}(\psi_Y P(i), \psi_Y P(0)) \leq 2M + 4$. It then follows that $d_{\mathcal{C}(Y)}(\Pi_Y P(0), \Pi_Y P(i)) \leq 4M + 14$. Therefore we may take $A = 4M + 15$. ☐

For nonannular subsurfaces, the constants for the Behrstock inequality (Equation 4.1) are a bit smaller. Indeed, for an overlapping pair of non-annular subsurfaces Y and Y', a version of Behrstock's lemma states that if C is a multicurve that cuts both Y and Y', and $\text{diam}_{C(Y)}(\psi_Y C \cup \psi_Y \partial Y') \geq 9$ then $\text{diam}_{\mathcal{C}(Y')}(\psi_{Y'} C \cup \psi_{Y'} \partial Y) \leq 4$ The proof, due to Chris Leininger, is as follows: we must have that $\text{diam}_{\mathcal{AC}(Y)}(\pi_Y C \cup \pi_Y \partial Y') \geq 5$, in particular there is a vertex x of $\pi_Y C$ and a vertex y of $\pi_Y \partial Y'$ such that $i(x, y) \geq 3$. This implies that there is an arc a of $\pi_{Y'} C$ which misses $\pi_{Y'} \partial Y$ and the conclusion follows [29]. We are interested in the case where C is a pants decomposition.

Proof of Lemma 4.4. Take $A = 33$. Suppose that Y and Y' overlap. We argue towards a contradiction. Set $Y'' = Y$ so that if $Z = Y'$ we have $Z' = Y'' = Y$. We claim that whenever $Z \in \{Y, Y'\}$ at most one $j \in \{0, i, n\}$ satisfies $\text{diam}_{\mathcal{C}(Z)}(\psi_Z P(j) \cup \psi_Z \partial Z') \leq 8$. Indeed, for if there is some $Z \in \{Y, Y'\}$ and two such elements j_1 and j_2, then we have $\text{diam}_{\mathcal{C}(Z)}(\psi_Z P(j_1) \cup \psi_Z P(j_2)) \leq 16$, implying that $d_Z(i) \leq 32$, a contradiction.

This implies that there exists $j' \in \{0, i, n\}$ such that whenever $Z \in \{Y, Y'\}$ we have that $\text{diam}_{\mathcal{C}(Z)}(\psi_Z P(j') \cup \psi_Z \partial Z') \geq 9$. This contradicts the statement of the Behrstock inequality given above where $C = P(j')$. ☐

4.2.3. *Assigning edges to subsurfaces.* We say that $P(i)$ is *active, non-tame, dominating, overpowering in* Y if $d_Y(i) \geq A + 6$, $A + 6 + 2J$, $A + 12 + 2J$, $A + 18 + 2J$ respectively. We say that $P(i)$ is *tame* in Y if it is not non-tame in Y.

We say that $P(i)$ and $P(i+1)$ are *moving in* Y if $\pi_Y P(i) \neq \pi_Y P(i+1)$.

Lemma 4.5. *Fix i. There are at most two minimal (with respect to nesting), non-annular subsurfaces Y such that $P(i)$ and $P(i+1)$ are moving and awake in Y.*

Proof. Write $\mathcal{S}$ for the set of such minimal and non-annular subsurfaces. The assumption that $P(i)$ and $P(i+1)$ are awake in Y allows us to use Lemma 4.4 to show that any pair of elements of $\mathcal{S}$ either miss or nest in one another. However the assumption that the subsurfaces are minimal implies that one cannot nest in another, so instead they must miss. Therefore we can realize each element of $\mathcal{S}$ on the surface

S simultaneously such that each pair of subsurfaces is disjoint. Now the rest of the argument will use the assumption that $P(i)$ and $P(i+1)$ are moving in each element of $\mathcal{S}$. The statement that there are at most two elements of $\mathcal{S}$ will now come from an Euler charactistic argument.

Suppose that the pants move occurs in a subsurface X, where $\omega(X) = 1$. Write Y_1, Y_2, Y_3 etc. for the elements of $\mathcal{S}$. We realize X and the Y_i on S such that ∂X and ∂Y_i are all in minimal position, and that no curve of ∂Y_i is a subset of X unless it is essential and non-peripheral in X.

We call a connected component R of $X \cap Y_i$ a *region*. For a region R of $X \cap Y_i$ we set $V(R)$ to be the number of intersection points of ∂X and ∂Y_i inside R; one can think of these points as corners of R. We set $\mathrm{index}(R) = \chi(R) - V(R)/4$. Likewise, we call the connected components of $X - \cup(Y_i)$ regions, and define their indices similarly. It is straightforward that the sum of the indices over all regions in X is equal to $\chi(X)$. Furthermore, there are no regions with positive index, indeed, because all boundary curves of the subsurfaces are in minimal position.

A *boundary circle* of a region R is a boundary component of R which is disjoint from every $X \cap Y_i$. A *bigon, square, hexagon* is a region R that is homeomorphic to a 2-disc and $V(R) = 2, 4, 6$ respectively. A *once-holed bigon* is a region R that is homeomorphic to an annulus and $V(R) = 2$. A region R is *exceptional* if R is a subset of some Y_i, and R is not a bigon, square, hexagon, or once-holed bigon with boundary circle in ∂X. Each Y_i must contain an exceptional region since $P(i)$ and $P(i+1)$ are moving in Y_i.

Now we argue that there are only at most two exceptional regions to complete the lemma. This is straightforward if $X = S_{1,1}$. Indeed, $\chi(X) = -1$, and any exceptional region contributes at most $-1/2$ to the Euler characteristic.

If $X = S_{0,4}$ then we are done if there are no once-holed bigons with boundary circle in some ∂Y_i—every other exceptional region contributes at most -1 to Euler characteristic. So suppose instead that there is such a region. Then there is some subsurface Y_1 with a curve in its boundary that is essential and non-peripheral in X. This separates X into two pairs of pants. The Euler characteristic of a pair of pants is equal to -1, so the only worry is that there are two exceptional, once-holed bigons in one pair of pants. However, each boundary circle of an exceptional, once-holed bigon must be essential and non-peripheral in X, so there can only be at most one such region in such a pair of pants. ☐

For each edge $P(i), P(i+1)$ there are at most two associated chains C_1 and C_2 (by Lemma 4.5) of nested, non-annular subsurfaces Y such that $P(i)$ and $P(i+1)$ are moving and awake in Y. The chains C_1 and C_2 need not be disjoint.

We assign the edge $P(i), P(i+1)$ to a non-annular subsurface Y if it is the minimal such subsurface in C_1 (or C_2) such that $P(i)$ and $P(i+1)$ are dominating in Y.

The reader is now given an explicit warning that this notion is not equivalent to assigning edges to minimal, non-annular subsurfaces Y such that $P(i)$ and $P(i+1)$ are moving and dominating in Y. For example, an edge can be moving and dominating in X and Y, but active and tame in Z, where Z and X both nest in Y but X and

Z miss. In this scenario there are two chains: the minimum of one is X and the minimum of the other is Z. The subsurface Y is contained in both chains. In the chain containing Z we have that Y is the minimal subsurface such that $P(i)$ and $P(i+1)$ are dominating in Y. So we wish to assign an edge to Y. However, Y is not minimal over all subsurfaces in which $P(i)$ and $P(i+1)$ are dominating, because of the existence of X.

4.2.4. *Sufficiently many edges per subsurface.* We say that $P(i)$ is *k-close to $P(j)$* in Z if $d_{\mathcal{C}(Z)}(\Pi_Z P(i), \Pi_Z P(j)) \leq k$.

Proposition 4.6 (enough edges in the middle)**.** *Suppose that $P(j)$ and $P(j+1)$ are moving and overpowering in a non-annular subsurface Z. Then there is some edge $P(i), P(i+1)$ that is assigned to Z such that $P(i)$ is 6-close to $P(j)$ in Z.*

The proof of Proposition 4.6 is technical. We have broken the proof into steps—including five statements. Upon first reading, it is beneficial to skip the proofs of statements (3) to (5) in order to grasp the strategy faster. Each of the terms awake, active, non-tame, dominating and overpowering are used in the proof below, and it seems that these are necessary in order for the inductive step to work.

Proof. Write $Y_0 = Z$ and $j_0 = j$. We are done if the edge $P(j_0), P(j_0+1)$ is assigned to Y_0. So suppose not. Associated to the edge $P(j_0), P(j_0+1)$ are at most two chains of nested, non-annular subsurfaces Y such that $P(j_0)$ and $P(j_0+1)$ are moving and awake in Y. Since the edge is not assigned to Y_0, it must be the case that there is some non-annular subsurface X such that $P(j_0)$ and $P(j_0+1)$ are moving and dominating in X, and X nests in Y_0. Now we pick Y_1 maximal and non-annular such that Y_1 nests in Y_0, $Y_1 \neq Y_0$, and $P(j_0)$ and $P(j_0+1)$ are moving and non-tame in Y_1. The existence of Y_1 is guaranteed since X exists.

Now by Lemma 4.2 (bounded jumps) there exists j_1 such that $P(j_1)$ and $P(j_1+1)$ are moving, active and tame in Y_1. We set $k = 1$.

Inductive step: Now we investigate whether the edge $P(j_k), P(j_k+1)$ is assigned to Y_0. If not, we shall construct Y_{k+1} and j_{k+1} in order to induct on k, as follows.

Suppose that we have constructed $j_1, ..., j_k$ and non-annular subsurfaces $Y_1, ..., Y_k$ (the base case is $k = 1$, and the statements (1) and (2) below shall follow from the construction of Y_1 and j_1 above) such that:-

(1) For $1 \leq a \leq k$ we have that Y_a is a maximal, non-annular subsurface such that $P(j_{a-1})$ and $P(j_{a-1}+1)$ are moving and non-tame in Y_a, Y_a nests in Y_{a-1} and $Y_a \neq Y_{a-1}$.
(2) For $1 \leq a \leq k$ we have that $P(j_a)$ and $P(j_a+1)$ are moving, active and tame in Y_a. On the other hand, we have that $P(j_0)$ and $P(j_0+1)$ are moving and overpowering in Y_0.

Then we can prove the following two statements.

(3) For $0 \leq a \leq b \leq k$ we have that $P(j_b)$ and $P(j_b+1)$ are moving and awake in Y_a.
(4) Suppose that Y_k nests in X, X nests in Y_0, and $X \neq Y_0$, then the statement that $P(j_k)$ and $P(j_k+1)$ are dominating in X is false.

Proof of statement (3). The moving statement is clear so now we prove the awakeness statement. This is done using induction: the base case is that $P(j_b)$ is awake in Y_b by statement (2) and the inductive step is as follows.

Suppose that $P(j_b)$ is awake in Y_a. By statement (1) we have that $P(j_{a-1})$ is awake in Y_a. By Lemma 4.3 (awake downstairs implies stationary upstairs) $P(j_b)$ is 6-close to $P(j_{a-1})$ in Y_{a-1}, because their images under $\Pi_{Y_{a-1}}$ are close to ∂Y_a. By statement (2) we have that $P(j_{a-1})$ is active in Y_{a-1} so $P(j_b)$ is awake in Y_{a-1}. This completes the inductive step and the proof is finished. The exact same argument works for $P(j_b + 1)$. □

Proof of statement (4). By statement (2) we may assume that $X \neq Y_k$. The remaining cases $X = Y_{a-1}$ for some $2 \leq a \leq k$, and otherwise, are treated separately below.

Suppose that $X = Y_{a-1}$ for some $2 \leq a \leq k$. Statement (1) implies that $P(j_{a-1})$ is awake in Y_a. By statement (3) we have that $P(j_k)$ is awake in Y_a. By Lemma 4.3 (awake downstairs implies stationary upstairs), $P(j_{a-1})$ and $P(j_k)$ are 6-close in $X = Y_{a-1}$ as they are close to ∂Y_a. We have $a - 1 \geq 1$ so by statement (2) we have that $P(j_{a-1})$ is tame in Y_{a-1} and hence $P(j_k)$ is not dominating in $X = Y_{a-1}$. Now we assume that $X \neq Y_a$ for $0 \leq a \leq k$.

We may assume that $P(j_k)$ and $P(j_k+1)$ are awake in X. By statement (3), $P(j_k)$ and $P(j_k + 1)$ are also awake in Y_a for $0 \leq a \leq k$. We have that Y_k nests in X so X and Y_a do not miss. By Lemma 4.4 either X nests in Y_a or Y_a nests in X. Therefore for some $1 \leq a \leq k$ we have that Y_a nests in X, X nests in Y_{a-1}, but we may assume that $X \notin \{Y_{a-1}, Y_a\}$ due to the earlier argument above. Statement (1) implies that $P(j_{a-1})$ and $P(j_{a-1} + 1)$ are awake in Y_a. By statement (3) we have that $P(j_k)$ and $P(j_k + 1)$ are awake in Y_a. By Lemma 4.3 (awake downstairs implies stationary upstairs), $P(j_{a-1})$, $P(j_{a-1}+1)$, $P(j_k)$ and $P(j_k+1)$ are 6-close in X as they are close to ∂Y_a. However by statement (1) it is false that $P(j_{a-1})$ and $P(j_{a-1}+1)$ are moving and non-tame in X. But they are moving in X since Y_a nests in X. Therefore they cannot both be non-tame, so one is tame, and hence one of $P(j_k)$ or $P(j_k + 1)$ is not dominating in X. □

Now we shall show:-

(5) For $1 \leq a \leq k$, $P(j_a)$ and $P(j_a + 1)$ are moving and dominating in Y_0. Furthermore, $P(j_a)$ and $P(j_0)$ are 6-close in Y_0.

Proof of statement (5). The moving statement is clear so now we prove the dominating statement. By statement (3), $P(j_a)$ is awake in Y_1 and by statement (1), $P(j_0)$ is awake in Y_1, so by Lemma 4.3, $P(j_0)$ and $P(j_a)$ are 6-close in Y_0. Now $P(j_0)$ is overpowering in Y_0 and so $P(j_a)$ is dominating in Y_0, and the exact same argument works for $P(j_a + 1)$. □

Now suppose that the edge $P(j_k), P(j_k + 1)$ is not assigned to Y_0. However by statement (3), $P(j_k)$ and $P(j_k + 1)$ are moving and awake in Y_a for $1 \leq a \leq k$. Furthermore by statement (5), $P(j_k)$ and $P(j_k+1)$ are moving and dominating in Y_0. Associated to the edge $P(j_k), P(j_k+1)$ are at most two chains of nested, non-annular subsurfaces Y such that $P(j_k)$ and $P(j_k + 1)$ are moving and awake in Y. At least one of these chains contains the sequence of nested subsurfaces $(Y_k, Y_{k-1}, ..., Y_1, Y_0)$. Since the edge is not assigned to Y_0, it must be the case that there is some non-annular subsurface X such that $P(j_k)$ and $P(j_k+1)$ are moving and dominating in X, where either X nests in Y_k, or, Y_k nests in X and X nests in Y_0. However the latter cannot be true by statement (4), so instead X nests in Y_k and $X \neq Y_k$. Now we pick Y_{k+1} maximal and non-annular such that Y_{k+1} nests in Y_k, $Y_{k+1} \neq Y_k$, and $P(j_k)$ and $P(j_k+1)$ are moving and non-tame in Y_{k+1}. The existence of Y_{k+1} is guaranteed since X exists. By construction, statement (1) holds in the new case $a = k+1$.

Now by Lemma 4.2 (bounded jumps) there exists j_{k+1} such that $P(j_{k+1})$ and $P(j_{k+1}+1)$ are moving, active and tame in Y_{k+1}. By construction, statement (2) holds in the new case $a = k+1$. Then we set k equal to $k+1$.

Now we repeat this procedure starting from the inductive step. It must terminate with some $k \leq \omega(S)$ and $P(j_k), P(j_k+1)$ is the required edge. □

The following lemma is the punchline of the proof of the main result in this section.

Lemma 4.7. *Suppose that P and P' are pants decompositions of S and that $(P(i))_i$ is a path connecting P and P' in the pants graph of S. Suppose that Y is non-annular and $d_{\mathcal{C}(Y)}(\psi_Y(P), \psi_Y(P')) \geq D$ then there are at least $\lfloor (D-2A-36-4J)/(J+6) \rfloor$ edges of $(P(i))_i$ assigned to the subsurface Y.*

Proof. Associated to Y is a geodesic G_Y in $\mathcal{C}(Y)$ whose length is equal to the integer $d_{\mathcal{C}(Y)}(\psi_Y(P), \psi_Y(P'))$. The set of vertices v such that $v \in G_Y$ and $d_{\mathcal{C}(Y)}(v, \{\Pi_Y P, \Pi_Y P'\}) \geq A + 18 + 2J$ we shall call *the overpowering interval of Y*.

