Morita Equivalences, Morita contexts and Hammers

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Outline

1 Definitions

2 Examples

3 Properties and Uses

4 Why do I care?
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2 Examples

3 Properties and Uses

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2 Examples
3 Properties and Uses
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1 Definitions
2 Examples
3 Properties and Uses
4 Why do I care?
A *Morita equivalence* is a lot like a hammer in that it is a tool and not an end in itself. This talk will describe to you what a hammer is, give some examples of hammers, explain how to use a hammer, and show some of the cool stuff you can build with them.
A *Morita equivalence* is a lot like a hammer in that it is a tool and not an end in itself. This talk will describe to you what a hammer is, give some examples of hammers, explain how to use a hammer, and show some of the cool stuff you can build with them.

**Definition: Hammer (Noun)**

A *hammer* is a tool characterized by a hard roughly cylindrical head adhered orthogonally to the top of a long shaft.
Notation

- $K$ a field (Though much of what I say will work for $K$ a commutative ring)
- $A$ and $B$ $K$-algebras with unit
- $M$, $N$, $P$ and $Q$ will always be modules.
- A left subscript of a ring will denote a left module and a right subscript a right module (i.e. $A_N B$ means $M$ is an $(A, B)$ bimodule)
Morita Equivalence

A *Morita equivalence* is an ordered sextuplet 
\((A, B, \_A P_B, \_B Q_A, f: \_A P_B \otimes_B \_B Q_A \to A, g: \_B Q_A \otimes_A \_A P_B \to B)\)

Where \(f\) and \(g\) are respectively \((A - B)\) and \(B - B\) bimodule maps making the diagrams below commute.
A Morita equivalence is an ordered sextuplet 
\((A, B, A P_B, B Q_A, f : A P_B \otimes_B B Q_A \rightarrow A, g : B Q_A \otimes_A A P_B \rightarrow B)\)

Where \(f\) and \(g\) are respectively \((A - B)\) and \(B - B\) bimodule maps making the diagrams below commute.

\[
\begin{align*}
(A P_B \otimes_B B Q_A) \otimes_A A P_B & \xrightarrow{f \otimes \text{Id}_P} A \otimes_A A P_Q \\
A P_B \otimes_B (B Q_A \otimes_A A P_B) & \xrightarrow{\text{Id}_P \otimes g} A P_B \otimes_B B \\
A P_B \otimes_B B & \xrightarrow{\text{Id}_P} P
\end{align*}
\]
Main definition

Definition: Morita equivalent

A and B are Morita equivalent if there exists a sextuplet: 
\((A, B, A P_B, B Q_A, f : A P_B \otimes_B B Q_A \to A, g : B Q_A \otimes_A A P_B \to B)\)

\(f\) and \(g\) are respectively \((A - B)\) and \(B - B\) bimodule isomorphisms satisfying the below ”associativity” diagrams.
Category theory definition

**Definition: $R$-mod**

Given a ring $R$ we can construct a category $R$-$mod$.

- Objects are left $R$ modules.
- Morphisms are left $R$-linear maps.
Definition: $R$-mod

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$A$ and $B$ Morita equivalent if there is an equivalence of categories between $A$-mod and $B$-mod.
Definition: \( R \)-mod

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- Objects are left \( R \) modules.  
- Morphisms are left \( R \)-linear maps.

Definition: Morita equivalent

\( A \) and \( B \) Morita equivalent if there is an equivalence of categories between \( A \)-mod and \( B \)-mod.

Definition: Equivalence of categories

Two categories \( \mathcal{C} \) and \( \mathcal{D} \) are equivalent if there exists functors \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{C} \) such that \( FG \cong \text{Id}_D \) and \( GF \cong \text{Id}_C \).
Sketch of proof of equivalence

Ring theory definition $\Rightarrow$ Category theory definition
Sketch of proof of equivalence

Ring theory definition ⇒ Category theory definition

- Define $F : A\text{-mod} \rightarrow B\text{-mod}$ by $A M \mapsto B Q_A \otimes_A A M$
  $h \mapsto (id_Q \otimes h)$
- Define $G : B\text{-mod} \rightarrow A\text{-mod}$ by $B N \mapsto A P_B \otimes_B B N$
  $h \mapsto (id_P \otimes h)$
Sketch of proof of equivalence

Ring theory definition \(\Rightarrow\) Category theory definition

- Define \(F : \text{A-mod} \to \text{B-mod}\) by 
  \[A M \mapsto B Q_A \otimes_A A M\]
  \[h \mapsto (\text{id}_Q \otimes h)\]

