On the adjoint representation of Hopf algebras

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Outline

1 Background and motivation

2 The Hopf annihilator of the adjoint representation

3 Conjugacy classes
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2 The Hopf annihilator of the adjoint representation

3 Conjugacy classes
Outline

1. Background and motivation

2. The Hopf annihilator of the adjoint representation

3. Conjugacy classes
Throughout the talk we will use the following notation.

- $\mathbb{K}$ will be a field with char $\mathbb{K} = p \geq 0$

- $(. )^* := \text{Hom}_\mathbb{K}(., \mathbb{K})$ will denote the $\mathbb{K}$-linear dual

- $G$ will denote a finite group

- $H$ will denote an arbitrary Hopf $\mathbb{K}$-algebra

- $(. )^+ := \ker \epsilon$ will denote the augmentation ideal
The adjoint representation of a group

A group $G$ acts on its self by conjugation.

$$gh = ghg^{-1} \quad (g, h \in G)$$

Extending $\mathbb{K}$-linearly gives an action of $\mathbb{K}G$ on itself.

**Definition.**
The group algebra equipped with this action will be called the *adjoint representation*, denoted $\text{ad} \mathbb{K}G$. 

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On the adjoint representation of Hopf algebras

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Motivation

Hopf annihilator

Conjugacy classes

The group picture

Theorem.

\[ \text{ad}_K G \text{ is completely reducible} \quad \leftrightarrow \quad G \text{ has a central Sylow } p\text{-subgroup} \]

\[ \text{ad}(K G / \text{rad } K G) \text{ is completely reducible} \quad \leftrightarrow \quad G \text{ has a normal Sylow } p\text{-subgroup} \]

Definition.

A module \( V \) has the Chevalley property if \( T(V) := \bigoplus_{n \in \mathbb{N}} V \otimes_n \) is completely reducible.
The group picture

Theorem.

- \( \text{ad}_K G \) is completely reducible
- \( \text{ad}(K^G/ \text{rad } K^G) \) is completely reducible
- \( G \) has a central Sylow \( p \)-subgroup
- \( G \) has a normal Sylow \( p \)-subgroup

Definition.

A module \( V \) has the Chevalley property if \( T(V) := \bigoplus_{n \in \mathbb{N}} V^\otimes n \) is completely reducible.
The group picture

Theorem.

<table>
<thead>
<tr>
<th>ad(\mathbb{K}G) is completely reducible</th>
<th>ad(\mathbb{K}G) has the Chevalley property</th>
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<tr>
<td>ad((\mathbb{K}G/ \text{rad} \mathbb{K}G)) is completely reducible</td>
<td>Completely reducible modules have the Chevalley property</td>
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Definition.

A module \(V\) has the *Chevalley property* if \(T(V) := \bigoplus_{n \in \mathbb{N}} V^{\otimes n}\) is completely reducible.
The group picture

**Theorem.**

\[
\begin{align*}
\text{ad} K G & \text{ is completely reducible} \\
\text{ad}(K G / \text{rad} K G) & \text{ is completely reducible} \\
\text{ad} K G & \text{ has the Chevalley property} \\
\text{Complete reducible modules have the Chevalley property}
\end{align*}
\]

**Classification Theorem**

Michler 86

**Definition.**

A module \( V \) has the *Chevalley property* if

\[
T(V) := \bigoplus_{n \in \mathbb{N}} V \otimes^n \text{ is completely reducible.}
\]
Components of the proof of the top implication

Sketch of proof

1. The largest Hopf ideal of $KG$ that annihilates $\text{ad}KG$ is:

$$KG(KZ(G))^+$$
Components of the proof of the top implication

Sketch of proof

1. The largest Hopf ideal of $\mathbb{K}G$ that annihilates $\text{ad} \mathbb{K}G$ is:
   
   $\mathbb{K}G(\mathbb{K}Z(G))^+$

2. $\text{ad} \mathbb{K}G$ completely reducible implies $p$ does not divide the order of any conjugacy class
Components of the proof of the top implication

**Sketch of proof**

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2. $\text{ad}\mathbb{K}G$ completely reducible implies $p$ does not divide the order of any conjugacy class.