If the edge $P(i), P(i+1)$ is assigned to the subsurface Y then we "colour" the vertex $\Pi_Y P(i)$ in G_Y. By Lemma 4.2 (bounded jumps) and Proposition 4.6 (enough edges in the middle), the set of coloured vertices in the overpowering interval is $(J+6)$-dense, provided that there are at least $J+7$ vertices in the overpowering interval. This implies the conclusion of the lemma. □

Theorem 4.8. *There are uniform constants C_0 and e_0 such that whenever P and P' are pants decompositions of S then*

$$e_0 \sum_{Y \in NA(S)} [d_{\mathcal{C}(Y)}(\psi_Y(P), \psi_Y(P'))]_{C_0} \leq d_{\mathcal{P}(S)}(P, P'),$$

where $NA(S)$ is the set of non-annular subsurfaces of S.

Proof. We take $C_0 = 4A + 84 + 10J$. By Lemma 4.7, there are at least

$$\lfloor (d_Y - 2A - 36 - 4J)/(J+6) \rfloor$$

edges assigned to any non-annular subsurface Y, where d_Y is the quantity $d_{\mathcal{C}(Y)}(\psi_Y(P), \psi_Y(P'))$. Therefore if $d_Y \geq C_0$ then

$$\begin{aligned} \lfloor (d_Y - 2A - 36 - 4J)/(J+6) \rfloor &\geq (d_Y - 2A - 36 - 4J - J - 6)/(J-6) \\ &\geq d_Y/(2J+12). \end{aligned}$$

However each edge is assigned to at most two non-annular subsurfaces. This provides the appropriate lower bound on the length of the path $(P(i))_i$. We may take $e_0 = 1/(4J + 24)$. □

Corollary 4.9. *Let $Y \subset S$ be a subsurface. Then there is a natural inclusion map $\mathcal{P}(Y) \to \mathcal{P}(S)$ that is a quasi-isometric embedding with constants only depending on Y. Furthermore, this still holds even when S is a surface homeomorphic to a pants decomposition with infinitely many pairs of pants.*

Proof. Take a pants decomposition of the complement of Y in S. Then any pants decomposition P of Y extends canonically to a pants decomposition $\bar{P}$ of S. This gives an injective map from $\mathcal{P}(Y)$ to $\mathcal{P}(S)$. We have

$$d_{\mathcal{P}(Y)}(P, P') \leq e \sum_{Z \in NA(Y)} [d_{\mathcal{C}(Z)}(\psi_Z(P), \psi_Z(P'))]_C + K,$$

where e, C and K depend only on Y. We may assume that $C \leq C_0$ where C_0 is given in Theorem 4.8. By Theorem 4.8 we obtain $d_{\mathcal{P}(Y)}(P, P') \leq \frac{e}{e_0} d_{\mathcal{P}(S)}(\bar{P}, \bar{P}') + K$ and these constants only depend on Y.

We are done even in the (exotic) case that S is a union of infinitely many pairs of pants because any such path of pants decompositions between $\bar{P}$ and $\bar{P}'$ has finite length and therefore occurs on a compact subsurface of S, so the lower bound on distance given above applies. □

Remark 4.10. The natural inclusion above is conjectured to be an isometric embedding in general ([3]) (in fact, convex/totally geodesic). Currently it is only known that simplicial embeddings of graph products of Farey graphs into the pants graphs are convex ([41]). Our Corollary 4.9 gives quasi-isometric embedding constants that only depend on Y and not on the ambient surface S.

We end this section with the following question.

Question 1. Is a similar result to Theorem 4.8 true for the marking graph?

The marking graph is more complicated because the subsurface projections of annuli need to be considered. For a given edge in the marking graph, there may be an arbitrary number of annuli Y_i such that the image under the projection π_{Y_i} changes significantly. The methods given in this section do not handle this technicality, since by Lemma 4.5, we only have two minimal subsurfaces to consider, rather than arbitrarily many. For the marking graph some extra analysis is required to understand how much progress can be made in all annular subsurface projections with one marking move.

5. The conjugacy problem for pseudo-Anosovs

Fix $S = S_{g,p}$ and let $f \in \mathrm{Mod}(S)$ be pseudo-Anosov with quasi-invariant tight geodesic axis $A_f \subset \mathcal{C}(S)$. That is, A_f is a biinfinite tight geodesic of $\mathcal{C}(S)$ such that for each $n \in \mathbb{Z}$, A_f and $f^n A_f$ 2δ–fellow travel in $\mathcal{C}(S)$. By Proposition 7.6 of [32], every pseudo-Anosov mapping class has a quasi-invariant tight geodesic axis in the curve graph.

Fix $\mu \in \mathcal{M}(S)$. For any $h \in \mathrm{Mod}(S)$, we use the notation $d(h)$ to denote the displacement $d_M(\mu, h\mu)$. Further, let $p_f : \mathcal{C}(S) \to A_f$ denote the nearest point retraction to the geodesic A_f, i.e. $p_f = \mathrm{near}_{A_f}$. For a marking μ, we modify the projection p_f slightly: first, consider μ as a bounded diameter subset of the curve graph. Then, let $p_f(\mu)$ be the collection of curves on A_f closest to the curves of μ enlarged so that $p_f(\mu)$ is a subsegment of A_f of diameter at least 3. Note that by uniform hyperbolicity, there is a uniform constant $D \geq 0$ so that $\mathrm{diam}_S(p_f(\mu)) \leq D$. In fact, we may take $D = 4\delta + 1$. Recall that by Lemma 7.7 of [32], for any $n \in \mathbb{Z}$ and $x \in \mathcal{C}(S)$, $d_{\mathcal{C}}(f^n(p_f(x)), p_f(f^n(x))) \leq 10\delta$.

If f has curve graph translation length $\ell_{\mathcal{C}}(f) \geq M + 2D + 10\delta + 4$, then we say that f is *well-displacing*.

Our first lemma is a simple exercise in hyperbolic geometry.

Lemma 5.1. *With the notation as above, if $f \in \mathrm{Mod}(S)$ is well-displacing, then $d_{\mathcal{C}}(\mu, p_f(\mu)) \leq \frac{d(f)+2D+2\delta}{2}$.*

The next lemma controls properties of the mapping class f using only the displacement of the fixed marking $\mu \in \mathcal{M}(S)$.

Lemma 5.2. *Suppose that $f \in \mathrm{Mod}(S)$ is well-displacing. For any proper subsurface $Y \subset S$ we have the following bounds:*

(1) $\mathrm{diam}_Y(p_f(\mu)) \leq 4M + d_Y(f^{-1}\mu, f\mu)$,
(2) $d_Y(p_f(\mu), p_f(f\mu)) \leq 4M + d_Y(f^{-1}\mu, f^2\mu) + 2 \cdot \mathrm{diam}_Y(p_f(\mu))$, *and*
(3) $d_Y(\mu, p_f(\mu)) \leq M + d_Y(\mu, f\mu) + d_Y(p_f(\mu), p_f(f\mu))$.

Hence, we get that

$$d_Y(\mu, p_f(\mu)) \leq 13M + d_Y(\mu, f\mu) + d_Y(f^{-1}\mu, f^2\mu) + d_Y(f^{-1}\mu, f\mu).$$

Proof. For any marking $\eta \in \mathcal{M}(S)$, set $\bar{\eta}_f = p_f(\eta)$. For the proof of (1), suppose that $\mathrm{diam}_Y(\bar{\mu}_f) \geq M + 1$. Then by the Bounded Geodesic Image Theorem (Theorem 3.2), $\bar{\mu}_f$ meets the 1-neighborhood of ∂Y in $C(S)$. We claim that each of the geodesics $[f^{-1}\mu, \overline{f^{-1}\mu}_f]$ and $[f\mu, \overline{f\mu}_f]$ miss $N_1(\partial Y)$, the 1-neighborhood of ∂Y in $\mathcal{C}(S)$. Therefore, their projection to $\mathcal{C}(Y)$ has diameter no more than M, again by the bounded geodesic image theorem. To see this, suppose that some vertex v on, say, $[f^{-1}\mu, \overline{f^{-1}\mu}_f]$ has distance ≤ 1 from ∂Y. Then since ∂Y as distance at most 1 from some curve $w \in \bar{\mu}_f \subset A_f$, $d_{\mathcal{C}}(v, w) \leq 2$. However, $\overline{f^{-1}\mu}_f$ are the vertices of A_f closest to $f^{-1}\mu$, hence

$$\begin{aligned}
d_{\mathcal{C}}(f^{-1}\bar{\mu}_f, \bar{\mu}_f) &\leq d_{\mathcal{C}}(\overline{f^{-1}\mu}_f, \bar{\mu}_f) + 10\delta \\
&\leq d_{\mathcal{C}}(\overline{f^{-1}\mu}_f, w) + 2D + 10\delta \\
&\leq d_{\mathcal{C}}(f^{-1}\bar{\mu}_f, v) + d_{\mathcal{C}}(v, w) + 2D + 10\delta \\
&\leq 4 + 2D + 10\delta.
\end{aligned}$$

This contradicts our assumption that f is well-displacing on $\mathcal{C}(S)$ and verifies the claim. Using the rectangle of geodesics (Figure 1) and tightness of the axis A_f, we conclude that

$$\operatorname{diam}_Y(p_f(\mu)) \leq 4M + d_Y(f^{-1}\mu, f\mu).$$

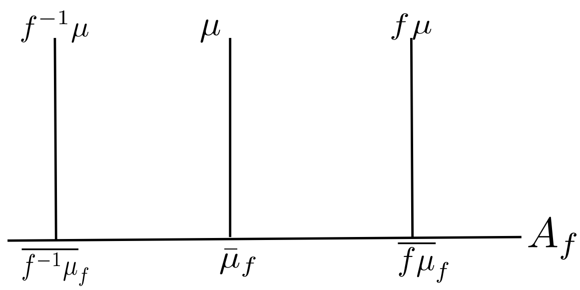

FIGURE 1. Avoiding ∂Y

The proof of (2) is similar. Suppose $d_Y(\bar{\mu}_f, \overline{f\mu_f}) \geq M+1$. Another application of the bounded geodesic image theorem gives that the portion of A_f between $\bar{\mu}_f$ and $\overline{f\mu_f}$ meets $N_1(\partial Y)$.

Consider the geodesic quadrilateral with sides $[f^{-1}\mu, f^2\mu]$, $[f^{-1}\mu, \overline{f^{-1}\mu_f}]$, $[f^2\mu, \overline{f^2\mu_f}]$ and $[\overline{f^{-1}\mu_f}, \overline{f^2\mu_f}] \subset A_f$. Exactly as in Part (1), we see that each of $[f^{-1}\mu, \overline{f^{-1}\mu_f}]$ and $[f^2\mu, f^2\bar{\mu}_f]$ miss $N_1(\partial Y)$. Therefore, their projection to $\mathcal{C}(Y)$ has diameter no more than M, again by the bounded geodesic image theorem. Using the indicated rectangle and the fact that A_f is tight, we see that

$$d_Y(p_f(\mu), p_f(f\mu)) \leq 4M + d_Y(f^{-1}\mu, f^2\mu) + 2 \cdot \operatorname{diam}_Y(p_f(\mu)).$$

To prove (3), first note that

$$d_Y(\mu, \bar{\mu}_f) \leq d_Y(\mu, f\mu) + d_Y(f\mu, \overline{f\mu_f}) + d_Y(\overline{f\mu_f}, \bar{\mu}_f).$$

Hence, it suffices to prove that if $d_Y(\mu, \bar{\mu}_f) \geq M+1$ then $d_Y(f\mu, \overline{f\mu_f}) \leq M$. If not, then both geodesic segment $[\mu, \bar{\mu}_f]$ and $[f\mu, \overline{f\mu_f}]$ meet $N_1(\partial Y)$. In this case, however, we see from the nearest point projection that $d_{\mathcal{C}}(\bar{\mu}_f, \overline{f\mu_f}) \leq D + 10\delta + 2$, a contradiction to the assumption that f is well-displacing. □

In light of Lemma 5.2, for each subsurface Y of S set

$$D_Y = \max\{d_Y(\mu, f\mu), d_Y(f^{-1}\mu, f^2\mu), d_Y(f^{-1}\mu, f\mu)\}.$$

Further, let $\mathbb{Y}_i$ $(i = 1,2,3)$ be the collection of subsurfaces for which the ith term in the definition of D_Y achieves the max. By Lemma 5.2, $d_Y(\mu, p_f(\mu)) \leq 13M + 3D_Y$ for each subsurface Y.

Corollary 5.3. *There is a constant $E \geq 0$ depending only on the complexity of S so that*

$$d_M(\mu, p_f(\mu)) \leq E \cdot d(f).$$

In particular, we may take $E = (400\omega(S))!$.

Proof. Pick $K \geq 13M$; note that, assuming the value of 100 for M, this is larger than $5M+15$, which is the cut-off obtained in Section 4.1 . Then there are constants $(200\omega(S))! \geq A, B \geq 1$ such that

$$d_M(\mu, p_f(\mu)) \leq A \sum_Y [d_Y(\mu, p_f(\mu))]_K + B.$$

Since $K \geq 13M$, Lemma 5.2 and Lemma 5.1 imply

$$\begin{aligned} d_M(\mu, p_f(\mu)) &\leq A \sum [13M + 3D_Y]_K + B \\ &\leq 6A \sum [D_Y]_K + B \\ &\leq 6A(\sum_{\mathbb{Y}_1} [D_Y]_K + \sum_{\mathbb{Y}_2} [D_Y]_K + \sum_{\mathbb{Y}_3} [D_Y]_K) + B \\ &\leq 6A(\sum_{\mathbb{Y}_1} [d_Y(\mu, f\mu)]_K + \sum_{\mathbb{Y}_2} [d_Y(f^{-1}\mu, f^2\mu)]_K + \sum_{\mathbb{Y}_3} [d_Y(f^{-1}\mu, f\mu)]_K) + B \\ &\leq 6A \cdot C(d_M(\mu, f\mu) + d_M(f^{-1}\mu, f^2\mu) + d_M(f^{-1}\mu, f\mu)) + B \\ &\leq 18A \cdot C \cdot d(f) + B \\ &\leq 18A \cdot B \cdot C \cdot d(f), \end{aligned}$$

where the last inequality holds because $d(f) \geq 1$, and $C = 40 \cdot \omega(S) - 120$ is the multiplicative error found in Section 4.1. □

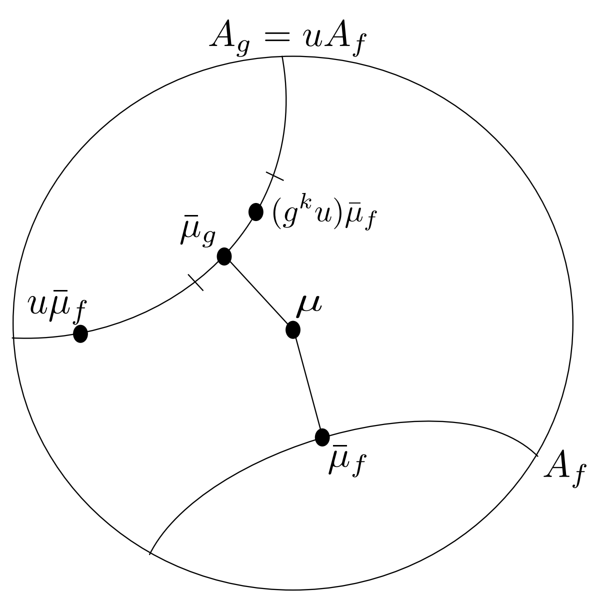

FIGURE 2. Axes and projections

Proposition 5.4 (Effective linear conjugator bound)**.** *Suppose that h_1 and h_2 are conjugate pseudo-Anosovs in* $\mathrm{Mod}(S)$*. Then there is a $w \in \mathrm{Mod}(S)$ such that $h_2 = wh_1w^{-1}$ and*

$$d(w) \leq c(S) \cdot (d(h_1) + d(h_2)).$$

Proof. First write $h_2 = uh_1u^{-1}$ for some $u \in \mathrm{Mod}(S)$. By [22], any pseudo-Anosov $h \in \mathrm{Mod}(S)$ satisfies $\ell_{\mathcal{C}}(h) \geq \frac{1}{200\cdot\omega(S)^2+133}$. Hence, if we set

$$N = (M + 2D + 10\delta + 4) \cdot (200 \cdot \omega(S)^2 + 133),$$

then $f = h_1^N$ is well-displacing. Set $g = h_2^N = ufu^{-1}$.