- Define \(G : \text{B-mod} \to \text{A-mod}\) by 
  \[B N \mapsto A P_B \otimes_B B N\]
  \[h \mapsto (\text{id}_P \otimes h)\]

- The natural transformation \(\eta : FG \cong \text{Id}_{\text{B-mod}}\) is given by 
  \[\eta_N = g \otimes_B \text{Id}_N : FG(N) = (B Q_A \otimes_A A P_B) \otimes_B B N \to N.\]
Sketch of proof of equivalence

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- Define $F : \text{A-mod} \rightarrow \text{B-mod}$ by $A M \mapsto B Q_A \otimes_A A M$
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- The associativity diagrams give naturality.

- That $g$ is an isomorphism gives that this is a natural isomorphism.
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- Define $F : A\text{-mod} \rightarrow B\text{-mod}$ by $A M \mapsto B Q_A \otimes_A A M$
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- The associativity diagrams give naturality.

- That $g$ is an isomorphism gives that this is a natural isomorphism.

- The proof follows symmetrically that $GF \cong \text{Id}_{A\text{-mod}}$. 
Sketch of proof of equivalence

Category theory definition $\Rightarrow$ Ring theory definition

- Take $F : A\text{-mod} \to B\text{-mod}$ and $G : B\text{-mod} \to A\text{-mod}$ to be functors and $\text{eta} : GF \to \text{Id}_A$ and $\mu : FG \to \text{Id}_B$ the natural transformations.
Sketch of proof of equivalence

Category theory definition $\Rightarrow$ Ring theory definition

- Take $F : A\text{-mod} \rightarrow B\text{-mod}$ and $G : B\text{-mod} \rightarrow A\text{-mod}$ to be functors and $\eta : GF \rightarrow \text{Id}_A$ and $\mu : FG \rightarrow \text{Id}_B$ the natural transformations.
- Define $BQ_A := F(A)$. 
Sketch of proof of equivalence

Category theory definition $\Rightarrow$ Ring theory definition

- Take $F : \text{A-mod} \to \text{B-mod}$ and $G : \text{B-mod} \to \text{A-mod}$ to be functors and $\eta : GF \to \text{Id}_A$ and $\mu : FG \to \text{Id}_B$ the natural transformations.
- Define $BQ_A := F(A)$. To get the right $A$-module structure on $F(A)$ $F$ is fully faithful gives the bijection below $A \cong \text{Hom}_{A-mod}(A, A) \cong \text{Hom}_{B-mod}(F(A), F(A))$ explicitly $a$ acts as the function $F(r_a)$
Sketch of proof of equivalence

Category theory definition $\Rightarrow$ Ring theory definition

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- We like likewise define $AP_B := G(B)$. 
Sketch of proof of equivalence

Category theory definition $\Rightarrow$ Ring theory definition

- Take $F : \text{A-mod} \to \text{B-mod}$ and $G : \text{B-mod} \to \text{A-mod}$ to be functors and $\eta : GF \to \text{Id}_\text{A}$ and $\mu : FG \to \text{Id}_\text{B}$ the natural transformations.

- Define $BQ_A := F(A)$. To get the right $\text{A-module}$ structure on $F(A)$ $F$ is fully faithful gives the bijection below $A \cong \text{Hom}_{\text{A-mod}}(A, A) \cong \text{Hom}_{\text{B-mod}}(F(A), F(A))$ explicitly $a$ acts as the function $F(r_a)$

- We like likewise define $AP_B := G(B)$. Now $f$ and $g$ can be defined as $\eta_A$ and $\mu_B$. 
Outline

1 Definitions

2 Examples

3 Properties and Uses

4 Why do I care?
# Examples of Hammers

## Examples

- Standard hammer
- Sledge hammer
- War Hammer

## Non-examples

- Warhammer 40K
- M.C. Hammer
- Captain Hammer
- A hammer headed shark
Examples of Hammers

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Examples of Morita Equivalences

**Example**

A is Morita equivalent to $M_n(A)$. 

Proof: Categorical

- Define $F: \text{A-mod} \rightarrow M_n(\text{A})-\text{mod}$ by $N \mapsto N^n$ with the standard matrix action on a vector.
- Define $G: M_n(\text{A})-\text{mod} \rightarrow \text{A-mod}$ by $M \mapsto E_{1,1}M$.
- $\eta: \text{Id}_{\text{A-mod}} = GF$ is given by $\eta_M(m) = E_{1,1}(m, m, \ldots, m)^T$.
- $\mu: \text{Id}_{M_n(\text{A})-\text{mod}} \rightarrow FG$ is given by $\eta_N(s) = (E_{1,1}s, E_{2,2}s, \ldots, E_{n,n}s)^T$. 