3. (2) implies $p$ does not divide $|G/Z(G)|$.
Components of the proof of the top implication

Sketch of proof

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4. (3) implies $\text{ad}\mathbb{K}G$ has the Chevalley property
## Components of the proof of the top implication

**Sketch of proof**

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4. (3) implies $\text{ad}\mathbb{K}G$ has the Chevalley property
A Hopf algebra $H$ acts on its self via the adjoint action.

$$hk = h_{(1)} k S(h_{(2)}) \quad (h, k \in H)$$

**Definition**

The Hopf algebra equipped with this action will be called the *adjoint representation*, denoted $\text{ad} H$. 

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- For $I \leq H$ an ideal, $H/I$ will denote the largest Hopf ideal contained in $I$. 
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The Hopf algebra equipped with this action will be called the *adjoint representation*, denoted $\text{ad} H$.

- For $I \leq H$ an ideal, $\mathcal{H} I$ will denote the largest Hopf ideal contained in $I$.
- For $A \subseteq H$ a subalgebra, $\mathcal{H} A$ will denote the largest Hopf subalgebra contained in $A$. 
A Hopf algebra $H$ acts on its self via the adjoint action.

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The Hopf algebra equipped with this action will be called the *adjoint representation*, denoted $\text{ad}H$.

- For $I \leq H$ an ideal, $\mathcal{H}I$ will denote the largest Hopf ideal contained in $I$.
- For $A \subseteq H$ a subalgebra, $\mathcal{H}A$ will denote the largest Hopf subalgebra contained in $A$.
- Let $\zeta(H)$ denoted the largest Hopf subalgebra contained in the center of $H$. 
The Hopf annihilator of the adjoint representation

Motivation

Theorem 1. (J.)

Let $H$ be a Hopf algebra that satisfies one of the following conditions:

1. $H$ is finite-dimensional or
2. the coradical of $H$ is cocommutative (e.g., $H$ is cocommutative or pointed).

Then the Hopf annihilator of the adjoint representation is given by $\mathcal{H}(\text{ann}^{\text{ad}}H) = H\zeta(H)^+$. 
The proof: Coinvariants I

- Let $\overline{H} = H/\mathcal{H}(\text{ann} \, \text{ad} \, H)$
The proof: Coinvariants I

• Let $\overline{H} = H / H(\text{ann}^{\text{ad}} H)$

• $H$ becomes a left $\overline{H}$-comodule via

$(\overline{\cdot} \otimes \text{Id}) \circ \Delta : H \to \overline{H} \otimes H$ i.e. $h \mapsto \overline{h}(1) \otimes h(2)$
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- Let $\overline{H} = H / \mathcal{H}(\text{ann} \, \text{ad} \, H)$

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- Let $\text{co} \overline{H} H := \{ h \in H | \overline{h}(1) \otimes h(2) = 1 \otimes h \}$
The proof: Coinvariants I

- Let \( \overline{H} = H / \mathcal{H}(\text{ann}^{\text{ad}} H) \)

- \( H \) becomes a left \( \overline{H} \)-comodule via
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(\ - \otimes \text{Id} \) \circ \Delta : H \to \overline{H} \otimes H \text{ i.e. } h \mapsto \overline{h}(1) \otimes h(2)
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- Let \( ^{co\overline{H}}H := \{ h \in H | \overline{h}(1) \otimes h(2) = 1 \otimes h \} \)

- \( \zeta(H)^+ \subseteq \mathcal{H}(\text{ann}^{\text{ad}} H) \) since
\[
z h = z(1) h S(z(2)) = z(1) S(z(2)) h = \epsilon(z) h = 0
\]
The proof: Coinvariants I