As above, let A_f denote a quasi-invariant tight axis of f in $\mathcal{C}(S)$ and set $A_g = uA_f$, which is an axis for the pseudo–Anosov g. Set $\bar{\mu}_f = p_f(\mu)$ and $\bar{\mu}_g = p_g(\mu)$, the nearest point retractions of μ to the geodesics A_f and A_g, respectively.

By construction $u \cdot \bar{\mu}_f$ lies on the axis A_g. Hence, by applying a power g^k we have that $d_{\mathcal{C}}(g^k u \cdot \bar{\mu}_f, \bar{\mu}_g) \leq \ell(g) \leq d(g)$. Set $w = g^k u$ and note that w still conjugates h_1 to h_2. By applying Corollary 5.3 to both f and g, we have the following:

(1) $d_M(\mu, \bar{\mu}_f) \leq E \cdot d(f)$,
(2) $d_M(\mu, \bar{\mu}_g) \leq E \cdot d(g)$.

Also, since $w\bar{\mu}_f$ lies within the portion of A_g between $\bar{\mu}_g$ and $g\bar{\mu}_g$ another application of the bounded geodesic image theorem gives $d_Y(\bar{\mu}_g, w\bar{\mu}_f) \leq 2M + d_Y(\bar{\mu}_g, g\bar{\mu}_g)$. Hence, just an in Corollary 5.3 we conclude that $d_M(\bar{\mu}_g, w\bar{\mu}_f) \leq E \cdot d(g)$.

We now compute,

$$\begin{aligned} d(w) &= d_M(\mu, w\mu) \\ &= d_M(\mu, \bar{\mu}_g) + d_M(\bar{\mu}_g, w\bar{\mu}_f) + d_M(w\bar{\mu}_f, w\mu) \\ &= d_M(\mu, \bar{\mu}_g) + d_M(\bar{\mu}_g, w\bar{\mu}_f) + d_M(\mu, \bar{\mu}_f) \\ &\leq 2E \cdot (d(f) + d(g)) \\ &\leq 2E \cdot N \cdot (d(h_1) + d(h_2)). \end{aligned}$$

Setting $C(S) = 2E \cdot N$ completes the proof. ∎

To finish the proof of Theorem 1.6, we appeal to the Milnor-Svarc lemma in order to use Proposition 5.4 to give bounds on the word length of w in a particular finite generating set, $\mathcal{S}$. Let $\mathcal{D} > 0$ be an upper bound for the diameter of the marking graph $\mathcal{M}(S)$, modulo the action of $\mathrm{Mod}(S)$. Then if $\mathcal{S}$ is the set of all mapping classes s so that $B_{\mathcal{D}}(\mu) \cap s \cdot B_{\mathcal{D}}(\mu) \neq \emptyset$, then $\mathcal{S}$ generates, and for any $g \in \mathrm{Mod}(S)$,

$$\frac{1}{\mathcal{D}} d(g) \leq \ell_{\mathcal{S}}(g) \leq d(g) + 1,$$

where $\ell_{\mathcal{S}}(g)$ denotes the word length of g in the generating set $\mathcal{S}$. Thus,

$$\ell_{\mathcal{S}}(g) \leq 2EN \cdot (d(h_1) + d(h_2)) \leq 2E \cdot N \cdot \mathcal{D} \cdot (\ell_{\mathcal{S}}(h_1) + \ell_{\mathcal{S}}(h_2)).$$

Theorem 1.6 then follows form a bound on $\mathcal{D}$. For this, let $\mathcal{D}_{\mathcal{P}}$ denote the diameter of $\mathcal{P}(S)$ modulo the action of $\mathrm{Mod}(S)$. Note that $\mathcal{D} \leq 2 \cdot \mathcal{D}_{\mathcal{P}} + 3g - 3 + p$. Indeed, if a pair of markings μ_1, μ_2 have the same base, one can be obtained from the other by applying a composition of twists and at most $|\mathrm{base}(\mu)|$ many half-twists, and none of the twists contribute to length in the quotient $\mathcal{M}(S)/\mathrm{Mod}(S)$. If μ_1, μ_2 have different bases, μ_1 can be transformed into a marking with the same base as μ_2 by performing flip moves and cleaning, perhaps half-twisting once between each such flip move; each flip move corresponds to an edge in $\mathcal{P}(S)$. Therefore, it suffices to give a bound for $\mathcal{D}_{\mathcal{P}}$. This has been obtained in [38] (see also [18]):

$$\mathcal{D}_{\mathcal{P}} \leq 2 \cdot (\omega(S)+1) \cdot (8\log_2(\omega(S)+1)-3).$$

Hence,

$$\begin{aligned} 2E \cdot N \cdot \mathcal{D} &\leq 824 \cdot (200 \cdot \omega(S)^2 + 133) \cdot (400\omega(S))! \cdot (\omega(S)+1) \cdot (8\log_2(\omega(S)) + 1 - 3) \\ &\leq (405\omega(S))! \end{aligned}$$

In particular, this yields the well-definedness of Calvez's algorithm for determining the Nielsen-Thurston type of braids (see Theorem 1.16). For example, if $n \leq 7$, there is a generating set $\mathcal{S}$ of the braid group B_n so that if $u, v \in B_n$ are conjugate pseudo-Anosov braids, then there is a conjugator w so that

$$\ell_{\mathcal{S}}(w) \leq 10000 \cdot (\ell_{\mathcal{S}}(u) + \ell_{\mathcal{S}}(v)).$$

6. Effectivizing Brock's theorem and applications

6.1. **Convex core bounds.** In this section, we use the effective Masur–Minsky machinery developed in Section 3 to prove an effective version of Brock's theorem [11]: that the volume of a quasi-fuchsian manifold is coarsely related to distance in the pants graph. We first address the upper bound on volume of the convex core by pants distance in the case that S is closed. As mentioned in the introduction, when S is closed and of genus at least 2 and $X, Y \in \mathcal{T}(S)$, Schlenker [39] has shown an upper bound on volume of the convex core of a quasi-fuchsian manifold $Q(X,Y)$ in terms of the Weil-Petersson distance $d_{WP}(X,Y)$:

$$\mathrm{vol}(\mathrm{core}(Q(X,Y))) \leq 3\sqrt{\pi(g-1)}\, d_{WP}(X,Y) + K_S,$$

for some constant K_S which grows at most linearly in g. As mentioned in the proof of Theorem 1.15,

$$d_{WP}(X,Y) \leq_{q(\omega)} d_{\mathcal{P}}(P_X, P_Y),$$

where P_X, P_Y are Bers short pants decompositions on X, Y respectively, and q is a quadratically growing function of $\omega(S)$. Chaining together these two inequalities proves the desired upper bound.

The remainder of this section is dedicated to proving the lower bound on volume of the convex core. Recall that for pants decompositions $P_I, P_T \in \mathcal{P}(S)$, the set of nonannular subsurfaces Y for which $d_Y(P_I, P_T) \geq K$ is denoted by $\Omega(P_I, P_T, K)$. Also, for $P \in \mathcal{P}(S)$ we set

$$V(P) := V_L(P) = \{X \in \mathcal{T}(S) \colon \ell_X(\alpha) \leq L \text{ for all } \alpha \in P\},$$

where L is the constant guaranteed to exist by Bers' theorem. Buser-Seppälä showed that L grows at most linearly in $\omega(S)$ ([15]).

Lemma 6.1. *Suppose that $P_1, P_2 \in \mathcal{P}(S)$ are pants decompositions such that $V_L(P_1)\cap V_L(P_2) \neq \emptyset$. Then*

$$\max\{i(\alpha_i, \alpha_{i+1})\colon \alpha_i \in P_i, \alpha_{i+1} \in P_{i+1}\} \leq I_\omega,$$

where $I_\omega = O(\exp(t(\omega)))$ for some fixed affine function t.

Proof. The collar lemma implies that a simple closed curve of length ℓ admits an embedded neighborhood of radius $\log(\coth(\ell/2)) =: R_\ell$. Thus, two curves of length $\leq \ell$ can only intersect at most ℓ/R_ℓ times. A calculation shows that this is eventually smaller than $\exp(2\ell)$, and thus the lemma follows from the fact that L grows at most linearly in $\omega(S)$. □

For $X \in \mathcal{T}(S)$ let P_X denote a shortest pants decomposition for X. Note that $X \in V(P_X)$.

Theorem 6.2 (Effective volume of convex core)**.** *There exists an affine function h of ω satisfying the following. Let M be a quasi-fuchsian 3-manifold homeomorphic to $S \times \mathbb{R}$ with conformal boundary components $X, Y \in \mathcal{T}(S)$. Let $\operatorname{core}(M)$ denote the convex core of M. Then*

$$\operatorname{vol}(\operatorname{core}(M)) \geq C_1(\omega) \cdot d_{\mathcal{P}}(P_X, P_Y) - C_2(\omega)$$

where $1/C_1(\omega), C_2(\omega) = O(h(\omega)^{h(\omega)})$.

Before proving Theorem 6.2, we require effective versions of several lemmas from Brock's [11].

Lemma 6.3 (Effective Lemma 4.3 of [11])**.** *Let $g = \{P_I = P_0, P_1, \ldots, P_N = P_T\}$ be a sequence of pants decompositions, such that*

$$\max\{i(\alpha_i, \alpha_{i+1})\colon \alpha_i \in P_i, \alpha_{i+1} \in P_{i+1}\} \leq I_\omega$$

for each $i = 0, \ldots, N-1$. Let $\mathcal{S}_g$ denote the image of g in $\mathcal{C}(S)$. Then,

$$d(P_I, P_T) \leq t(\omega)^{t(\omega)} \cdot 80\omega(2\log(I_\omega) + 2) \cdot |\mathcal{S}_g|,$$

where $t(\omega)$ is the function of $\omega(S)$.

Proof. Since $i(\alpha_i, \alpha_{i+1}) \leq I_\omega$ for each $\alpha_i \in P_i$ and $\alpha_{i+1} \in P_{i+1}$, we have that for any nonannular subsurface $Y \subset S$, $d_Y(P_i, P_{i+1}) \leq k_\omega$, for $k_\omega = 2\log(I_\omega) + 2$. Let $K = 5M + 3k_\omega + 25$, and for each $Y \in \Omega = \Omega(P_I, P_T, K)$, define $J_Y = [i_Y, t_Y] \subset [0, N] \cap \mathbb{Z}$ as follows: i_Y is the largest index r with $d_Y(P_I, P_r) \leq 2M + k_\omega + 9$ and t_Y is the smallest index s greater than i_Y with $d_Y(P_s, P_T) \leq 2M + k_\omega + 9$. Denote by $\operatorname{diam}(J_Y)$ the diameter of J_Y as a subset of $\mathbb{R}$ and set $\operatorname{diam}_Y(J_Y) = \max\{d_Y(P_l, P_m)\colon l, m \in J_Y\}$.

Following the notation of Brock, for $\alpha \in \mathcal{S}$ with $\pi_Y(\alpha) \neq \emptyset$ define $J_Y(\alpha) \subset J_Y$ to be collection of integers $i \in J_Y$ for which $\alpha \in P_i$. As in Brock, it is clear that $\operatorname{diam}_Y(J_Y(\alpha)) \leq 4$, and that each integer in J_Y is contained in $J_Y(\alpha)$ for some $\alpha \in \mathcal{S}$. Hence, if we set $\mathcal{S}_Y = \{\alpha \in \mathcal{S}\colon J_Y(\alpha) \neq \emptyset\}$, then

$$\operatorname{diam}_Y(J_Y) \leq 4k_\omega \cdot |\mathcal{S}_Y|.$$

Claim: If $Y, Z \in \Omega(P_I, P_T, K)$ with $Y \pitchfork Z$, then for any $\alpha \in \mathcal{C}^0(S)$ either $J_Y(\alpha) = \emptyset$ or $J_Z(\alpha) = \emptyset$. Hence, $\mathcal{S}_Y \cap \mathcal{S}_Z = \emptyset$.

Proof. Toward a contradiction, suppose that $k \in J_Y(\alpha) \subset J_Y$ and $m \in J_Z(\alpha) \subset J_Z$. Since Y and Z overlap we have either $d_Y(Z, P_I) \leq M$ or $d_Z(Y, P_I) \leq M$ by the Behrstock inequality. Assume the former; the latter case is proven by exchanging the occurrences of Y and Z in the proof. By the triangle inequality,

$$\begin{aligned} d_Y(Z, P_k) &\geq d_Y(P_I, P_k) - d_Y(Z, P_I) \\ &\geq 2M + 9 - M \geq M + 1. \end{aligned}$$

So again by the Behrstock inequality, we have $d_Z(Y, P_k) \leq M$. Combining this with the fact that

$$d_Z(P_k, P_m) \leq d_Z(P_k, \alpha) + d_Z(\alpha, P_m) \leq 8,$$

we see that

$$\begin{aligned} d_Z(Y, P_T) &\geq d_Z(P_k, P_T) - d_Z(Y, P_k) \\ &\geq d_Z(P_m, P_T) - d_Z(Y, P_k) - 8 \\ &\geq 2M + 9 - M - 8 \\ &\geq M + 1. \end{aligned}$$

Hence we conclude, using the Behrstock inequality once more, that $d_Y(Z, P_T) \leq M$. This, together with our initial assumption, implies

$$d_Y(P_I, P_T) \leq d_Y(P_I, Z) + d_Y(Z, P_T) \leq 2M < K$$

contradicting that $Y \in \Omega(P_I, P_T, K)$. ☐

We now return to the proof of the lemma. Since the size of a set of nonoverlapping subsurfaces of S is bounded by $2\omega(S)$, we have

$$\sum_{Y \in \Omega} \mathrm{diam}_Y(J_Y) \leq 4(2\omega(S))k_\omega \cdot |\mathcal{S}|.$$

Note that this follows since if $\alpha \in \mathcal{S}_Z \cap \mathcal{S}_Y \neq \emptyset$ then $J_Z(\alpha) \cap J_Y(\alpha) \neq \emptyset$ and so Y and Z are nonoverlapping on S (by our claim). By the definition of J_Y, we observe

$$d_Y(P_I, P_T) \leq \mathrm{diam}_Y(J_Y) + 4M + 2k_\omega + 18.$$

Since, for each $Y \in \Omega(P_I, P_T, K)$, $d_Y(P_I, P_T) \geq 5M + 3k_\omega + 25$ we have $\frac{1}{5} \cdot d_Y(P_I, P_T) \leq \mathrm{diam}_Y(J_Y)$ and so putting this with the inequality above

$$\begin{aligned} \sum_{Y \in \Omega} d_Y(P_I, P_T) &\leq 5 \cdot \sum_{Y \in \Omega} \mathrm{diam}_Y(J_Y) \\ &\leq 20(2\omega(S))k_\omega \cdot |\mathcal{S}|. \end{aligned}$$

Now combining this with the effective distance formula for pants distance and observing that $K \leq j(\omega(S))$ for j some affine function of ω, gives

$$\begin{aligned} d(P_I, P_T) &\leq t(\omega)^{t(\omega)} \cdot \sum_{Y \subseteq S} [[d_Y(P_I, P_T)]]_K + t(\omega)^{t(\omega)} \\ &\leq t(\omega)^{t(\omega)} \cdot 20(2\omega(S))k' \cdot |\mathcal{S}| + t(\omega)^{t(\omega)}, \end{aligned}$$

where $t(\omega)$ is an affine function depending on $j(\omega)$ that can be explicitly bounded from above using the proof of Proposition 3.5. Since $\mathcal{S} \neq \emptyset$ whenever $P_I \neq P_T$, we conclude

$$d(P_I, P_T) \leq t(\omega)^{t(\omega)} \cdot 40(2\omega(S))k_\omega \cdot |\mathcal{S}|. \qquad \square$$

Following Brock, we now show how large pants distance produces many curves in $Q(X,Y)$ whose lengths are less than L. This, via Brock's packing argument, will give a lower bound on the volume of the convex core of $Q(X,Y)$.

Lemma 6.4 (Effective Lemma 4.2 of [11])**.** *Let $Q(X,Y)$ denote the quasi-fuchsian manifold with conformal boundary $X, Y \in \mathcal{T}(S)$. Let $P_X, P_Y \in \mathcal{P}(S)$ with $X \in V(P_X)$ and $Y \in V(P_Y)$. Then for some constant K to be determined in the course of the proof,*

$$d_{\mathcal{P}}(P_X, P_Y) \leq K(|\mathcal{S}_L| + 1),$$

where $\mathcal{S}_L$ is the set of geodesics in $Q(X,Y)$ of length no greater than L.