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Examples of Morita Equivalences

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Morita Context

Some times a hammer is to rigid and one need a softer tool (A rubber mallet if you will). For a Morita equivalence the equivalent of a rubber mallet is a Morita context.
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**Definition: Morita context**

A *Morita context* between $A$ and $B$ if there is an sextuplet $(A, B, _AP_B, _BQ_A, f : _AP_B \otimes_B _BQ_A \to A, g : _BQ_A \otimes_A _AP_B \to B)$ Satisfying all the condition of a Morita equivalence expect $f$ and $g$ are no longer required to be isomorphisms.
Morita context first example

Morita context example

Take $V \in A\text{-}mod$ then there is a Morita context between $A$ and $B := \text{Hom}_A(V, V)$.
Morita context first example

Morita context example

Take $V \in A\text{-mod}$ then there is a Morita context between $A$ and $B := \text{Hom}_A(V, V)$

Sketch of Proof

- $V$ is a $(A - B)$ bimodule where $v \cdot h = h(v)$
Morita context first example

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Take $V \in A\text{-mod}$ then there is a Morita context between $A$ and $B := \text{Hom}_A(V, V)$

Sketch of Proof

• $V$ is a $(A - B)$ bimodule where $v \cdot h = h(v)$
• $V^\wedge := \text{Hom}_A(V, A)$ this has bimodule structure given by $< h \mapsto \phi \leftarrow a, v > = < \phi, h(a \cdot v) >$. 

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Morita context first example

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- $f : V \otimes_B V^* \to A$ is given by $g(v \otimes_B \phi) = \phi(v)$
- $g : V^* \otimes V \to B$ is given by $\phi \otimes_A V = w \mapsto \phi(w).v$
Morita context second example

Idempotents

Take $e$ and idempotent in $A$ (i.e. $ee = e$) then $eAe$ is a ring and there is a Morita context between $A$ and $eAe$. If we also have $AeA = A$ then this is a Morita equivalence.
Morita context second example

**Idempotents**

Take $e$ and idempotent in $A$ (i.e. $ee = e$) then $eAe$ is a ring and there is a Morita context between $A$ and $eAe$. If we also have $AeA = A$ then this is a Morita equivalence.

**Sketch of Proof**

- $AP_{eAe} = Ae$ and $eAe Q_A = eA$ with actions given by multiplication.
Idempotents

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Sketch of Proof

- $A_{P_{eAe}} = Ae$ and $eAe_{Q_A} = eA$ with actions given by multiplication.
- Define $f : Ae \otimes_{eAe} eA \rightarrow A$ by $aA \otimes_A be \mapsto aeb$ (Surjectivity of this map requires $AeA = A$)
Idempotents

Take $e$ and idempotent in $A$ (i.e. $ee = e$) then $eAe$ is a ring and there is a Morita context between $A$ and $eAe$. If we also have $AeA = A$ then this is a Morita equivalence.

Sketch of Proof

- $_{A}P_{eAe} = Ae$ and $_{eAe}Q_{A} = eA$ with actions given by multiplication.
- Define $f : Ae \otimes_{eAe} eA \to A$ by $ae \otimes_{A} be \mapsto aeb$ (Surjectivity of this map requires $AeA = A$)
- Define $g : eA \otimes_{A} Ae$ be $ea \otimes_{A} be \mapsto eabe$
- The associativity condition holds since multiplication is associative.
Outline

1. Definitions
2. Examples
3. Properties and Uses
4. Why do I care?
A hammer has many applications some are listed below

- Securing nails in wood
- Removing nails for wood
- Decomposing large objects into multiple smaller objects via the application of blunt force
- Self defense
- Using as a metaphor for a complex mathematical tool
What does a Morita equivalence preserve?

**Properties preserved**
- Simple
- Semisimple
- Left (right) Noetherian
- Left (right) Artinian
- Prime
- Semiprime
- Primitive
- Semiprimitive

**Definition**
A is simple if $A$ has no two sided ideals.
What does a Morita equivalence preserve?

<table>
<thead>
<tr>
<th>Properties preserved</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple</td>
<td>$A$ is semisimple if every $A$-module is projective.</td>
</tr>
<tr>
<td><strong>Semisimple</strong></td>
<td></td>
</tr>
<tr>
<td>left (right) Noetherian</td>
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<tr>
<td>left (right) Artinian</td>
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<td>Prime</td>
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What does a Morita equivalence preserve?