- Let $\overline{H} = H / \mathcal{H}(\text{ann} \ ad \ H)$

- $H$ becomes a left $\overline{H}$-comodule via $(\ - \otimes \text{Id} ) \circ \Delta : H \rightarrow \overline{H} \otimes H \ \text{i.e.} \ h \mapsto \overline{h}(1) \otimes h(2)$

- Let $\text{co}^{\overline{H}}H := \{ h \in H | \overline{h}(1) \otimes h(2) = 1 \otimes h\}$

- $\zeta(H)^+ \subseteq \mathcal{H}(\text{ann} \ ad \ H)$ since 
  $z \ h = z(1) h S(z(2)) = z(1) S(z(2)) h = \epsilon(z) h = 0$

- Now $\zeta(H) \subseteq \text{co}^{\overline{H}}H$ since 
  $\overline{Z}(1) \otimes Z(2) = (Z(1) - \epsilon(Z(1))1 + \epsilon(Z(1))1) \otimes Z(2)$
  $= \epsilon(Z(1))1 \otimes Z(2) = 1 \otimes z$
The proof: Coinvariants II

- $co^H H \subseteq \mathcal{E}(H)$ since

$$ch = c_{(1)} h \epsilon(c_{(2)}) = c_{(1)} h S(c_{(2)}) c_{(3)} = c_{(1)} hc_{(2)} = \overline{c}_{(1)} hc_{(2)} = \overline{T} hc = hc$$
The proof: Coinvariants II

- \( \text{co}H \subseteq \mathcal{L}(H) \) since

\[
ch = c_{(1)} h\epsilon(c_{(2)}) = c_{(1)} hS(c_{(2)})c_{(3)} = c_{(1)} hc_{(2)} = \overline{c}_{(1)} hc_{(2)} = 1hc = hc
\]

- \( \text{co}H \) is a right subcomodule of \( H \), thus:

\[
\Delta(\text{co}H) \subseteq \text{co}H \otimes H \subseteq \mathcal{L}(H) \otimes H
\]
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Theorem. (Chirvasitu, Kasprzak. preprint)

\[
\zeta(H) = \{ h \in H | \Delta(h) \in \mathcal{L}(H) \otimes H \}
\]
The proof: Coinvariants II

- \( coH \subseteq \mathcal{L}(H) \) since

\[
ch = c(1)h\epsilon(c(2)) = c(1)hS(c(2))c(3)
\]

\[
= c(1)hc(2) = \bar{c}(1)hc(2) = \bar{1}hc = hc
\]

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\]

**Theorem.** (Chirvasitu, Kasprzak. preprint)

\[
\zeta(H) = \{ h \in H | \Delta(h) \in \mathcal{L}(H) \otimes H \}
\]

- Giving \( coH \subseteq \zeta(H) \) and so \( coH = \zeta(H) \)
Recall the assumption of Theorem 1 that:

1. \( H \) is finite-dimensional or
2. the coradical of \( H \) is cocommutative.
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Either imply $H$ is a faithfully coflat $\overline{H}$-comodule. Thus:
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Either imply \( H \) is a faithfully coflat \( \overline{H} \)-comodule. Thus:

- \( H \) is a faithfully flat \( \zeta(H) \)-module
- \( H \) is a faithfully coflat \( H/H\zeta(H)^+ \)-comodule
The proof: an equivalence

**Theorem.** (Takeuchi 79)

We have the following inverse maps:

\[
\begin{align*}
\{ & A \mid \text{a left } H\text{-comodule algebra} \\
& H \text{ faithfully flat over } A \} \quad \overset{\text{coH}/I}{\leftrightarrow} \quad \{ & I \mid \text{I left } H\text{-module coideal} \\
& H \text{ faithfully coflat over } H/I \} 
\end{align*}
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The proof: an equivalence

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& \{ I \mid \text{I left } H\text{-module coideal} \\
& \text{H faithfully coflat over } H/I \} \\
\end{align*}
\]

The result follows from the diagram below:

\[
\begin{array}{ccc}
\zeta(H) & \xleftarrow{\quad} & H_{\zeta}(H)^+ \\
& \parallel & \\
& \quad \xrightarrow{\quad} & H(ann^{ad}H)
\end{array}
\]
Consequences

For the remainder assume $H$ is finite-dimensional.
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**Theorem.** (Rieffel 67)

For $V$ an $H$-module:

$$\text{ann } T(V) = \mathcal{H}(\text{ann } V)$$

Thus $V$ has the Chevalley property iff $H/(\mathcal{H} \text{ ann } V)$ is semisimple.
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**Corollary 1.** (J.)