Proof. First, recall that by Bers, the $Q(X,Y)$–length of the curves in P_X and P_Y are no greater than $2L$. Let Z_X and Z_Y be pleated surfaces with pleating locus containing P_X and P_Y, respectively. Let P_I and P_T be pants decompositions such that $Z_X \in V(P_I)$ and $Z_Y \in V(P_T)$.

Hence, both P_X and P_I are $2L$–short on Z_X and so we compute D such that $d_{\mathcal{P}}(P_X, P_I) \leq D$. In general, let I_L denote the maximum number of intersections between L–short pants decompositions on a hyperbolic surface.

With this set up, replace Z_X and Z_Y with very close simplicial hyperbolic surfaces that also map P_X and P_Y to their geodesic representatives, respectively. This can be achieved by Thurston's spinning construction. By very close, we mean that with respect to the uniformizations of these simplicial hyperbolic surfaces the pants decompositions P_I and P_T are still $L+\epsilon$–short, for some very small ϵ. The point is that we are now in the situation of Brock who, following Canary [17], builds a one parameter family of simplicial hyperbolic surfaces $k_t : S_t \to Q(X,Y)$ with $k_0 = Z_X$ and $k_1 = Z_Y$. Uniformizing these maps provides a 1-parameter family of 1-Lipschitz maps $h_t : Z_t \to Q(X,Y)$, from hyperbolic surface $Z_t \in \mathcal{T}(S)$ such that:

- $\ell_{Z_0}(P_X) \leq 2(L+\epsilon)$,
- $\ell_{Z_1}(P_Y) \leq 2(L+\epsilon)$.

As in Brock, we may use this family of 1-Lipschitz maps to produce a sequence of pants decompositions $P_0, \ldots, P_N \in \mathcal{P}(S)$ such that:

(1) $Z_0 \in V_{2L}(P_X) \cap V(P_0)$,
(2) $Z_1 \in V_{2L}(P_Y) \cap V(P_0)$,
(3) $V(P_i) \cap V(P_{i+1}) \neq \emptyset$ for $i = 0, \ldots N-1$,
(4) $Z_t \in \cup_i V(P_i)$ for all $t \in [0,1]$, and each $V(P_i)$ meets $(Z_t)_{t\in[0,1]}$.

By (4) and Lemma 6.1, no curve in P_i intersects any curve in P_{i+1} more than I_L times. Since the maps are 1-Lipschitz, we have that the length of each curve in P_i has length no greater than L in $Q(X,Y)$ for each $0 \leq i \leq N$. Hence the sequence

$g = \{P_0, \dots P_N\}$ satisfies the conditions of Lemma 6.3 and we conclude that

$$d(P_0, P_N) \leq t(\omega)^{t(\omega)} \cdot 40(2\omega - 6)(2\log(I_L) + 2) \cdot |\mathcal{S}_L|,$$

where we have used that $\mathcal{S}_g \subset \mathcal{S}_L$. To complete the argument, we use the intersection bound between P_I and P_0 (and P_T and P_N) bound the pants distance between P_I and P_0. As L grows linearly in ω, the intersection bound grows exponentially in ω. It follows that all non-annular projections between P_I and P_0 are bounded above by an affine function of ω. Then using the distance formulas developed in section 3, we obtain a bound on $d_{\mathcal{P}}(P_I, P_0)$ on the order of $t(\omega)^{t(\omega)}$, where we have replaced t with a larger affine function if necessary. □

Finally, we state the following result of Brock:

Lemma 6.5 (Lemma 4.8 of [11]). *Let M be a geometrically finite hyperbolic 3-manifold with ∂M incompressible, and let $\mathrm{vol}(M)$ denote its convex core volume. Then there is a constant $C_1 > 1$ depending only on L and $C_2 > 0$ depending only on $\chi(\partial M)$ for which*

$$\frac{|\mathcal{S}_L|}{C_1} - C_2 < \mathrm{vol}(M).$$

We refer to Brock for a proof of Lemma 6.5, however, we make a few remarks on the constants C_1 and C_2, as computed in Brock's proof. First, if we denote by V_R the volume of an R-ball in $\mathbb{H}^3$, then

$$C_1 = \frac{V_{L+2\epsilon}}{V_{\epsilon/2} \cdot V_{\epsilon/4}},$$

where ϵ is the 3-dimensional Margulis constant. The constant C_2 is defined to be the difference between the volume of an ϵ-neighborhood of the core of M and the volume of the core itself. This difference is bounded by twice the volume of an ϵ-neighborhood of a pleated surface in M representing the convex core boundary. This is computed in Proposition 8.12.1 of Thurston's notes [42], where the following upper bound is obtained:

$$\frac{-2\pi\chi(S) \cdot V_{\epsilon+a}}{A_a}.$$

Here, a is the length of the shortest curve on ∂M which is homotopically trivial in M, and A_a is the area of a disk of radius a in $\mathbb{H}^2$. Since in our setting, no curve in S is trivial in the 3-manifold, we are free to choose any value for a. We set $a = 1$ to obtain the desired bound.

6.2. **Mapping tori.** We now prove Theorem 1.10. Indeed, it follows readily from Brock's original argument in [12]; the constants that appear in that argument are all either the same as, or are controlled by, those in the argument above for the convex core case. In the case that S is closed, we also present an alternative proof suggested in [12], which uses Theorem 1.8 and a recent result of Brock–Bromberg [14].

Given $X \in \mathcal{T}(S)$ and $\psi \in \mathrm{Mod}(S)$ pseudo-Anosov, let $Q_{n,X}$ denote the quasifuchsian manifold $Q(\psi^{-n}(X), \psi^n(X))$. The relevant result of Brock–Bromberg states that the quantity $|\mathrm{vol}(\mathrm{core}(Q_{n,X})) - 2n \cdot \mathrm{vol}(M_\psi)|$ is uniformly bounded for all $n \in \mathbb{N}$. Combining this with Theorem 1.8 yields

$$d_{\mathcal{P}}(\psi^{-n}(P_X), \psi^n(P_X)) \le_{(h(\omega))!} \mathrm{vol}(\mathrm{core}(Q_{n,X})) \le 2n \cdot \mathrm{vol}(M_\psi) + K,$$

where P_X is a Bers pants decomposition on X, and $K \in \mathbb{N}$ exists by the Brock–Bromberg result. Then dividing by $2n$ and taking a limit as n goes to infinity yields the desired inequality:

$$\overline{\tau}_{\mathcal{P}}(\psi) \le_{h(\omega)^{h(\omega)}} \mathrm{vol}(M_\psi).$$

7. Covering maps and Pants graphs

In this section, we prove Theorem 1.13. First, we describe how a cover $p : S \to S'$ gives rise to a map between pants graphs $\tilde{p} : \mathcal{P}(S') \to \mathcal{P}(S)$. As mentioned in the introduction, a pants decomposition on S' will lift to a multicurve on S, but not necessarily to one that is maximal. Thus to define $\tilde{p}$, we must make a choice of how to extend the lift of a pants decomposition on S' to a full pants decomposition on S.

To do this, first choose a hyperbolic metric σ on S', which we can lift via p to a metric $p^*\sigma$ on S'. Then given a pants decomposition $P \in \mathcal{P}(S')$, $p^*(P)$ is a multicurve on S, which we extend to a shortest possible pants decomposition with respect to $p^*\sigma$. This yields a map $\tilde{p}$ from $\mathcal{P}(S')$ to $2^{\mathcal{P}(S)}$. We first show that $\tilde{p}$ is coarsely well-defined.

Given P a pants decomposition of S', if P_1, P_2 are two pants decompositions of S obtained by extending $p^*(P)$ to a full pants decomposition, then by the collar lemma (and as stated in Lemma 6.1),

$$i(P_1, P_2) \le \exp(t_1(\omega(S))),$$

for some affine function t_1.

Lemma 7.1. *Let P_1, P_2 be pants decompositions on a surface S. Then*

$$d_{\mathcal{P}}(P_1, P_2) \le_A \prod_{j=1}^{\omega(S)} [2 \cdot \log_2(4 \cdot i(P_1, P_2) + 2) + 2],$$

for $A = (200 \cdot \omega(S))!$.

Proof. Proposition 3.5 implies the desired inequality, so long as the sum of sufficiently large non-annular subsurface projections $d_Y(P_1, P_2)$ is bounded from above by the right hand side. Indeed, for any $Y \subseteq S$, the projection of P_1 can intersect the projection of P_2 no more than $4 \cdot i(P_1, P_2) + 2$ times. Thus, using the basic bound on curve graph distance in terms of intersection number, $d_Y(P_1, P_2)$ is bounded above by $2 \cdot \log_2(4 \cdot i(P_1, P_2) + 2) + 2$. □

We now turn to the proof of Theorem 1.13. Lemma 7.1 implies that for P_1, P_2 as above,

$$\begin{aligned} d_{\mathcal{P}(S)}(P_1, P_2) &\le (200 \cdot \omega)! \cdot (t_2(\omega))^{\omega(S)} \\ &\le \omega(S)^{t_3(\omega(S))}, \end{aligned}$$

for t_2, t_3 some fixed affine functions. This proves the coarse well-definedness of $\tilde{p}$.

Next, suppose that $P_1, P_2 \in \mathcal{P}(S')$ differ by a single elementary move. Then there is some affine t_5 satisfying

$$i(p^*(P_1), p^*(P_2)) \leq t_5(\omega(S));$$

indeed, $i(P_1, P_2) = 1$, and this intersection will have $\deg(p)$ many pre-images, which is bounded by $\omega(S)$. Furthermore, we can extend $p^*(P_1)$ and $p^*(P_2)$ to full pants decompositions of S' by appending the same multicurve to both. Thus Lemma 7.1 together with the proof of the coarse well-definedness of $\tilde{p}$ implies that

$$d_{\mathcal{P}(S)}(\tilde{p}(P_1), \tilde{p}(P_2)) \leq \omega(S)^{t_6(\omega(S))},$$

for some affine t_6.

It remains to show that for any $P_1, P_2 \in \mathcal{P}(S')$,

$$d_{\mathcal{P}(S)}(\tilde{p}(P_1), \tilde{p}(P_2)) \geq_{\omega^{t(\omega)}} d_{\mathcal{P}(S')}(P_1, P_2),$$

where t is some affine function to be determined, and chosen to be larger than $\max_{i=1,\ldots,6} t_i$.

To show this, let $Q(P_1, P_2)$ be a quasi-fuchsian manifold homotopy equivalent to S', chosen so that P_1 is Bers short on one conformal structure at infinity and P_2 is Bers short on the other. Then applying Theorem 1.8,

$$d_{\mathcal{P}(S')}(P_1, P_2) \leq_{(h(\omega))!} \mathrm{vol}(\mathrm{core}(Q(P_1, P_2))).$$

The covering map p induces a covering $\tilde{p} : Q(p^*(P_1), p^*(P_2)) \to Q(P_1, P_2)$ between quasi-fuchsian manifolds, and the volume of the convex core of $Q(p^*(P_1), p^*(P_2))$ is at least $\mathrm{vol}(\mathrm{core}(Q(P_1, P_2)))$.

Moreover, applying Theorem 1.14, the volume of the convex core of $Q(p^*(P_1), p^*(P_2))$ is at most $3\sqrt{\pi(g-1)}d_{WP}(X(p^*(P_1)), X(p^*(P_2))) + K_S$, where $X(p^*(P_i))$ denotes the conformal structures at infinity. As mentioned in the proof of Theorem 1.15, Weil-Petersson distance is bounded above by an affine function of pants distance whose coefficients depend quadratically on ω, and therefore we conclude that

$$\mathrm{vol}(\mathrm{core}(Q(p^*(P_1), p^*(P_2)))) \leq_{q(\omega)} 3\sqrt{\pi(g-1)}d_{\mathcal{P}(S)}(p^*(P_1), p^*(P_2)) + K_S,$$

for some quadratic function $q(\omega)$. Also as mentioned in the proof of Theorem 1.15, the constant K_S grows at most linearly in $\omega(S)$, and therefore after collecting all constants and combining inequalities, we obtain

$$d_{\mathcal{P}(S')}(P_1, P_2) \leq_{\omega^{t(\omega)}} d_{\mathcal{P}(S)}(p^*(P_1), p^*(P_2)),$$

for some affine t, as desired.

8. Poorly behaving hierarchies and optimal bounds

In this final section, we provide examples of "bad hierarchies," i.e. hierarchies having pathological behavior which becomes worse as complexity grows. These examples establish, for example, the lower bound on $C(\omega)$ in Theorem 1.2.

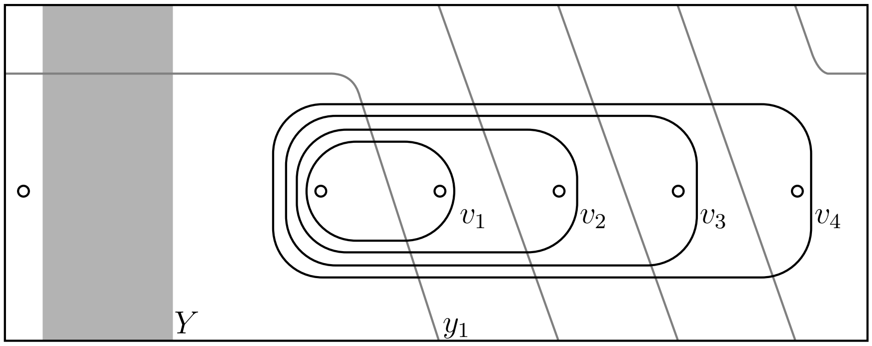

FIGURE 3. The surface $S_{1,n+2}$ where $n = 4$. The horizontal sides and vertical sides are suitably identified.

8.1. **Linearly large links.** The main motivation for this section is to illustrate that the large link lemma has no uniform constant, despite there being a uniform constant for the bounded geodesic image theorem (Theorem 3.2). This indicates that finding an exponential upper bound on the multiplicative and additive errors for the distance estimates (in the marking/pants graph) is a difficult problem.

We provide linear lower bounds in $\omega(S)$ for the constants in the large link lemma (Lemma 3.4 in Section 3 above) and the sigma projection lemma (Lemma 3.3 above) of Masur and Minsky [32] in the case where the subsurface Y is an annulus. At the end of the section we briefly describe how to obtain logarithmic lower bounds in $\omega(S)$ when the subsurface Y is non-annular. Since the large link lemma is a corollary of the sigma projection lemma, we focus only on the large link lemma constants, and then deduce similar lower bounds for the sigma projection lemma constants.

Let v and w be multicurves. Recall that the subsurface filled by v and w is denoted by $F(v, w)$. We begin with the following lemma:

Lemma 8.1. *Let n be a positive integer. There is an annular subsurface Y of $S = S_{1,n+2}$ and there are curves $y_1, ..., y_{n+1}, v_1, ..., v_n$ of S with the following properties.*

- *The curves $v_1, ..., v_n$ miss Y.*
- *The curves $y_1, ..., y_{n+1}$ cut Y.*
- *The curves $v_1, ..., v_n$ form a multicurve $v_1 \cup ... \cup v_n$.*
- *There exist curves $u_2, ..., u_{n+1}$ such that we have $\partial F(v_1, y_1) = y_2 \cup u_2$, and inside a component domain of (S, v_{i-1}) we have $\partial F(v_i, y_i) = y_{i+1} \cup u_{i+1}$.*
- $\mathrm{diam}_{\mathcal{C}(Y)}(\pi_Y(y_1) \cup \pi_Y(y_{n+1})) \geq n$.

Proof. On S there exists an annulus Y and a curve y_1 such that y_1 intersects the core curve of Y exactly once. We then define the curves $v_1, ..., v_n$ as in Figure 3. It is clear that each v_i separates S into two components. We set $W_1 = S$ and define W_{i+1} to be the component domain of (S, v_i) that contains v_{i+1} (hence also $v_{i+1}, ..., v_n$). It is clear that $W_1 \supset W_2 \supset ... \supset W_n$. We deduce that W_{i+1} is also a component domain of (W_i, v_i). The other component domain of (W_i, v_i) is homeomorphic to a pair of pants.

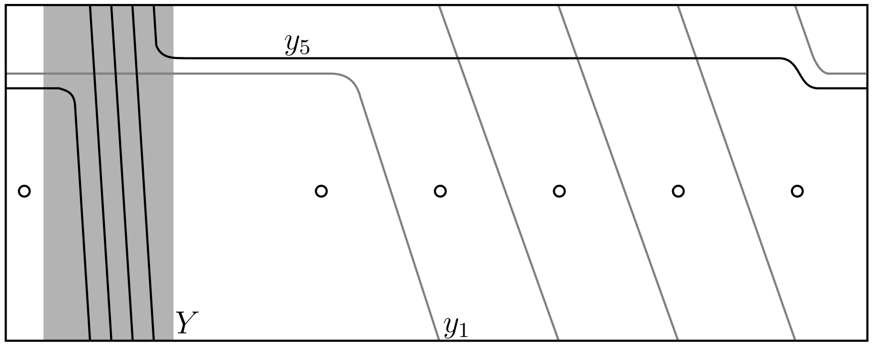

FIGURE 4. The curves y_1 and y_{n+1} are shown where $n = 4$. The curves are in minimal position. Notice the twisting about the annulus Y.