Properties preserved
- Simple
- Semisimple
- left (right) Noetherian
- left (right) Artinian
- Prime
- Semiprime
- Primitive
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Definition
A is left (right) Noetherian if every ascending chain of left (right) ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq ...$ stabilizes after finitely many steps.
What does a Morita equivalence preserve?

### Properties preserved

- Simple
- Semisimple
- Left (right) Noetherian
- Left (right) Artinian
- Prime
- Semiprime
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### Definition

A is left (right) Artinian if every descending chain of left (right) ideals $I_1 \supseteq I_2 \supseteq I_3 \supseteq ...$ stabilizes after finitely many steps.
What does a Morita equivalence preserve?

**Properties preserved**
- Simple
- Semisimple
- left (right) Noetherian
- left (right) Artinian
- Prime
- Semiprime
- Primitive
- Semiprimitive

**Definition**

A is prime if the zero idea is prime i.e. for $I$, $J$ ideals in $A$ $IJ = 0 \implies I = 0$ or $J = 0$
What does a Morita equivalence preserve?

**Properties preserved**
- Simple
- Semisimple
- Left (right) Noetherian
- Left (right) Artinian
- Prime
- Semiprime
- Primitive
- Semiprimitive

**Definition**

A is semiprime if the zero ideal is semiprime i.e. 
\[ J^k = 0 \Rightarrow J = 0. \]
What does a Morita equivalence preserve?

**Properties preserved**
- Simple
- Semisimple
- left (right) Noetherian
- left (right) Artinian
- Prime
- Semiprime
- Primitive
- Semiprimitive

**Definition**
A is called primitive if the zero ideal is annihilator of a left module i.e. \( A \) has a faithful left module.

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What does a Morita equivalence preserve?

**Properties preserved**
- Simple
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- Left (right) Artinian
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- Semiprimitive

**Definition**

A is semiprimitive if the Jacobson radical is 0. Recall the Jacobson radical of $A$ is the two sided idea that annihilates all simple left (equivalently right) modules.
What does a Morita equivalence preserve?

**Properties not preserved**
- Commutative
- Reduced
- Goldie
- Frobenius

**Definition**
I hope you know this one on your own.
### Properties not preserved

- Commutative
- Reduced
- Goldie
- Frobenius

### Definition

A is reduced if it has no zero divisors.
What does a Morita equivalence preserve?

**Properties not preserved**
- Commutative
- Reduced
- Goldie
- Frobenius

**Definition**
A is Goldie if it has finite uniform dimension and satisfies the ACC on left (right) annihilator subsets.
What does a Morita equivalence preserve?

**Properties not preserved**
- Commutative
- Reduced
- Goldie
- Frobenius

**Definition**
A is Frobenius if $A \cong A^*$ as left (equivalently right) $A$ modules. Here $\langle a \leftarrow f, b \rangle = \langle f, ba \rangle$. 

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4 Why do I care?
Hammers have benefited my life in many ways some are listed below.

- Construction of shelter
- Stress relief
- Adhering items to walls for aesthetic purposes
- Paper weight
Why do I care about hammers?

Hammers have benefited my life in many ways some are listed below.

- Construction of shelter
- Stress relief
- Adhering items to walls for aesthetic purposes
- Paper weight

As useful as hammers may be they are of little use in acquiring a PhD in math (In fact the recent construction has shown them to be quite the opposite) fortunately Morita contexts are a better tool in this regard.
Background

Take $G$ a finite subset of $\text{Aut}(A)$. Then $\mathbb{K}G$ acts on $A$ in the natural way.
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Take $G$ a finite subset of $\text{Aut}(A)$. Then $\mathbb{K}G$ acts on $A$ in the natural way.

**Definition: Invariants**

The $G$ invariants of $A$ will be denoted $A^{\mathbb{K}G} := \{a \in A | g(a) = a \forall g \in G\}$
Take $G$ a finite subset of Aut($A$). Then $\mathbb{K}G$ acts on $A$ in the natural way.

