Linear Algebra 15th February 2013

\[ L_m \subset L_{m+1} \]
\[ K_m \subset K_{m+1} \]
\[ \chi_m = \chi_0 + \mathbf{z} \quad \mathbf{z} \in K_m \]
\[ \chi_m \in \chi_0 + K_m \]
\[ r_m = b - A\chi_m \perp L_m \]

We had two examples:
\[ L_m = A K_m \]
\[ L_m = K_m \]

\[ K_m(A,v) = \text{span}\{v, Av, \ldots, A^{m-1}v\} = \left\{ \sum_{i=0}^{m-1} \mathbf{x}_i A^i v \right\} \]
\[ = \left\{ p(A)v \mid p \in \mathbb{P}_{m-1} \right\} \]

Today an implementation of this: GMRES

No also GCR of H. Elman's basis, M. Schultz

GMRES developed by Y. Saad, M. Schultz

Before \( n \times k \):
\[ \mathbf{r}_m = K_m \iff \| r_m \| = \min_{\chi \in \chi_0 + K_m} \| b - A\chi \| \]
\[ x_m = x_0 + z \quad \text{for} \quad z \in K_m(A, r_0) \quad \text{or} \quad 0 \]

Notice that the matrix basis $A_k x_k$ approximately satisfies the power method, so the vectors become nearly linearly dependent as they converge to the dominant eigenvector.

Want an ON basis of $[r_0 A r_0 A^2 r_0 \ldots A^{m-1} r_0]$

Want $V_m = [v_1 \ldots v_m]$ which spans successively the same subspaces, in ON columns

Rather than MGS = QR

Arnoldi:
\[ v_1 = r_0 / \| r_0 \| \]

MGS $w_2 = A r_0 - \langle A r_0, v_1 \rangle v_1$

Arnoldi $w_2 = A v_1 - \langle A v_1, v_1 \rangle v_1$

\[ v_2 = w_2 / \| w_2 \| \]

\[ w_3 = A v_2 - \langle A v_2, v_1 \rangle v_1 - \langle A v_2, v_2 \rangle v_2 \]

\[ v_3 = w_3 / \| w_3 \| \]

So Arnoldi avoids working on an extremely bad basis as MGS on $[r_0 A r_0 \ldots A r_0]$ does since $A^k \approx A^{k+1}$

End $A v_k = A v_k$
We can save the normalization coefficients \( \| w_k \| \) and write

\[
A v_k = \sum h_{kj} v_j
\]

\[
h_{kj} = \| w_k \|^{-1}
\]

\[\tag{3.3} A V_k = V_{k+1} H_{k+1,k} \]

\[
\begin{bmatrix}
I_n & 0 \\
0 & H_{n,k}
\end{bmatrix} = \begin{bmatrix}
I_n & 0 \\
0 & \alpha_{n,k}
\end{bmatrix}
\]

H will be upper Hessenberg = upper triangular + first subdiagonal

\[\tag{3.4} = V_k H - h_{k+1,k} V_{k+1} e_k^* \]

Both (3.3) and (3.4) are called "the Arnoldi relation"

\[ V_k^* A V_k = H_k \neq 0 \]

\[
\begin{bmatrix}
I_n & 0 \\
0 & 0
\end{bmatrix}
\]

So \( H_k \) is the restriction of \( A \) in the proper subspace \( \mathbf{K} \) basis.
\[ x_m = v_0 + V_m y_m \]

\[ b - Ax_m = b (A v_0 + V_m y_m) \]
\[ = b - Ax_0 - AV_m y_m \]
\[ = r_0 - AV_m y_m \]
\[ = r_0 - V_{m+1} H_{m+1} y_m \]
\[ = \beta y_{m+1} - V_{m+1} H_{m+1} y_m \]
\[ = V_{m+1} (\beta e_i - H_{m+1} y_m) \]

To take the norm:
\[ ||b - Ax_m|| = ||V_{m+1} (\beta e_i - H_{m+1} y_m)|| \]
\[ \text{Orthogonal SP} \]
\[ = ||\beta e_i - H_{m+1} y_m|| \]

(4.10)

minimum over
\[ \min ||r_0 - AV_m y|| \text{ in } n \times m \times p \text{ problem} \]
\[ = \min ||\beta e_i - H_{m+1} y_m|| \text{ in } (m+1)\times m \times p \text{ problem} \]
\[ m < n \]

So we have a dramatic dimension reduction for the problem of solving the least squares minimization.
2. Arnoldi's got $V_m$, upper Hessenberg
2. Solve $(m+1) \times m$ least squares problem, get $y_m$

Obtain $x_m = x_0 + V_m y_m$

\[ V_{m-1} \] submatrix of $V_m$
\[ H_{m,m-1} \] submatrix of $H_{m+1,m}$

so we can form this matrix.

Write the least squares solve by $QR$

\[
\begin{bmatrix} Q & R \end{bmatrix}
\]

We can do the $QR$ incrementally - row by row!

\[
\begin{align*}
\text{(4.10)} & = \min_{y_k} \| \beta e_1 - Q_{m+1,k} [R_{1,1} y_k] \| \\
& = \min_{y_k} \| Q_m^* \beta e_1 - [R_{1,1} y_k] \| \\
\end{align*}
\]

\[
\text{So, } y = R_m \backslash Q_m^* \beta e_1
\]

\[ y_k \]

\[ \text{Monday - how to do } QR \text{ incrementally} \]

\[ \text{NB. residual is absolute value of } \beta_0 \text{ value at bottom} \]

\[ f = \Box \text{ "computed residual" Review Givens rotations} \]