The reader should now check inductively that the curve y_i intersects v_i twice but v_i bounds a pair of pants in W_i, and so $F(v_i, y_i)$ is a four holed sphere of which exactly two boundary components are essential, non-peripheral and non-isotopic in W_i (we point out that this is where we require $n+2$ punctures when $i = n$). Exactly one of these boundary components intersects v_{i+1} twice, and we set y_{i+1} to be this curve, the other curve we call u_{i+1}. We deduce that $y_{i+1} \cup u_{i+1}$ is a multicurve in W_{i+1}. We have defined $y_1, ..., y_{n+1}$.

Finally we show that $\operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y(y_1) \cup \pi_Y(y_{n+1})) \geq n$. See Figure 4. There are subarcs $a \subset y_1$ and $a' \subset y_{n+1}$ such that $a, a' \subset Y$ and $|a \cap a'| = n-1$. Given a cover of S corresponding to Y, we pick lifts of a and a' within the homeomorphic lift of Y. This proves that there are lifts of y_1 and y_{n+1} that intersect at least $n-1$ times in the cover. By equation (2.1) we have that the distance between these arcs of $\pi_Y(y_1)$ and $\pi_Y(y_{n+1})$ is at least n. □

Using Lemma 8.1 throughout, for the surface $S = S_{1,n+2}$ we shall construct a partial hierarchy H which restricts to a hierarchy H' on a subsurface properly containing Y so that Y is not a domain of H'. Therefore H, and any extension of H to a hierarchy, does not have a geodesic whose support is Y. On the other hand, we shall have $\operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y\mathbf{I}(H) \cup \pi_Y\mathbf{T}(H))$ bounded from below in terms of n. This provides a lower bound on the constant in the first clause of the large link lemma of Masur and Minsky.

The construction: There is a component domain S' of (S, v_n), homeomorphic to $S_{1,2}$, that contains the annulus Y. For each n, the curves y_{n+1} and u_{n+1} are essential and non-peripheral in S', but their homeomorphism classes in S' are simultaneously independent of n (they only differ by twisting about the core curve of Y). Therefore for each n we may pick a hierarchy K between some $\mu = \mathbf{I}(K)$ and $\mathbf{T}(K) = y_{n+1} \cup u_{n+1}$ such that the support of K is S', Y is not the support of any geodesic in K, and μ cuts Y. Therefore $\operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y\mathbf{I}(K) \cup \pi_Y\mathbf{T}(K))$ is uniformly bounded because the surface S' is homeomorphic to $S_{1,2}$. The forward and backward sequences $\Sigma^{\pm}(Y, K)$ are non-empty and share the same top geodesic by Theorem 4.7 of [32].

Now we use an inductive procedure to describe some of the geodesics of the required (partial) hierarchy H. We use the W_i from the proof of Lemma 8.1. The base case is the main geodesic g_1, where $D(g_1) = W_1 = S$, the tight sequence of g_1 is $(v_1, y_2 \cup u_2, y_1)$, $\mathbf{I}(g_1) = \mu \cup v_1 \cup ... \cup v_n$ where $\mu = \mathbf{I}(K)$ above, and, $\mathbf{T}(g_1) = y_1$.

The inductive step is as follows. Assume that $1 \leq i \leq n-1$, so in particular $i+1 \leq n$. We have $D(g_i) = W_i$, the tight sequence of g_i is $(v_i, y_{i+1} \cup u_{i+1}, y_i)$, $\mathbf{I}(g_i) = \mu \cup v_i \cup ... \cup v_n$, and, $\mathbf{T}(g_i) = y_i \cup u_i$, unless $i = 1$ where we have $\mathbf{T}(g_1) = y_1$. To complete the inductive step we are required to construct g_{i+1} and $W_{i+1} = D(g_{i+1})$. There is a component domain W_{i+1} of (W_i, v_i) that contains $\mu, v_{i+1}, ..., v_n, y_{i+1}$ and u_{i+1}. Therefore W_{i+1} is directly forwards and backwards subordinate to g_i. We have $\mathbf{I}(W_{i+1}, g_i) = \mu \cup v_{i+1} \cup ... \cup v_n$ and $\mathbf{T}(W_{i+1}, g_i) = y_{i+1} \cup u_{i+1}$. Now we may define g_{i+1} to be a tight geodesic whose tight sequence is $(v_{i+1}, y_{i+2} \cup u_{i+2}, y_{i+1})$, and $D(g_{i+1}) = W_{i+1}$, with the required initial and terminal markings relative to g_i. We have completed the inductive step and hence constructed $g_1, ..., g_n$. These geodesics will belong to our (partial) hierarchy H.

We have that S' is directly forwards and backwards subordinate to g_n: $\mathbf{I}(S', g_n) = \mu = \mathbf{I}(K)$ and $\mathbf{T}(S', g_n) = y_{n+1} \cup u_{n+1} = \mathbf{T}(K)$.

Now we include the geodesics of K in the partial hierarchy H. Note that Y is not the support of any geodesic of H. This is because $\Sigma^+(Y, H) = \Sigma^+(Y, K) \cup \{g_n, ..., g_1\}$, $\Sigma^-(Y, H) = \{g_1, ..., g_n\} \cup \Sigma^-(Y, K)$, and K is a hierarchy not containing Y as a domain.

By Lemma 8.1 we have $\mathrm{diam}_{\mathcal{C}(Y)}(\pi_Y y_1 \cup \pi_Y y_{n+1}) \geq n$. Therefore

$$\mathrm{diam}_{\mathcal{C}(Y)}(\pi_Y \mathbf{T}(K) \cup \pi_Y \mathbf{T}(H)) \geq n.$$

However $\mathrm{diam}_{\mathcal{C}(Y)}(\pi_Y \mathbf{I}(K) \cup \pi_Y \mathbf{T}(K))$ and $\mathrm{diam}_{\mathcal{C}(Y)}(\pi_Y \mathbf{I}(H) \cup \pi_Y \mathbf{I}(K))$ are uniformly bounded, and so we deduce that

$$\mathrm{diam}_{\mathcal{C}(Y)}(\pi_Y \mathbf{I}(H) \cup \pi_Y \mathbf{T}(H))$$

grows linearly in n, but Y is not the support of a geodesic in H. This provides a linear lower bound in n for the constant in the large link lemma.

We remark that the above construction can be easily extended to *complete hierarchies* using Theorem 4.6 of [32] which extends a partial hierarchy to a hierarchy. We have shown the following:

Proposition 8.2. *There are hierarchies H_n between markings in $S_{1,n}$ and annular subsurfaces Y_n such that $d_{Y_n}(\boldsymbol{I}(H), \boldsymbol{T}(H)) \succ n$ yet Y_n is not a domain in H_n.*

8.2. **Lower bounds for non-annular subsurfaces.** We shall briefly describe a method that shows that for each fixed non-annular subsurface Y the large link constant is at least logarithmic in $\omega(S)$.

All that is required is a version of Lemma 8.1, namely, some curves $y_1, ..., y_{n+1}$ and $v_1, ..., v_n$ to go with the subsurface Y with all the properties of Lemma 8.1, except that this time $\mathrm{diam}_{\mathcal{C}(Y)}(\pi_Y(y_1) \cup \pi_Y(y_{n+1}))$ will be bounded from below by a logarithmic function in n (depending on $\xi(Y)$), rather than a linear function of n. The constructions of the hierarchies will follow as they did for the case when Y is an annulus. We now construct the desired curves.

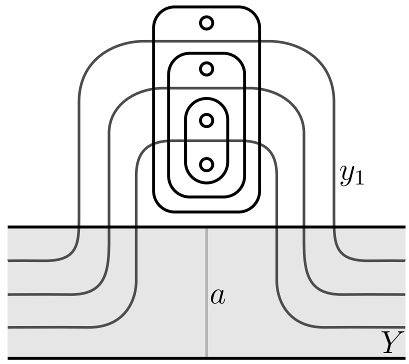

FIGURE 5. A general construction. The non-annular subsurface Y is shaded. The curves $v_1, ..., v_n$ are darkest with v_1 innermost. In this figure we have $n = 3$.

Fix a surface S' of which Y is a subsurface (S' assumed to be minimal if desired). The construction begins with a curve y inside the subsurface Y. Take any essential arc a of Y. We assume that a cuts y in Y. We set $n = i(a, y)$. Isotope a and y into minimal position. Drag the curve y along a, in one direction, outside of the subsurface Y, see Figure 5. This process produces bigons between the representative of y and ∂Y. We then puncture $n+1$ times the surface S', so that the representative of y no longer shares a bigon with ∂Y, but of course this is an entirely new curve on a new surface which we call y_1 and S respectively. See Figure 5. We add curves $v_1, ..., v_n$ as illustrated.

Tightening y_1 and v_1 produces two curves y_2 and u_2. The curve y_2 will intersect ∂Y fewer times than y_1, but its image under π_Y will intersect a. Indeed, we may inductively define y_i similarly as before. Informally, the curve y_i is the result of passing $i-1$ strands of the curve y_1 past the additional punctures and back into Y. We have that y_{n+1} represents the original curve y in Y. These are the required curves $y_1, ..., y_{n+1}$ and $v_1, ..., v_n$.

We wish to bound $\operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y(y_1) \cup \pi_Y(y_{n+1}))$ from below. The key observation is that a misses $\pi_Y(y_1)$. Therefore it is enough to consider $d_{\mathcal{C}(Y)}(\sigma(a), \pi_Y(y_{n+1}))$, or equivalently, $d_{\mathcal{C}(Y)}(\sigma(a), y)$. We have $i(\sigma(a), y) = 2n$. It is understood that there is a logarithmic upper bound for $d_{\mathcal{C}(Y)}(\sigma(a), y)$ in terms of $i(\sigma(a), y)$, so this construction cannot possibly provide a linear lower bound for $\operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y(y_1) \cup \pi_Y(y_{n+1}))$ in $\omega(S)$.

But these logarithmic bounds on distance in $\mathcal{C}(Y)$ in terms of intersection number are sharp by considering pseudo-Anosov mapping classes. For instance, we may fix a pseudo-Anosov f of Y and fix the arc a, and consider the pair a and $f^i(y)$ for large i. Then $i(a, f^i(y))$ is exponential in i, and so $\omega(S)$ would be exponential in i, yet $\operatorname{diam}_{\mathcal{C}(Y)}(\pi_Y(y_1) \cup \pi_Y(y_{n+1}))$ would be linear in i. Since the latter quantity is directly related to the large link constant we are done.

8.3. **Hierarchies with small subsurface projections.** This section culminates in Theorem 8.11: we construct hierarchies which are exponentially long as a function of both the complexity of the underlying surface and the length of the main geodesic, and so that all subsurface projections are bounded by a linear function in co-complexity.

We first show the existence of an affine $\mathbb{N}$-valued function K, and a complete hierarchy $H_{0,p}$ on $S_{0,p}$ satisfying the following:

(1) $d_Y(\mathbf{I}(H), \mathbf{T}(H)) \leq K(p - \omega(Y))$;
(2) $|H| \geq 2^p$;
(3) For k any tight geodesic of H, every vertex of k is a simple closed curve (as opposed to being a multicurve).
(4) If α and β are the initial and terminal vertices of the main geodesic of H, then both α and β bound 3-holed spheres on one side.

The strategy is to induct on p. When $p = 4$, pick a length 3 geodesic.

To construct the desired partial hierarchy $H_{0,p}$ on $S_{0,p}$, we assume by induction the existence of $H_{0,p-1}$ on $S_{0,p-1}$, which we identify with the subsurface S': one side of a curve v on $S_{0,p}$ which bounds a 3-holed sphere on the other side of S'. Let $\alpha^{(p-1)}, \beta^{(p-1)}$ denote the initial and terminal vertices of the main geodesic of $H_{0,p-1}$; by induction, both bound 3-holed spheres on one side.

We will also not assume that $H_{0,p-1}$ is complete, only that it is a partial hierarchy. From this, we will obtain a partial hierarchy $H_{0,p}$ and we will prove the desired subsurface projection bounds for its initial and terminal markings. Once we construct such a partial hierarchy for each p, we can then complete each one, yielding a complete hierarchy satisfying the desired properties on each planar surface.

We can assume, by including $S_{0,p-1}$ into $S_{0,p}$ in the appropriate way, that both $\alpha^{(p-1)}$ and $\beta^{(p-1)}$ bound 3-holed spheres on one side in $S_{0,p}$; then the curve v is the relative boundary of $S' = S_{0,p-1}$. Henceforth, we will refer to $\alpha^{(p-1)}$ as α (resp. $\beta^{(p-1)}$ as β). Denote by $S' \subset S_{0,p}$ the copy of $S_{0,p-1}$ on which the hierarchy $H_{0,p-1}$ is defined. We assume that $d_{\mathcal{C}(S')}(\alpha, \beta) = 3$.

Note that β and v are disjoint and in the same $\mathrm{Mod}(S_{0,p})$ orbit. Let $S'' := S \setminus (\beta \cup v)$.

Proposition 8.3. *There exists a pants decomposition P of S'' such that the following holds.*

(1) *For every curve c of P and for every arc a of $\pi_{S''}(\alpha)$ we have that $i(a, c) \leq 2$.*
(2) *There is a mapping class ϕ of S that preserves P and interchanges β and v such that $\pi_{S''}\alpha$ and $\phi(\pi_{S''}\alpha)$ have uniformly bounded combinatorics in S''. That is, for any subsurface Y of S'' (including S'') we have $d_Y(\pi_Y \alpha, \pi_Y \phi(\alpha))$ is at most* 6.

Proof. If S is a 5-holed the proposition holds trivially because S'' is a 3-holed sphere, and ϕ is the unique (up to isotopy) map which interchanges β and v. Now assuming $p > 5$, we construct P. Set $X(0) = S''$, $i = 1$ and $c_0 = \beta$. We proceed inductively. The pants decomposition P will be of the form $c_1, ..., c_i$.

Inductive step: by induction we have that α only intersects one boundary component of $X(j-1)$, namely c_{j-1}. Furthermore, $X(j-1)$ is not a pair of pants. We have that $\pi_{X(j-1)}\alpha$ fills $X(j-1)$ and that $X(j-1)$ is a planar surface. By considering outermost arcs of $\pi_{X(j-1)}\alpha$, there is an arc a_j of $\pi_{X(j-1)}\alpha$ that cuts off an annulus in $X(j-1)$, one of whose boundary components is a boundary component of S.

We surger the arc a_j with c_{j-1} to construct the curve c_j. Note that c_{j-1} and c_j cut off a pair of pants in S. Because c_{j-1} intersects each arc of $\pi_{S''}\alpha$ at most twice, we have that c_j intersects each of arc $\pi_{S''}\alpha$ at most twice.

If $j = \omega(S'')$ then $c_1, ..., c_j$ is a pants decomposition P of Z and we stop. If not then $j < \omega(S'')$ and we set $X(j)$ to be the unique subsurface of $X(j-1)$ that is complementary to c_j and is not a pair of pants and then we repeat the inductive step. This concludes the construction of P. Note that we have proved also the first clause of the proposition, namely, the existence of $P = c_1, ..., c_i$. We now show the second clause.

Each c_j and c_{j+1} of P cuts off a pair of pants in S. Moreover, c_1 cuts off a pair of pants in S'' that has boundary $\{c_1, \beta, \delta_1\}$ for δ_1 some boundary component of S, and c_i cuts off a pair of pants in S'' with boundary $\{c_i, v, \delta_i\}$ for δ_i some boundary component of S. Consider the dual tree $\mathcal{T}(P)$ to the pants decomposition P of S''; that is $\mathcal{T}$ has one vertex on the interior of each complementary region of P and an edge connecting two vertices when the corresponding pairs of pants are adjacent. Then since each pair of pants is bounded by two of the $c_i's$ and one boundary component of the full surface, there is some $n \in \mathbb{N}$ so that $\mathcal{T}$ is obtained by taking $n-2$ disconnected vertices $v_1, ..., v_{n-2}$, a path on n vertices $w_0, w_1, ..., w_{n-1}$ and connecting v_i to w_i by an edge.

It follows that there is a homeomorphism of S'' that interchanges β and v, and preserves P. Indeed, the desired homeomorphism corresponds to an automorphism of $\mathcal{T}$ interchanging the two end vertices of the path. Because β and v cut off pairs of pants in S, such a homeomorphism can easily be extended to a homeomorphism $\phi' \colon S \to S$.

