Linear Algebra 4 February 2013

Mixed the beginning of class

Examples of matrices from regular splitting, etc.

$A$ $n \times n$

$A \rightarrow G(A)$

Graph of $A$

$V = \{1, 2, \ldots, n\}$

$(i, j) \in E \iff a_{ij} \neq 0$

In general, ignore self-loops

Note: a symmetric matrix is equivalent to an undirected graph

E.g.

\[ 
\begin{array}{c}
\begin{array}{c}
\text{line}
\end{array}
\end{array}
\]

\[ 
\begin{array}{c}
\begin{array}{c}
\text{star graph}
\end{array}
\end{array}
\]

Of course, a permutation of rows and columns is equivalent to a relabeling of the graph

A triangular graph has edges only towards "lower" nodes

A block triangular graph has subgraphs with a similar relationship

A path between two nodes $i, j$ is a sequence of edges in the graph $i, k_1, k_2 \ldots, k_{n-1}, j$
Def. A matrix $A$ is irreducible iff any two nodes in the graph have a path connecting $i$ to $j$.

A matrix $A$ is reducible iff there is a permutation such that it is block lower triangular.

\[ P^T A P = \begin{bmatrix} * & * & \cdots & * \\ * & \ddots & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ * & * & * & * \end{bmatrix} \]

Now, $A \succeq 0$

$(A)_{ij} \neq 0$ iff path of length 1 (an edge)

$(A^2)_{ij} \neq 0$ iff path of length 2 in $A$

Thus, \( \rho(A) < \infty \land \)

\[ (\sum A^k) > 0 \] iff $A$ is irreducible.

Notice, for $m$ matrices, inverse is Neumann series that is positive iff irreducible.
PERRON-FROBENIUS THEOREM

A square matrix

a) \( A > 0 \) then \( \rho(A) \) is a simple eigenvalue, strictly dominant, i.e. all eigenvalues strictly less

b) \( A > 0 \) and irreducible

The \( \rho(A) \) is a simple eigenvalue, and all other eigenvalues of the same modulus are also simple multiples of \( \rho(A) \) via roots of unity, e.g. \( h \) of them

are hence \( \rho(A)^h \) i.e. the \( h \)th roots of unity

The eigenvector corresponding to the eigenvalue can be taken to be positive.

Moreover, any other positive eigenvector is a multiple of this eigenvector.

**Corollary**

If \( A > 0 \), \( \rho(A) \in \Lambda(A) \) and \( \exists v \geq 0 \) \( \text{Av} = \rho(A)v \)

proof via filling in with \( e \)

**Lemma** let \( T > 0 \) and \( y \neq 0 \) s.t. \( y \frac{d}{dt} Ty \)

then \( \forall \delta \leq \rho(T) \)

**Proof** Assume not. Then \( \alpha > \rho(T) \)

\[
(\alpha I - T)^{-1} = \sum_{n=0}^{\infty} \frac{1}{\alpha^n} T^n
\]

\[
\geq \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{1}{\alpha^n} T^n \geq 0
\]

inverses via Neumann series
\[ y = (qI - T)^{-1}(qI - T)y \geq 0 \]  
\[ \Rightarrow 0 \geq 0 \]  
\[ \text{contradiction} \]

**Corollary**

Let \( T \geq 0, \ x \geq 0, \ x \neq 0 \) s.t. \( Tx = qx \). Then \( \rho(T) \geq 0 \)

\[ \text{if } Tx > qx \text{ then } \rho(T) > 0 \]

Proof:

\[ -x = y \neq 0 \]

\[ Ty = -Tx \leq -qx = qy \Rightarrow qy \leq \rho(T) \]

**Corollary**

\[ x \geq 0, \ Tx \leq qx \text{ then } \rho(T) \leq 0 \]

Proof:

\[ T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \ x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ Tx = 0 \leq ex \Rightarrow y \leq 0 \]

but \( \rho(T) = 1 > 0 \)

Note

If \( O \leq A \leq B \text{ then } \rho(A) \leq \rho(B) \)

Let \( v \) be the Perron vector of \( A \). \( Bv > Av = \rho(A)v \)

\[ Bv \geq \rho(A)v \Rightarrow \rho(B) \geq \rho(A) \]

I am worried about this claim.
Comparison theorem

\[ A = M_1 - N_1 = M_2 - N_2 \] for regular splittings of \( A \), with \( A^{-1} \geq 0 \). If \( M_1 \geq M_2 \), then \( \rho(A^{-1}N_1) \leq \rho(A^{-1}N_2) = \rho_2 \) then \( A^{-1} \geq 0 \) and \( M_1 \geq M_2 \) then \( \rho_1 \leq \rho_2 \)

Proof

\[ M_1 \geq M_2 \quad \Rightarrow \quad N_1 \geq N_2 \quad \Rightarrow \quad 0 \leq A^{-1}N_1 \leq A^{-1}N_2 \]

\[ \Rightarrow \quad \rho(A^{-1}N_1) \leq \rho(A^{-1}N_2) \]

\[ A = M - N = M(I - M^{-1}N) \quad \Rightarrow \quad A^{-1}N = (I - M^{-1}N)^{-1}M^{-1}N \]

Let \( \{v_i\} \quad A^{-1}Nv_i = \tau_i v_i = (I - M^{-1}N)^{-1}M^{-1}Nv_i \]

I have straight up lost the thread here! Can't brain today, so sick.

\[ \rho(M_1^{-1}N_1) = \frac{\rho(A^{-1}N_1)}{1 + \rho(A^{-1}N_1)} \geq \frac{\rho(A^{-1}N_2)}{1 + \rho(A^{-1}N_2)} = \rho(M_2^{-1}N_2) \]

Today:

- Defined irreducibility & graph of a matrix
- Proved Perron-Frobenius
- Several inequality lemmas
- A Comparison theorem

HW Next week: want some graphs of convergence of various stationary methods.