**Definition: Invariants**

The $G$ invariants of $A$ will be denoted $A^G := \{ a \in A | g(a) = a \forall g \in G \}$

**Definition: $A\#\mathbb{K}G$**

$A\#\mathbb{K}G$ is a ring which as a group is $A \otimes \mathbb{K}G$ with multiplication defined by $(a \# g)(b \# h) = ag \cdot b \# gh$. 
A more Hopfy example

Despite my best efforts I was unable to define a Hopf algebra structure on the set of all hammer so the parallels must sadly end here.
A more Hopf algebra example

**Group example**

There is a Morita context between $A^KG$ and $A#^KG$ (In fact we can replace $KG$ with any finite dimensional Hopf algebra and this statement remains true.)

**Structure of the context**

- Denote $e = \frac{1}{|G|}\sum_{g \in G} g$ it is easy to check that $e \in KG$ is an idempotent and $1#e \in A#H$ is an idempotent.
A more Hopfy example

Group example

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- $A^KG \cong e.A$
There is a Morita context between $A^\mathbb{K}G$ and $A\#\mathbb{K}G$ (In fact we can replace $\mathbb{K}G$ with any finite dimensional Hopf algebra and this statement remains true.)

Structure of the context

- Denote $e = \frac{1}{|G|} \sum_{g \in G} g$ it is easy to check that $e \in \mathbb{K}G$ is an idempotent and $1\#e \in A\#H$ is an idempotent.
- $A^\mathbb{K}G \cong e.A$
- $(1\#e)(A\#H)(1\#e) = (e.A\#e\mathbb{K}Ge) = A^H \#e \cong A^H$
A more Hopfy example

Group example

There is a Morita context between $A^\mathbb{K}G$ and $A^\#\mathbb{K}G$ (In fact we can replace $\mathbb{K}G$ with any finite dimensional Hopf algebra and this statement remains true.)

Structure of the context

- Denote $e = \frac{1}{|G|} \sum_{g \in G} g$ it is easy to check that $e \in \mathbb{K}G$ is an idempotent and $1^\#e \in A^\#H$ is an idempotent.
- $A^\mathbb{K}G \cong e.A$
- $(1^\#e)(A^\#H)(1^\#e) = (e.A^\#e\mathbb{K}Ge) = A^H \#e \cong A^H$
- Thus the Morita context holds and is an equivalence if $(A^\#\mathbb{K}G)(1^\#e)(A^\#\mathbb{K}G) = A^\#\mathbb{K}G$
**Definition: H-Spec**

For $I$ an ideal in $A$ we say $I$ is an *$H$-prime ideal* or $I \in \text{H-Spec}(A)$ if $I$ is stable under the action of $KG$ and for $J_1, J_2$ $KG$ stable ideals then $J_1 J_2 \subseteq I \Rightarrow J_1 \subseteq I$ or $J_2 \subseteq I$. 
**Definition: H-Spec**

For \( I \) an ideal in \( A \) we say \( I \) is an \emph{H-prime ideal} or \( I \in H\text{-Spec}(A) \) if \( I \) is stable under the action of \( KG \) and for \( J_1 \) \( J_2 \) \( KG \) stable ideals then \( J_1 J_2 \subseteq I \Rightarrow J_1 \subseteq I \) or \( J_2 \subseteq I \).

**Theorem [3]**

For \( P \in \text{Spec}(A) \), \( I \in H\text{-Spec}(KG) \) and \( N \) the intersection of all prime ideals minimal over \( I \) the following facts are known.

1. \( P \) minimal over \( I \) \( \iff (P : H) = I \)
2. The number of primes minimal over \( I \) is bounded by \(|G|\)
3. \( N^{\dim_K(H)} \subseteq I \)
**Application**

**Definition: H-Spec**

For $I$ an ideal in $A$ we say $I$ is an *H-prime ideal* or $I \in \text{H-Spec}(A)$ if $I$ is stable under the action of $\mathbb{K}G$ and for $J_1$, $J_2 \mathbb{K}G$ stable ideals then $J_1 J_2 \subseteq I \Rightarrow J_1 \subseteq I$ or $J_2 \subseteq I$.

**Theorem [3]**

For $P \in \text{Spec}(A)$, $I \in \text{H-Spec}(\mathbb{K}G)$ and $N$ the intersection of all prime ideals minimal over $I$ the following facts are known.

1. $P$ minimal over $I \iff (P : H) = I$
2. The number of primes minimal over $I$ is bounded by $|G|$
3. $N^{\text{dim}_{\mathbb{K}(H)}} \subseteq I$

This remains and other similar questions remain open when $\mathbb{K}G$ is replaced with a semisimple Hopf algebra.