Now we prove that for any such homeomorphism $\phi' \colon S \to S$ we have the following. For any subsurface $Y \subset S''$ that is not an annulus with core curve in P, we have that $d_Y(\pi_Y\alpha, \pi_Y\phi'(\alpha)) \leq 6$, that is, the distance is uniformly bounded. Indeed, P must cut such Y, therefore for any curve c of P and for any arcs $a \in \pi_{S''}\alpha$ and $a' \in \pi_{S''}g'(\alpha)$ that cut Y, we have $i(a,c) \leq 2$ and $i(c,a') \leq 2$. The intersection bound now implies the distance bound.

Finally, to deal with subsurface projections where Y is an annulus with core curve of P, we simply postcompose ϕ' with Dehn twists about each core curve of P so that $d_Y(\pi_Y\alpha, \pi_Y\phi'(\alpha)) \leq 2$. This constructs the required mapping class ϕ of S. □

Lemma 8.4. *For any pants decomposition $\mathcal{R}$ of S'', there exists an essential simple closed curve γ such that γ intersects each element of $\mathcal{R}$ essentially, and exactly twice.*

Proof. Let Γ be the dual tree to $\mathcal{R}$ on S''. Let $N(\Gamma)$ be a small regular neighborhood of Γ, and let γ' be the boundary of this neighborhood. Note that γ' is a simple closed curve. Then define γ to agree with γ' in each pair of pants of $S'' \setminus \mathcal{R}$ which is not associated to a univalent vertex of Γ. If P in $\mathcal{R}$ is associated to a univalent vertex, P has one boundary component b and two punctures; we require that within P, γ separates the two punctures from one another and intersects b twice.

Then γ is a simple closed curve intersecting each curve in $\mathcal{R}$ exactly twice, and γ intersects exactly once the simple arc separating the two boundary components contained in any pair of pants corresponding to a univalent vertex. As there must be at least two univalent vertices, γ intersects two of these simple arcs, each exactly once and these arcs are in distinct pairs of pants. it follows that γ is homotopically non-trivial and non-peripheral, and hence essential. □

In accordance with Lemma 8.4, let ρ be a simple closed curve in S'' intersecting every element of P exactly twice, and let ϵ denote the right Dehn twist of ρ about each curve in P.

Lemma 8.5. *$\rho \cup \epsilon$ fills S'', and for $Y \subseteq S''$ any essential subsurface,*

$$d_Y(\rho, \epsilon) < 5.$$

Proof. That $\rho \cup \epsilon$ fills S'' follows from Proposition 3.5 of [21]. Given $Y \subseteq S''$, suppose first that some curve $c \in P$ projects to Y. Then since both ρ and ϵ intersect c twice,

$$d_Y(\rho, \epsilon) \leq d_Y(\rho, c) + d_Y(c, \epsilon) < 5.$$

If no curve in P projects to Y, since P is a pants decomposition, it follows that Y must be an annulus whose core curve is homotopic to an element of $\mathcal{P}$. By construction, the arc $\pi_Y(\epsilon)$ is obtained from $\pi_Y(\rho)$ by applying one twist, and therefore in this case,

$$d_Y(\rho, \epsilon) \leq 2.$$

□

We require the following proposition:

Proposition 8.6. *Let α, β be simple closed curves which fill a surface S so that $d_Y(\alpha, \beta) < K$. Then if f is the pseudo-Anosov $f = T_\beta^{-B} T_\alpha^B$, where $B = 2M + 3$ for M the bounded geodesic image theorem constant, then f admits a geodesic axis A in $\mathcal{C}(S)$, and $\mathrm{diam}_Y(A) \leq 4M + 4K + 8$ for any proper essential subsurface Y.*

Proof. We begin by constructing the geodesic axis A for f.

Let $d_{\mathcal{C}}(\alpha, \beta) = d + 2$ and let $[\alpha, \gamma_0, ..., \gamma_d, \beta]$ be a geodesic segment in $\mathcal{C}(S)$. Here, square brackets denote an oriented path of adjacent vertices; set p to be the subpath between γ_0 and γ_d. Applying T_β^{-B} to the geodesic p and reversing orientation, we obtain the geodesic segment

$$T_\beta^{-B}(\overline{p}) = [\gamma_d, \gamma_{d+1} \ldots, \gamma_{2d}] = [T_\beta^{-B}(\gamma_d), T_\beta^{-B}(\gamma_{d-1}), \ldots, T_\beta^{-B}(\gamma_0)].$$

Set

$$A = \cup_{n \in \mathbb{Z}} f^n(p \cdot T_\beta^{-B}(\overline{p})),$$

where $p \cdot T_\beta^{-B}(\overline{p})$ denotes the length $2d$ path obtained by concatenating the paths at $\gamma_d = T_\beta^{-B}(\gamma_d)$.

Since $f^{n+1}(\gamma_0) = f^n(T_\beta^{-B}(\gamma_0)) = f^n(\gamma_{2d})$, A is a path of adjacent vertices that is f-invariant. For convenience, set $\gamma_{2id} = f^i(\gamma_0)$ and $\gamma_{(2i+1)d} = f^i(\gamma_d)$ so that we may label A as the biinfinite path

$$A = \ldots, \gamma_{-1}, \gamma_0, \gamma_1, \ldots.$$

Note that γ_{2id} is adjacent to $f^i(\alpha)$ and $\gamma_{(2i+1)d}$ is adjacent to $f^i(\beta)$ in $\mathcal{C}(S)$; we say that these translates of α and β are *associated* to the corresponding vertices of A.

Lemma 8.7. *A is a geodesic axis for f.*

Proof. We define vertex paths B_k that begin and end on the f-orbit of α and β and show that these paths are geodesics. Each of these paths begins at a vertex in the orbit, moves to the associated vertex of A, follows a subpath of A, and finally moves to an adjacent vertex in the f-orbit of α or β. We use square brackets to denote the path between vertices using this construction.

For $i \in \mathbb{Z}_{\geq 0}$, define $B_{2i} = [f^{-i}(\alpha), \alpha]$ and $B_{2i+1} = [f^{-i}(\alpha), \beta]$. Note that in each case $\ell(B_k) = kd + 2$ and so

$$\ell(B_{k+1}) = \ell(B_k) + d(\alpha, \beta) - 2.$$

We show that B_k is a geodesic path by induction on k. When $k = 1$, B_1 is the original geodesic of $\mathcal{C}(S)$ between α and β of length $d + 2$. Now suppose that B_j is a geodesic for $j \leq k$. The proof that B_{k+1} is geodesic depends on the parity of k. First assume that $k = 2i$ so that $B_k = [f^{-i}(\alpha), \alpha]$ is a geodesic of length $2di + 2$, and $B_{k+1} = [f^{-i}(\alpha), \beta]$ is a path of length $(2i+1)d + 2$. By the induction hypothesis, $f^{-1}B_{k-1} = [f^{-i}(\alpha), f^{-1}(\beta)]$ is a geodesic segment and it is clear that each vertex of this path intersects α (e.g. using that B_k is a geodesic). Hence, $\mathrm{diam}_\alpha([f^{-i}(\alpha), f^{-1}(\beta)]) \leq M$ by the bounded geodesic image theorem (Theorem 3.2). Then

$$\begin{aligned} d_\alpha(f^{-i}(\alpha), \beta) &\geq d_\alpha(f^{-1}(\beta), \beta) - \mathrm{diam}_\alpha([f^{-i}(\alpha), f^{-1}(\beta)]) \\ &\geq d_\alpha(T_\alpha^{-(2M+3)}(\beta), \beta) - M \\ &\geq M + 1. \end{aligned}$$

So again by the bounded geodesic image theorem, any geodesic from $f^{-i}(\alpha)$ to β must pass through the 1-neighborhood of α. Hence,

$$d(f^{-i}(\alpha), \beta) \geq \ell(B_k) + d(\alpha, \beta) - 2 = \ell(B_{k+1}).$$

This implies that B_{k+1} is a geodesic segment in the case when k is even.

The proof when k is odd is similar and is included for completeness. Assume $k = 2i + 1$. By the induction hypothesis $B_{k-1} = [f^{-i}(\alpha), \alpha]$ is a geodesic of length $2di + 2$ and each vertex of this path meets β. As above, this implies that $\mathrm{diam}_\beta(B_{k-1}) \leq M$ and so

$$d_\beta(f^{-i}(\alpha), f(\alpha)) \geq M + 1.$$

Since this implies that any geodesic from $f^{-i}(\alpha)$ to $f(\alpha)$ passes through a 1-neighborhood of β,

$$d(f^{-i}(\alpha), f(\alpha)) \geq \ell(B_k) + d(\beta, f(\alpha)) - 2 = \ell(B_k) + d(\alpha, \beta) - 2$$

Hence, $B_{k+1} = [f^{-(i+1)}(\alpha), \alpha] = f^{-1}([f^{-i}(\alpha), f(\alpha)])$ is a geodesic segment of length $(2i+1)d + 2$, as required.

Now to see that A is a geodesic, observe that any finite subpath is a subpath of some B_k. Since A is invariant under f by construction, A is a geodesic axis for f. □

To complete the proof of the proposition, we will need the standard (bigon) train tracks for α and β, as constructed in [36]. Our convention will be that for positive dehn twists, transverse curves "turn left." Define the train track τ to be the track obtained by smoothing intersections of α and β. Let τ' be the track obtained by using the other smoothing. Then, τ is an invariant track for f and τ' is an invariant track for f^{-1}.

Let Y be a proper subsurface of S. If every vertex of A meets Y, then $\mathrm{diam}_Y(A) \leq M$ by the bounded geodesic image theorem. So for the remainder of the proof we assume that this is not the case.

Suppose first that Y is not an annulus corresponding to a translate of α or β and there is an $n \in \mathbb{Z}$ such that Y misses either $f^n(\alpha)$ or $f^n(\beta)$.

Suppose that Y misses $f^n(\alpha)$; the case for a translate of β is similar. Note that since A is an axis of f,

$$\mathrm{diam}_Y(A) = \mathrm{diam}_{f^{-n}(Y)}(A)$$

so we may replace Y with $f^{-n}(Y)$ and assume that Y missed α. Let $X = S \setminus \alpha$ and let ϵ be any curve in the orbit of α or β, except α. We show that every arc of $\epsilon \cap X$ is isotopic to an arc of $\beta \cap X$. Since, in this case, $Y \subset X$, this proves that the Y-diameter of the set of orbits of α and β is less than or equal to 3. As these orbits are distance 1 from A, we have

$$\mathrm{diam}_Y(A) \leq 5.$$

To this end, note that the claim is obvious if $\epsilon = \beta$, so assume that $\epsilon = f^n(\alpha)$ or $f^n(\beta)$ for $n \neq 0$. If $n > 0$ then $\epsilon \prec \tau$ and if $n < 0$ then $\epsilon \prec \tau'$. In either case, realize ϵ in a small tie neighborhood N of the track and realize X so that either track intersects X in precisely $\beta \cap X$. With ϵ and X in this position, our assertion that each arc of $\epsilon \cap X$ is isotopic to an arc of $\beta \cap X$ is clear since ϵ crosses every branch of either τ or τ'. It only remains to show that these representatives of ϵ and ∂X are in minimal position. For this, assume without loss of generality that $\epsilon \prec \tau$.

First note that any bigon in the complement of ϵ and ∂X implies that there is a bigon in $S \setminus (\partial X \cup \tau)$. To see this, pass to the universal cover $\tilde{S} \to S$ and let $\tilde{\tau}, \tilde{\partial X}$, and $\tilde{\epsilon}$ be the complete preimages of τ, X and ϵ, respectively. Then any bigon B lifts to a bigon $\tilde{B}$ between $\tilde{\epsilon}$ and $\tilde{\partial X}$. Since $\tilde{\epsilon}$ is a train route, i.e. it makes only legal turns on τ, the complement $\tilde{B} \setminus \tilde{\tau}$ must contain an outer most bigon. A standard argument now implies that there is a bigon in S in the complement of ∂X and τ.

By inspection, no such bigon between τ and ∂X can be contained in the annular neighborhood $S \setminus X$ of α. Further, any bigon contained in X would correspond to a bigon between α and β, a contradiction. This completes the argument that every arc of $\epsilon \cap X$ is isotopic to an arc of $\beta \cap X$.

Suppose next that Y meets every translate of α and β.

By assumption, the distance between the subsurface projections of α and β is bounded by K for any proper subsurface of S. This bounds the projection between consecutive vertices in the orbits of α and β to the subsurface Y:

$$d_Y(f^i(\alpha), f^i(\beta)) = d_{f^{-i}(Y)}(\alpha, \beta) \leq K$$

and

$$d_Y(f^{i+1}(\alpha), f^i(\beta)) = d_{f^{-i}(Y)}(T_\beta^{-B}(\alpha), \beta) = d_{T_\beta^B f^{-i}(Y)}(\alpha, \beta) \leq K.$$

Now suppose that γ_l is the first vertex of A that does not meet Y. Then γ_{l+4} meets Y, as does every vertex after it. Set i to be the largest integer less than l/d and j be the smallest integer greater then $(l+4)/d$. Noe that $j - i \leq 4$.

Let A^- denote the ray $\ldots, \gamma_{d(i-1)}, \gamma_{di}$ and let A^+ denote the ray $\gamma_{dj}, \gamma_{dj+1}, \ldots$, both contained in A. Then

$$\operatorname{diam}_Y(A) \leq \operatorname{diam}_Y(A^-) + d_Y(\gamma_{di}, \gamma_{dj}) + \operatorname{diam}_Y(A^+).$$

By the bounded geodesic image theorem, $\operatorname{diam}_Y(A^-), \operatorname{diam}_Y(A^+) \leq M$. Also note that $d(f^i(x), \gamma_{di}), d(f^j(x'), \gamma_{dj}) \leq 1$ for $x, x' \in \{\alpha, \beta\}$ (depending on the parity of i and j). Since $j - i \leq 4$, the bounds above on the subsurface distances between translates of α and β imply that $d_Y(\gamma_{id}, \gamma_{jd}) \leq 6 + 4K$. We conclude that in this case

$$\operatorname{diam}_Y(A) \leq 2M + 4K + 6.$$

Finally, suppose Y is an annulus with core $f^n(\alpha)$ or $f^n(\beta)$ for some $n \in \mathbb{Z}$. As before, we are free to assume that Y is an annulus with core, say, α. Then the geodesic rays $\ldots, \gamma_{-2}, \gamma_{-1}$ and $\gamma_1, \gamma_2, \ldots$ each have projection to $\mathcal{C}(Y)$ bounded by M. Moreover,

$$d_Y(\gamma_{-1}, \gamma_1) = d_\alpha(T_\alpha^{-B}(\gamma_1), \gamma_1) \leq B + 1.$$

In this case, we conclude that $\operatorname{diam}_Y(A) \leq 2M + B + 1 = 4M + 4$.

This exhausts all cases and completes the proof of the proposition. □

Using Proposition 8.6, it follows that the map $F = T_\rho^B T_\epsilon^{-B}$ satisfies

$$d_Y(\epsilon, F^n(\epsilon)) + d_Y(\rho, F^n(\rho)) < C,$$

where C is independent of n, Y, and S. We say that F is the pseudo-Anosov with uniformly bounded projections that is *adapted* to the pants decomposition P.

We now define $H_{0,p}$, a collection of tight geodesics as follows:

As a collection of vertices in various curve graphs,

$$H_{0,p} := H_{0,p-1} \cup F^{19}\phi(H_{0,p-1}),$$

As a collection of tight geodesics, $H_{0,p}$ consists of the geodesic $\{\alpha, v, \beta, F^{19}\phi(\alpha)\}$, and all geodesics of $H_{0,p-1}$ and their images under $F^{19}\phi$. Thus the domains appearing in $H_{0,p}$ are either domains of $H_{0,p-1}$, their images under $F^{19}\phi$, or the vertex v.

Lemma 8.8. *The distance between α and $F^{19}\phi(\alpha)$ in $\mathcal{C}(S_{0,p})$ is 3. Furthermore, let $\mathcal{D}(\beta)$ be the domains Y of $H_{0,p-1}$ which support geodesics, and so that ∂Y is disjoint from β. Then for each $Y \in \mathcal{D}(\beta)$, we have*

$$d_{\mathcal{C}(S'')}(\mathcal{D}(\beta), F^{19}\phi(\partial Y)) > 1.$$

That is, for each such Y, ∂Y meets the boundary of $F^{19}\phi(Z)$ for all $Z \in \mathcal{D}(\beta)$.

