Linear Algebra 30 January 2013

\[ A x = b \]
\[ A = M - N \]
\[ x_{n+1} = M^{-1} N x_n + M^{-1} b \]

Convergent if \( T \rightarrow 0 \) if \( \rho(T) < 1 \)

Choose \( M \) st. \( M x = v \) is easy to solve.

Recall (1.4) \( \rho(T) \leq \| T \| \) any matrix norm

(1.5) \( \forall \varepsilon > 0 \) \( \exists \| T \| \) st. \( \| T \| < \rho(T) + \varepsilon \)

Lemma (1.6) Let \( A \) be nonsingular and \( T \) such that
\[ I - T \] is nonsingular. Then there exists a unique splitting
\[ T = M^{-1} N \]

Proof let \( M = A (I - T)^{-1} \) nonsingular
\[ N = M^{-1} A \]

Then
\[ M^{-1} N = (I - T) A^{-1} (M - A) = (I - T) A^{-1} A (I - T)^{-1} - (I - T) \]
\[ = T \]

\[ M^{-1} N = T \]
\[ A = M^{-1} N \]
\[ M^{-1} M = A = M^{-1} N \] so \( M = M^{-1} N \) unique

Notice
\[ M^{-1} A = I - T \] so \( T = I - M^{-1} A \) connected to (shifted) left preconditioning.
Splittings which are convergent

Recall \( B \geq 0 \) if every entry is \( \geq 0 \)
\( B > 0 \) if every entry is \( > 0 \)

For vectors we call this the hyperoquadric hyperoquadric of nonnegative vectors \( \mathbb{R}^n_+ \)

\( B \geq 0 \iff \mathbb{R}^n_+ \to \mathbb{R}^n_+ \)

This is an example of a pointed cone, i.e. a sol which maps into itself under multiplication by positive scalars.

N.B. This is distinct from the notion of positive (in)definite, often denoted \( A \succ 0 \)

A splitting \( A = M - N \) is a regular splitting if \( M \succ 0 \), \( N \geq 0 \)

weak regular splitting if \( M \succ 0 \), \( N > 0 \)

weak splitting if \( M - N > 0 \)

Ex. Find examples of weak but not weak regular, \( A \), not regular

let \( A = M - N \) weak regular splitting

Def. \( A \) is monotone if \( A^{-1} \geq 0 \)

Theorem \( \text{Ex} \) \( \Rightarrow \mathbb{R}^n \)

(2.8) \( A^{-1} \geq 0 \iff \rho (M^{-1}N) < 1 \)

Note

A singular \( T \), \( I + T \) invertible, then all the splittings (not unique) can be found explicity.
Def. A is called an M-matrix (Minkowski) if

\[ A = s I - B \]

\[ B \geq 0, \quad \rho(B) < s \]

If \( \rho(B) = s \), A is singular.

If \( \rho(B) < s \), A is nonsingular.

A is a nonsingular M-matrix if A is monotone, \( A^{-1} \geq 0 \)

\[ A = s I - B = s \left( I - \frac{1}{s} B \right) \]

\[ A^{-1} = \frac{1}{s} \left( I - \frac{1}{s} B \right)^{-1} = \frac{1}{s} \sum_{k=0}^{\infty} B^k \]

\[ \text{Nevumann Series} \]

\[ (I - C)^{-1} = \sum_{k=0}^{\infty} C^k \]

\( (I + C + C^2 + \ldots + C^k)(I - C) = I - C^{k+1} \)

But spectral radius of C < 1 so limit is I

So, back to Prop (2.8)

A nonsingular, \( A = M - N \) weak splitting \( A^{-1} \geq 0 \)

\[ A^{-1} \geq 0 \iff \rho(M^{-1}N) < 1 \]

\[ T = M^{-1}N \geq 0 \]

\[ M - N \]

\[ A = M(I - T) \]

\[ M^{-1} = (I - T)A^{-1} \]

(3.11)

\[ (I + T + T^2 + \ldots + T^k)(I - T)A^{-1} = (I + T + \ldots + T^k)M^{-1} \]
\[(3.11) = (I - T^{A+1}) A^{-1} = A^{-1} - T^{A+1} A^{-1} \leq A^{-1}\]

\[\text{if} \ A^{-1} \Rightarrow \rho(T) < 1 \]

\[\Rightarrow \sum_{k=0}^{\infty} T^k \text{ convergent} \]

\[\text{Since } T > 0 \Rightarrow T^k \rightarrow 0 \Rightarrow \rho(T) < 1\]

Thus, shown \((\Rightarrow)\) shown

\((\Leftarrow)\) trivial \(\rho(T) < 1 \Rightarrow I - T \text{ invertible} \ A^{-1} = (I + T^{-1})^{-1}

\[\text{A nonsingular } M \text{-matrix}\]

That if we can get a weak regular splitting, it is convergent

Note - different treatment from the book thus far.

We write \[\square\] \[A = D - L - U\]

JACOBI splitting \[M = D \quad N = L + U\]

Ex. Monotone matrix which is not an M-matrix \[A = 5 I - B\]

GAUSS-SEIDEL \[M = D - L \quad N = U\]

Stein-Rosenberg Theorem

\[T_g = I = D^{-1} (L + U)\]
\[T_g = G = (D - L)^{-1} U\]

1. Either \(\rho(G) < \rho(J) < 1\)

2. \(\rho(G) > \rho(J) > 1\)

3. \(\rho(G) = \rho(J) = 1\)

Weak assumptions on \(A\) (e.g. nonsingular)
\((Ax^k)_i = a_{ii} x_i + \sum_{j=1}^{i-1} a_{ij} x_j + \sum_{j=i+1}^{n} a_{ij} x_j\)

Subtract from RMS to get:

\[x_{k+1}^i = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij} x_j^{k+1} \right)\]

Symmetric over relaxation:

\[x_{k+1}^i = (1 - \omega) x_i^k + \frac{\omega}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{k+1} - \sum_{j=i+1}^{n} a_{ij} x_j^{k+1} \right)\]

\(0 < \omega \leq 2\) chosen to improve the spectral radius.

\[D x^{k+1} - \omega L x^{k+1} = (1 - \omega) D x^k + \omega b\]

\[\frac{1}{\omega} (D - \omega L) x^{k+1} = \frac{1}{\omega} \left( (1 - \omega) D + \omega b \right) x^k + b\]

\[M x^{k+1} = N x^k + b\]

This is the so-called SOR splitting.

\[\omega\]

Typical shape of \(\omega\) vs. \(k\).

A 2-matrix is one with negative off-diagonal entries.

\textbf{Example: Find an example of a monotone matrix \(A\) with a weak splitting that is not convergent.}

Turn in these exercises.