$\text{ad } H$ has the Chevalley property iff $H/H\zeta(H)^+$ is semisimple.
Consequences

For the remainder assume $H$ is finite-dimensional.

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**Corollary 1.** (J.)

$\text{ad } H$ has the Chevalley property iff $H / H\zeta(H)^+$ is semisimple.

**Corollary 2.** (J.)

$\text{ad } H$ has the Chevalley property implies $H$ is unimodular.
Now $\text{ad} H$ can be viewed as a right $H$-comodule with structure map $\Delta$. With this $\text{ad} H$ becomes a Yetter-Drinfeld module, thus it is natural to consider the Drinfeld double.
Review: Drinfeld double

Now $\text{ad} H$ can be viewed as a right $H$-comodule with structure map $\Delta$. With this $\text{ad} H$ becomes a Yetter-Drinfeld module, thus it is natural to consider the Drinfeld double.

Definition.

The Drinfeld double of $H$ is the Hopf algebra $D(H)$. The coalgebra structure of $D(H)$ is given by:

$$D(H)^{\text{coalg}} \cong H^{*\text{cop}} \otimes H$$

The element $f \otimes h$ is denoted $f \triangleright \triangleleft h$. The multiplication is given by:

$$(f \triangleright h)(g \triangleright k) = f(h(1) \rightarrow g \leftarrow S^{-1}(h(3))) \triangleright h(2)k$$
Conjugacy class definition

The Drinfeld double acts on $H$ via the action below:

$$(f \Join h).k = (^h k) \leftarrow S^{-1}(f) \quad (f \in H^* h, k \in H)$$
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$$(f \otimes h).k = (h^k) \leftarrow S^{-1}(f) \quad (f \in H^* h, k \in H)$$

**Definition.** (Cohen, Westreich 2010)

If $H$ is a completely reducible $D(H)$-module then we say a **conjugacy class** is a simple $D(H)$-submodule of $H$. 
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**Definition.** (Cohen, Westreich 2010)

If $H$ is a completely reducible $D(H)$-module then we say a *conjugacy class* is a simple $D(H)$-submodule of $H$.

**Example: group algebras**

The action of $D(\mathbb{K}G)$ on $\mathbb{K}G$ is completely reducible. The conjugacy classes, as defined above, are the modules arising from $D(\mathbb{K}G)$ acting as above on the $\mathbb{K}$-span of the classical conjugacy classes.
Results on conjugacy classes

Proposition 1. (J.)

For $H$ a finite-dimensional Hopf algebra:

1. If $H$ is a completely reducible $D(H)$-module then $H$ is cosemisimple.
2. If $H$ is cosemisimple and $\text{ad} H$ is a completely reducible then $H$ is a completely reducible $D(H)$-module.
Results on conjugacy classes

Proposition 1. (J.)

For $H$ a finite-dimensional Hopf algebra:

1. If $H$ is a completely reducible $D(H)$-module then $H$ is cosemisimple.
2. If $H$ is cosemisimple and $\text{ad} H$ is a completely reducible then $H$ is a completely reducible $D(H)$-module.

Theorem 3. (J.)

Let $H$ be a cosemisimple, involutory Hopf algebra with $\mathbb{K} = \overline{\mathbb{K}}$ then $\text{ad} H$ completely reducible implies char $\mathbb{K}$ does not divide the dimension of any of the conjugacy classes.