Remark 8.9. If the lemma is true, β can not be in $\mathcal{D}(\beta)$, and indeed it is not because the annulus β does not support a geodesic in $H_{0,p-1}$.

Proof. First note that $\{\alpha, v, \beta, F^{19}\phi(\alpha)\}$ constitutes a length 3 path in $\mathcal{C}(S'')$, so to prove the first part of Lemma 8.8 it suffices to show that $d_{\mathcal{C}(S'')}(\alpha, F^{19}\phi(\alpha)) \geq 3$. Since F has translation length 2 in $\mathcal{C}(S'')$ and $\pi_{S''}(\alpha)$ is at most 6 from $\pi_{S''}(\phi(\alpha))$,

$$d_{\mathcal{C}(S'')}(\pi_{S''}(\alpha), F^{19}\pi_{S''}(\phi(\alpha))) > 5.$$

Then let w be any simple closed curve on S. If $w = v$ or β, then w intersects either α or $F^{19}\phi(\alpha)$. Otherwise, w projects to S'', and therefore it must intersect either $\pi_{S''}(\alpha)$ or $F^{19}\pi_{S''}\phi(\alpha)$, and thus $d_{\mathcal{C}(S_{0,p})}(\alpha, F^{19}\phi(\alpha)) = 3$ as desired.

As for the second claim, recall that $S' = S_{0,p-1}$ and we can identify S'' with $S' \setminus \beta$. Let $\left\{\alpha, v^{(p-1)}, u, \beta\right\}$ be the main geodesic of $H_{0,p-1}$. By induction, there is some map J on $S_{0,p-1}$ (identified with $S' = S_{0,p} \setminus v$) so that $J(\alpha) = \beta$ and J interchanges u and $v^{(p-1)}$.

Also by induction, for any domain Y of $H_{0,p-1}$, one of the following three must hold:

(1) Y is a domain of $H_{0,p-2}$;
(2) Y is the image of a domain in $H_{0,p-2}$ under the map J;
(3) $Y = v^{(p-1)}$.

If Y is a domain of $H_{0,p-2}$, then ∂Y is disjoint from $v^{(p-1)}$. In the case of (2), ∂Y is disjoint from u, which is itself disjoint from $v^{(p-1)}$. Thus

$$\operatorname{diam}_{\mathcal{C}(S'')}\mathcal{D}(\beta) \leq 10, \tag{8.1}$$

and

$$d_{S''}(\alpha, \partial Y) \leq 10, \tag{8.2}$$

since both $v^{(p-1)}$ and u project to S'', and the projection map is coarse Lipschitz.

Then using (8.2),

$$d_{S''}(\partial Y, \phi(\partial Y)) \leq d_{S''}(\partial Y, \alpha) + d_{S''}(\alpha, \phi(\alpha)) + d_{S''}(\phi(\alpha), \phi(\partial Y)) \leq 26.$$

Here we are using the fact that ϕ fixes $S'' = S_{0,p-1} \setminus \beta$.

Then by the triangle inequality in $\mathcal{C}(S'')$,

$$\begin{aligned} d_{S''}(\mathcal{D}(\beta), F^{19}\phi(\partial Y)) &\geq d_{S''}(\partial Y, F^{19}\partial Y) - \operatorname{diam}_{S''}(\mathcal{D}(\beta)) - d_{S''}(F^{19}\partial Y, F^{19}\phi(\partial Y)) \\ &\geq 38 - 10 - 26 > 1, \end{aligned}$$

as desired.

□

We now show that $H_{0,p}$ satisfies the axioms of being a partial hierarchy (as seen at the beginning of the proof of Theorem 4.6 in [32]:)

(1) There is a distinguished main geodesic g whose domain is the full surface $S_{0,p}$, and whose initial and terminal markings are equal to $\mathbf{I}(H)$ and $\mathbf{T}(H)$, respectively;
(2) If b, f are tight geodesics in H and $Y \subseteq S$, with Y directly backwards subordinate to b and directly forwards subordinate to f, then H contains at most one tight geodesic k such that $D(k) = Y$, and k is directly forward subordinate to f and directly backwards subordinate to b;
(3) For every geodesic k in H other than g, there exists geodesics b, f in H such that k is directly forward subordinate to f, and directly backwards subordinate to b.

Let g denote the main geodesic; we must check that g is tight. Indeed, $\alpha \cup \beta$ fill S', which is the complement of v, and $v \cup F^n\phi(\alpha)$ fill $S \setminus \beta$ for the same reason.

To check (2), it suffices to show that given any domain Y of $H_{0,p-1}$, Y can not be expressed as $F^{19}\phi(W)$, for some W which is a domain of $H_{0,p-1}$. Note that if ∂W intersects β, then $F^{19}\phi(W)$ must intersect v, and therefore it can not be equal to a domain in S'. Thus it suffices to consider domains $W \in \mathcal{D}(\beta)$. By Lemma 8.8, for any such W, $F^{19}\phi(W) \notin \mathcal{D}(\beta)$. However, $F^{19}\phi(W) \subset S''$ and therefore it is disjoint from β, hence if $F^{19}\phi(W)$ was a domain of $H_{0,p-1}$, it would have to be in $\mathcal{D}(\beta)$.

For (3), by induction $H_{0,p-1}$ is a partial hierarchy, and therefore (3) is immediate for geodesics of $H_{0,p}$ which are geodesics of $H_{0,p-1}$. Then (3) follows, since any geodesic of $H_{0,p}$ which is not the main geodesic is a homeomorphic image of some geodesic of $H_{0,p-1}$.

Therefore $H_{0,p}$ is indeed a partial hierarchy. By construction,

$$|H_{0,p}| \geq 2|H_{0,p-1}|.$$

It remains to show that for any subsurface $Y \subseteq S_{0,p}$,

$$d_Y(\alpha, F^{19}\phi(\alpha)) < K(p, Y).$$

We first remark that the desired bound for $Y = \beta$ or $Y = v$ is vacuous as $\alpha, F^{19}\phi(\alpha)$ do not both project to these domains.

Note also that $d_{S''}(\alpha, F^{19}\phi(\alpha)) < 100$ follows immediately from the fact that the projection of α and $\phi(\alpha)$ to S'' are at most 6 apart, that F has translation length 2 on $\mathcal{C}(S'')$, α is close by to the axis for F, and that $\pi_{S''}$ is coarse Lipschitz.

We next address subsurfaces $Y \subsetneq S''$. Note that α must project to Y; indeed, α intersects ∂Y non-trivially since $i(\partial Y, \beta) = 0$ and α fills $S \setminus v$ with β. Similarly, $\phi(\alpha)$ projects to Y by applying the same argument after interchanging β and v via ϕ. Therefore, using Proposition 8.6 to bound $d_{F^{-19}Y}(F^{-19}\alpha, \alpha)$, and part (2) of Proposition 8.3 to bound $d_{F^{-19}Y}(\alpha, \phi(\alpha))$, we obtain

$$\begin{aligned} d_Y(\alpha, F^{19}\phi(\alpha)) &= d_{F^{-19}Y}(F^{-19}\alpha, \phi(\alpha)) \\ &\leq d_{F^{-19}Y}(F^{-19}\alpha, \alpha) + d_{F^{-19}Y}(\alpha, \phi(\alpha)) \end{aligned}$$

$$\leq 428 + 6 = 434.$$

It remains to consider the following three cases:

(1) Both v and β project to Y;
(2) β projects to Y but v does not;
(3) v projects to Y but β does not.

In the first case, we have

$$d_Y(\alpha, F^{19}\phi(\alpha)) \leq d_Y(\alpha, v) + d_Y(v, \beta) + d_Y(\beta, F^{19}\phi(\alpha)) \leq 10.$$

In the second case,

$$d_Y(\alpha, F^{19}\phi(\alpha)) \leq d_Y(\alpha, \beta) + d_Y(\beta, F^{19}\phi(\alpha)).$$

Since Y is disjoint from v and is therefore a subsurface of S', by the induction hypothesis,

$$d_Y(\alpha, \beta) < K(p - 1 - \omega(Y)).$$

Furthermore, β and $F^{19}\phi(\alpha)$ are disjoint, and therefore

$$d_Y(\alpha, F^{19}\phi(\alpha)) < K(p - 1 - \omega(Y)) + 6.$$

Finally in the third case, v projects to Y, and therefore

$$d_Y(\alpha, F^{19}\phi(\alpha)) \leq d_Y(\alpha, v) + d_Y(v, F^{19}\phi(\alpha)),$$

and note that

$$d_Y(v, F^{19}\phi(\alpha)) = d_{F^{-19}Y}(v, \phi(\alpha)) = d_{\phi^{-1}F^{-19}Y}(\beta, \alpha).$$

Since $F^{-19}Y$ is disjoint from β, its image under ϕ^{-1} is disjoint from v, and it is therefore contained in S'. Hence again by the induction hypothesis,

$$d_Y(\alpha, F^{19}\phi(\alpha)) \leq K(p - 1 - \omega(Y)) + 6.$$

Therefore, we define $K(p - \omega(Y)) := \max(K(p - 1 - \omega(Y)) + 6, 10 + K_1 + C)$.

Remark 8.10. We can complete $H_{0,p}$ to a hierarchy with or without annuli, and therefore this construction applies to both pants decompositions and markings.

The existence of the hierarchies $H_{0,p}$ implies exponential growth for the additive error in the inequality bounding hierarchy length above by sums of subsurface projections. We conclude this section by promoting the previous construction to hierarchies with arbitrarily long main geodesic, which will establish exponential growth of the multiplicative error as well. We record this in the following theorem:

Theorem 8.11. *Given $k, p \in \mathbb{N}$, there exists a hierarchy $H_{0,p}^{(k)}$ on $S_{0,p}$ satisfying the following properties:*

(1) *the main geodesic of $H_{0,p}^{(k)}$ has length at least k and at most $2k$;*
(2) $|H_{0,p}^{(k)}| \geq 2^{p+k}$;
(3) *There is a uniform constant Q so that for any essential subsurface $Y \subset S_{0,p}$,*

$$d_Y(\boldsymbol{I}(H^{(k)}), \boldsymbol{T}(H^{(k)})) < 3K(p - \omega(Y)) + Q.$$

Let h be an oriented tight geodesic segment; given $k \in \mathbb{N}$ we call a tight geodesic $\tilde{h}$ a *left* (resp. right) *tight k-extension* of h if $\tilde{h}$ is obtained by concatenating a length k tight geodesic segment to the left (resp. right) endpoint of h. Let $g(p)$ denote the main geodesic of the hierarchy $H_{0,p}$. We show:

Lemma 8.12. *For any $k \in \mathbb{N}$, there exists a right tight k-extension of $g(p)$.*

Proof. Label the vertices of $g(p)$ by $g(p) = \{\alpha, v_0, v_1, \beta\}$; recall that by construction, both α and β bound 3-holed spheres on one side. Let γ be any simple closed curve on $S_{0,p}$ satisfying the following properties:

(1) γ is disjoint from β;
(2) γ bounds a 3-holed sphere on one side;
(3) $d_{S\setminus\beta}(\gamma, \alpha) > M + 3$, for M the bounded geodesic image theorem constant.

To find such a γ, start with a pants decomposition $\mathcal{Q}$ of $S \setminus \beta$ intersecting each component of $\pi_{S\setminus\beta}(\alpha)$ at most twice, as in Lemma 8.3. There must be some curve $\delta \in \mathcal{Q}$ which bounds a 3-holed sphere on one side. Then let γ be the image of δ under a high power of the pseudo-Anosov supported on $S\setminus\beta$ with uniformly bounded projections, adapted to $\mathcal{Q}$.

We claim that $g' := \{\alpha, v_0, v_1, \beta, \gamma\}$ is a tight geodesic. Indeed, any geodesic h connecting α to γ must pass through the 1-neighborhood of β by Theorem 3.2. However, since β bounds a 3-holed sphere on one side, h must actually pass through β; otherwise every vertex of h projects non-trivially to $S \setminus \beta$, which contradicts the fact that $d_{S\setminus\beta}(\alpha, \gamma)$ is large. Therefore g' is geodesic.

To check that g' is tight, we need only show that

$$d_{S\setminus\beta}(v_1, \gamma) \geq 3.$$

Suppose not; again since β bounds a 3-holed sphere on one side, v_1 projects non-trivially to $S \setminus \beta$, and therefore $d_{S\setminus\beta}(v_1, \gamma) < 3$ implies that $d_{S\setminus\beta}(\alpha, \gamma) < M + 3$ by Theorem 3.2, which is a contradiction.

This argument can be repeated indefinitely, and hence there exists a (left or right) k-tight extension of $g(p)$ for any $k \in \mathbb{N}$. □

Remark 8.13. In general, it is not the case that a geodesic segment can be extended to a larger geodesic, let alone one that is tight (see [6]). However, we avoid the pathology of 'dead-ends' by choosing our geodesic such that all vertices bound 3-holed spheres on one side.

Remark 8.14. Since γ was produced by taking a high power of a pseudo-Anosov on $S \setminus \beta$ with bounded subsurface projections, we remark that γ can also be chosen to satisfy the following additional property:

For $Y \subseteq S \setminus \beta$ any proper essential subsurface, $d_Y(\alpha, \gamma) < E$, where E is a uniform constant.

Let $\{\alpha, v_0, v_1, \beta, \gamma_1, \gamma_2\} = \{g(p), \gamma_1, \gamma_2\}$ be the right tight 2-extension of $g(p)$ whose existence is guaranteed by Lemma 8.12. We will choose this extension so that the

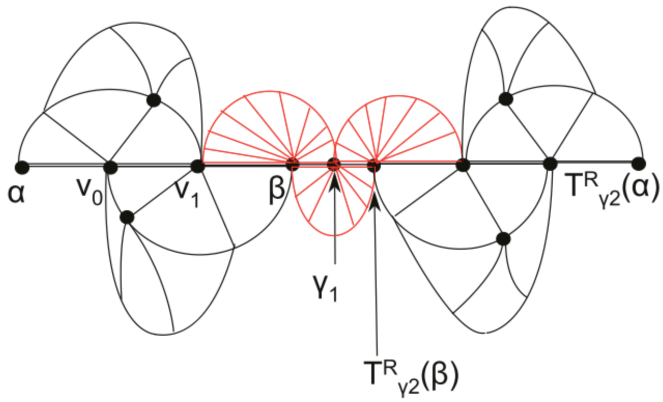

FIGURE 6. A schematic picture of $\tilde{H}^{(2)}$. $H_{0,p}$, and its image under $T^B_{\gamma_2}$ are pictured in black.

third property listed in the proof of Lemma 8.12 is replaced by a lower bound of $2M+3$ on projections to $S \setminus \beta$. Then consider the following path $\tilde{g}(p)$ in $\mathcal{C}(S_{0,p})$:

$$\tilde{g}(p) := \left\{\alpha, v_0, v_1, \beta, \gamma_1, T^B_{\gamma_2}(\beta), T^B_{\gamma_2}(v_1), T^B_{\gamma_2}(v_0), T^B_{\gamma_2}(\alpha)\right\},$$

where B is the constant from Proposition 8.6.

We claim that $\tilde{g}$ is a tight geodesic. To show this, it suffices to check tightness at the node γ_1. By tightness of $\{g(p), \gamma_1, \gamma_2\}$, the choice of γ_1, γ_2, and the same argument used in the proof of Lemma 8.12,

$$d_{S\setminus\gamma_1}(\beta, \gamma_2) \geq 3.$$

By choice of B and applying Theorem 3.2, any geodesic in $\mathcal{C}(S \setminus \gamma_1)$ connecting β to $T^B_{\gamma_2}(\beta)$ must pass through the 1-neighborhood of γ_2, and therefore

$$d_{S\setminus\gamma_1}(\beta, T^B_{\gamma_2}(\beta)) > 3.$$

We now define the partial hierarchy $\tilde{H}^{(2)} := \tilde{H}^{(2)}_{0,p}$ as follows. The main geodesic of $\tilde{H}^{(2)}$ is $\tilde{g}(p)$; if k is any other geodesic of $\tilde{H}^{(2)}$, then k satisfies one of the following:

(1) k is a geodesic of $H_{0,p}$;
(2) k is the image of a geodesic in $H_{0,p}$ under the map $T^B_{\gamma_2}$;
(3) k is an arbitrarily chosen geodesic with endpoints β and $T^B_{\gamma_2}(\beta)$ in $\mathcal{C}(S\setminus\gamma_1)$, or a geodesic with endpoints v_1 and γ_1 in $\mathcal{C}(S\setminus\beta)$ or an image of such a geodesic under $T^B_{\gamma_2}$;

Note that (3) describes three possible types of geodesics, each type determined by the endpoints: either β and $T^B_{\gamma_2}(\beta)$, v_1 and γ_1, or $T^B_{\gamma_2}(v_1)$ and γ_1; these three geodesics are represented by the three red pin-wheels in Figure 8.3. We require that there is exactly one such geodesic of $\tilde{H}^{(2)}$ for each of these three types.

Let $\mathbf{D}(H_{0,p})$ denote the set of domains supporting tight geodesics of $H_{0,p}$. To check that $\tilde{H}^{(2)}$ is a partial hierarchy, we now show that no domain appears more than once in the list

$$\left\{\mathbf{D}(H_{0,p}), T^B_{\gamma_2}(\mathbf{D}(H_{0,p}))\right\}.$$

If there is a repetition, then there is some $Y, W \in \mathbf{D}(H_{0,p})$ such that

$$T^B_{\gamma_2}(Y) = W.$$

Any domain Y of $H_{0,p}$ must be disjoint from some vertex of $g(p)$. Similarly, $T^B_{\gamma_2}(Y)$ must be disjoint from some vertex of $\tilde{g}(p)$ which is the image of an original vertex of $g(p)$. Suppose that Y is disjoint from either α, v_0, or v_1. Note that the distance in the full curve complex between v_1 and the image of any domain of $H_{0,p}$ is at least 2, and therefore if Y is disjoint from v_1 its distance to the image of any domain must be at least 1, so in particular it can not equal any such image. If Y is disjoint from either v_0 or α, then the distance between it and any image of a domain of $H_{0,p}$ is at least 2 and so the same argument applies. Finally, if Y is disjoint from β, it must also be disjoint from some other vertex of $g(p)$ since β is not a central vertex of any pin-wheel in $H_{0,p}$ (see Figure 8.3– β is the central vertex of a red geodesic, which does not appear as an original geodesic of $H_{0,p}$), and so again the same argument applies.

Therefore $\tilde{H}^{(2)}$ is indeed a partial hierarchy, and by construction its total length is more than twice that of $H_{0,p}$. We now show that for any $Y \subset S$,

$$d_Y(\alpha, T^B_{\gamma_2}(\alpha)) \leq 3K(p - \omega(Y)) + 2\max(E, 10, M).$$

We first prove the inequality in the case that Y is a domain of $\tilde{H}^{(2)}$. If the geodesic k supported on Y is a geodesic of $H_{0,p}$ (or an image of such a geodesic under $T^B_{\gamma_2}$), then by the large link lemma, Lemma 3.4, and the construction of $H_{0,p}$, k has length at most $2K(p, Y)$; applying Lemma 3.4 once more yields the desired result.

If k is the geodesic of $\tilde{H}^{(2)}$ supported on $S \setminus \gamma_1$ connecting β to $T^B_{\gamma_2}(\beta)$, or the geodesic in $\mathcal{C}(S \setminus \beta)$, note that in the construction of a 1-tight extension, the quantity $d_{S\setminus\beta}(v_1, \gamma_1)$ can be bounded from above by choosing γ_1 appropriately. Namely, we can choose γ_1 such that

$$d_{S\setminus\beta}(\gamma_1, \alpha) < 2\max(5, M).$$

Since v_0 and v_1 both project to $S \setminus \beta$, it follows that

$$d_{S\setminus\beta}(v_1, \gamma_1) \leq 2(1 + \max(5, M)).$$

A similar argument works for bounding $d_{S\setminus\gamma_1}(\beta, \gamma_2)$, and therefore we can also bound the length of the geodesic supported on $S \setminus \gamma_1$, and so we obtain

$$\begin{aligned} d_{S\setminus\gamma_1}(\beta, T^B_{\gamma_2}(\beta)) &\leq d_{S\setminus\gamma_1}(\beta, \gamma_2) + d_{S\setminus\gamma_1}(\gamma_2, T^B_{\gamma_2}(\beta)) \\ &\leq 2d_{S\setminus\gamma_1}(\beta, \gamma_2). \end{aligned}$$

Next, suppose that Y is not a domain of $\tilde{H}^{(2)}$. We consider the following three possibilities:

(1) γ_1 and β project to Y;
(2) γ_1 projects but β does not;
(3) γ_1 does not project to Y.

First observe that in all cases, we can assume that γ_2 projects, for if not, the projection of $T^B_{\gamma_2}(\alpha)$ to Y will have arcs in common with the projection of α to Y.

In case (1), note that

$$\begin{aligned} d_Y(\alpha, \gamma_1) &\leq d_Y(\alpha, \beta) + d_Y(\beta, \gamma_1) \\ &\leq K(p - \omega(Y)) + 1. \end{aligned}$$

Therefore by the triangle inequality and the fact that γ_1 is fixed by $T^B_{\gamma_2}$,

$$d_Y(\alpha, T^B_{\gamma_2}(\alpha)) \leq 2K(p - \omega(Y)) + 2.$$

In case (2), $Y \subseteq S \setminus \beta$, and therefore using Remark 8.14,

$$d_Y(\alpha, \gamma_1) < \max(E, 10, 2M),$$

and the desired bound follows since γ_1 is disjoint from γ_2. In case (3), Y is in the complement of γ_1, so by applying Remark 8.14 to the second stage of the right-tight construction, we obtain

$$d_Y(\alpha, \gamma_2) < \max(E, 10, 2M),$$

and again we have the desired inequality by applying the triangle inequality. Thus, all projections are bounded; we can then complete the partial hierarchy to a hierarchy $H^{(2)}$.

We next observe that this process can be repeated indefinitely; that is, we inductively define a partial $\tilde{H}^{(k)}$, with completion $H^{(k)}$, as follows:

Let z_1, z_2 be the two right-most endpoints of a right 2-tight extension of the main geodesic g of $\tilde{H}^{(k-1)}$; let z denote the right-most endpoint of g. Then define $\tilde{H}^{(k)}$ to be the partial hierarchy whose geodesics h satisfy one of the following:

(1) h is a geodesic of $\tilde{H}^{(k-1)}$;
(2) h is an image of a geodesic of $\tilde{H}^{(k-1)}$ under the map T^B_z;
(3) h is a geodesic supported on $S \setminus z$ or on $S \setminus z_1$.

By the same argument used for $\tilde{H}^{(2)}$, $\tilde{H}^{(k)}$ is a partial hierarchy, whose length is at least twice that of $\tilde{H}^{(k-1)}$. Furthermore, we claim that for any subsurface Y,

$$d_Y(\alpha, T^B_z(\alpha)) < 3K(p - \omega(Y)) + 2\max(E, 10, M),$$

whenever it is defined.

To show this, first suppose that Y is a domain of $H^{(j)}$ for some $j < k$. Then either Y is a domain of the partial hierarchy $\tilde{H}^{(j)}$, or Y is a domain that must be added to the partial hierarchy to complete it. In either case, it follows from the process by which a partial hierarchy is completed to a hierarchy that Y must also be a domain of $H^{(j+1)}$. Moreover, the geodesic supported on Y does not change as j increases.

Either Y is a domain of $H_{0,p}$, or there exists some $j < k$ such that Y is a domain of $H^{(j)}$ but Y is not a domain of $H^{(j-1)}$. If Y is a domain of $H_{0,p}$, then the geodesic supported on Y is at most $2K(p, Y)$, and therefore by Lemma 3.4,

$$d_Y(\alpha, T_z^B(\alpha)) < 3K(p - \omega(Y)) + 1.$$

If, on the other hand, there exists j such that Y is not a domain of $H^{(j-1)}$, let $z_1^{(j-1)}$ denote the last vertex of the main geodesic of $\tilde{H}^{(j-1)}$, and let $z_2^{(j-1)}, z^{(j-1)}$ denote the pennultimate and final vertex, respectively, of the right 2-tight extension of the main geodesic of $\tilde{H}^{(j-1)}$. Then at least one of the following holds:

(1) $z_1^{(j-1)}$ and $z_2^{(j-1)}$ project to Y;
(2) $z_2^{(j-1)}$ projects to Y but $z_1^{(j-1)}$ does not;
(3) $z_2^{(j-1)}$ does not project to Y.

The argument is similar to the argument used above to bound the projection to a subsurface Y between α and γ_2 when Y is not a domain of $\tilde{H}^{(2)}$: In case (1), by Lemma 3.4, since Y is not a domain of $H^{(j-1)}$ it follows that

$$d_Y(\alpha, z_1^{(j-1)}) < K(p - \omega(Y))$$
$$\Rightarrow d_Y(\alpha, T^B_{z_2^{(j-1)}}(\alpha)) \leq K(p - \omega(Y)) + 1.$$

Therefore, the geodesic of $H^{(j)}$ supported on Y has length at most $2K(p-\omega(Y))+1$. Thus by applying the large link lemma and noting that the geodesic of $H^{(k)}$ supported on Y has the same length as the geodesic of $H^{(j)}$ supported on Y, we obtain

$$d_Y(\alpha, T_z^B(\alpha)) \leq 3K(p - \omega(Y)) + 1.$$

In case (2), $Y \subseteq S \setminus z_1^{(j-1)}$, and therefore using Remark 8.14,

$$d_Y(\alpha, z_2^{(j-1)}) < \max(E, 10, 2M).$$

Thus by the triangle inequality and the fact that $z_2^{(j-1)}$ is fixed by $T^B_{z^{(j-1)}}$, we have

$$d_Y(\alpha, T^B_{z^{(j-1)}}(\alpha)) \leq 2\max(E, 10, M),$$

and thus the geodesic supported on Y in $H^{(j)}$ has length at most $K(p - \omega(Y)) + 2\max(E, 10, M)$. Then the large link lemma implies

$$d_Y(\alpha, T_z^B(\alpha)) \leq 2K(p - \omega(Y)) + 2\max(E, 10, M).$$

The final case again follows by Remark 8.14 of the right-tight extension, the triangle inequality and the large link lemma and the argument is completely analogous to the third case for $\tilde{H}^{(2)}$ above.

It remains to bound the projection to a subsurface Y which is not a domain of $H^{(j)}$ for any $j < k$. Whether Y is a domain of $\tilde{H}^{(k)}$ or not, a completely analogous argument to the one used directly above implies the desired inequality.

This completes the construction.

References

1. Ian Agol, *Small 3-manifolds of large genus*, Geometriae Dedicata **102** (2003), no. 1, 53–64.
2. Tarik Aougab, *Uniform hyperbolicity of the graphs of curves*, Geometry & Topology **17** (2013), no. 5, 2855–2875.
3. Javier Aramayona, Cyril Lecuire, Hugo Parlier, Kenneth J Shackleton, et al., *Convexity of strata in diagonal pants graphs of surfaces*, Publicacions matemàtiques **57** (2013), no. 1, 219–237.
4. Jason A Behrstock, *Asymptotic geometry of the mapping class group and teichmüller space*, Geometry & Topology **10** (2006), no. 3, 1523–1578.
5. Lipman Bers, *Simultaneous uniformization*, Bull. Amer. Mat. Soc. **66** (1960), no. 2, 94–97.
6. Joan S Birman and William W Menasco, *The curve complex has dead ends*, Geometriae Dedicata **177** (2015), no. 1, 71–74.
7. Brian Bowditch, *Uniform hyperbolicity of the curve graphs*, Pacific Journal of Mathematics **269** (2014), no. 2, 269–280.
8. Brian H Bowditch, *Intersection numbers and the hyperbolicity of the curve complex*, Journal für die reine und angewandte Mathematik (Crelles Journal) **2006** (2006), no. 598, 105–129.
9. ———, *The ending lamination theorem*, preprint (2011).
10. Martin Bridgeman, *Average bending of convex pleated planes in hyperbolic three-space*, Inventiones mathematicae **132** (1998), no. 2, 381–391.
11. Jeffrey Brock, *The weil-petersson metric and volumes of 3-dimensional hyperbolic convex cores*, Journal of the American Mathematical Society **16** (2003), no. 3, 495–535.
12. Jeffrey Brock, *Weil-petersson translation distance and volumes of mapping tori*, Comm. Anal. Geom. **11** (2003), 987–999.
13. Jeffrey Brock and Kenneth Bromberg, *Geometric inflexibility and 3-manifolds that fiber over the circle*, Journal of Topology (2011), jtq032.
14. ———, *Inflexibility, weil-petersson distance, and volumes of fibered 3-manifolds*, Math. Res. Lett. (to appear) (2014).
15. Peter Buser, Mika Seppälä, et al., *Symmetric pants decompositions of riemann surfaces*, Duke Math. J **67** (1992), no. 1, 39–55.
16. Matthieu Calvez, *Dual garside structure and reducibility of braids*, Journal of Algebra **356** (2012), no. 1, 355–373.
17. Richard D Canary, *A covering theorem for hyperbolic 3-manifolds and its applications*, Topology **35** (1996), no. 3, 751–778.
18. William Cavendish and Hugo Parlier, *Growth of the weil–petersson diameter of moduli space*, Duke Mathematical Journal **161** (2012), no. 1, 139–171.
19. Matt Clay, Kasra Rafi, and Saul Schleimer, *Uniform hyperbolicity of the curve graph via surgery sequences*, Algebraic & Geometric Topology **14** (2015), no. 6, 3325–3344.
20. Benson Farb and Dan Margalit, *A primer on mapping class groups (pms-49)*, Princeton University Press, 2011.
21. ———, *A primer on mapping class groups (pms-49)*, Princeton University Press, 2011.
22. Vaibhav Gadre and Chia-Yen Tsai, *Minimal pseudo-anosov translation lengths on the complex of curves*, Geometry & Topology **15** (2011), no. 3, 1297–1312.
23. É. Ghys and P. de la Harpe (eds.), *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics, vol. 83, Birkhäuser Boston, Inc., Boston, MA, 1990, Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988. MR 1086648 (92f:53050)
24. Mikhael Gromov, *Hyperbolic groups*, Springer, 1987.
25. Sebastian Hensel, Piotr Przytycki, and Richard CH Webb, *Slim unicorns and uniform hyperbolicity for arc graphs and curve graphs*, arXiv preprint arXiv:1301.5577 (2013).
26. Nikolai V Ivanov, *Automorphism of complexes of curves and of teichmüller spaces*, International Mathematics Research Notices **1997** (1997), no. 14, 651–666.

27. Mustafa Korkmaz, *Automorphisms of complexes of curves on punctured spheres and on punctured tori*, Topology Appl. **95** (1999), no. 2, 85–111. MR 1696431
28. Feng Luo, *Automorphisms of the complex of curves*, Topology **39** (2000), no. 2, 283–298.
29. Johanna Mangahas, *A recipe for short-word pseudo-anosovs*, American Journal of Mathematics **135** (2013), no. 4, 1087–1116.
30. Dan Margalit, *Automorphisms of the pants complex*, Duke Mathematical Journal **121** (2004), no. 3, 457–480.
31. Howard A Masur and Yair N Minsky, *Geometry of the complex of curves i: Hyperbolicity*, Inventiones mathematicae **138** (1999), no. 1, 103–149.
32. ———, *Geometry of the complex of curves ii: Hierarchical structure*, Geometric and Functional Analysis **10** (2000), no. 4, 902–974.
33. Yair Minsky, *The classification of kleinian surface groups, i: Models and bounds*, Annals of Mathematics (2010), 1–107.
34. Yair N Minsky, *Teichmüller geodesics and ends of hyperbolic 3-manifolds*, Topology **32** (1993), no. 3, 625–647.
35. ———, *Kleinian groups and the complex of curves*, Geometry & Topology **4** (2000), no. 1, 117–148.
36. Robert C Penner, *A construction of pseudo-anosov homeomorphisms*, Transactions of the American Mathematical Society **310** (1988), no. 1, 179–197.
37. Kasra Rafi and Saul Schleimer, *Covers and the curve complex*, Geometry & Topology **13** (2009), no. 4, 2141–2162.
38. Kasra Rafi, Jing Tao, et al., *The diameter of the thick part of moduli space and simultaneous whitehead moves*, Duke Mathematical Journal **162** (2013), no. 10, 1833–1876.
39. Jean-Marc Schlenker, *The renormalized volume and the volume of the convex core of quasifuchsian manifolds*, Math. Res. Lett. **20** (2013), no. 4, 773–786.
40. Robert Tang, *The curve complex and covers via hyperbolic 3-manifolds*, Geometriae Dedicata **161** (2012), no. 1, 233–237.
41. Samuel J Taylor and Alexander Zupan, *Products of farey graphs are totally geodesic in the pants graph*, J. Topol. Anal. (to appear) (2013).
42. William P Thurston, *Geometry and topology of 3-manifolds*, Princeton lecture notes (1979).
43. Richard CH Webb, *Uniform bounds for bounded geodesic image theorems*, Journal für die reine und angewandte Mathematik (Crelles Journal) (2013).

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