Numerical Analysis 2 April 2012

Stability of Single-Step Methods

Consider the explicit single-step method of form

\[ y_{n+1} = y_n + h P(t_n, y_n, h) \]

Assume that the explicit single-step method has a \( p \)-th order accuracy globally, i.e.,

\[ y^{(p+1)}(t_n) - y_{n+1} = O(h^{p+1}) \]

almost always, i.e.,

\[ h P(t_n, y_n, h) - (y^{(p+1)}(t_n) - y_{n+1}) = O(h^p) \]

If \( P \) is Lipschitz-\( y \), i.e.,

\[ |P(t, y, h) - P(t', y', h)| \leq L_p |y - y'| \]

and \( y_0 = y(t_0) \) then the global truncation error is

\[ y(t_n) - y_n = O(h^p) \]

**Def.** If \( P(t, y, h) = f(t, y) \) then the single-step method is consistent and the method has at least first order accuracy.

**Thm.** Consider the IVP \( \{ \frac{dy}{dt} = f(t, y) \} \) on \([0, T]\) \((T \text{ fixed})\).

Let \( h = \frac{T}{N} \), \( t_n = nh \). If the single step explicit method is consistent and \( P \) satisfies the Lipschitz condition then \( y_n \), then we have

\[ \lim_{h \to 0} \frac{y(t_n) - y_n}{h} = 0 \]

i.e., the solution converges.

Stability of Multi-step Methods

**Def.** A polynomial satisfies the root condition if all its roots are lie in the unit circle in the complex plane, and those on the circle have multiplicity one.

Consider the multi-step methods of the form

\[ \sum_{j=0}^{p} \alpha_j y_{n+j} = h \sum_{j=0}^{p} \beta_j y_{n+j} \]

A multi-step method is zero-stable if \( P(s) = \sum_{j=0}^{p} \beta_j s^j \) satisfies the root condition.
Suppose that a multistep method is assigned with starting points:

\[ y_0, y_1, \ldots, y_{k-1}, \text{ satisfying } \lim_{h \to 0} y_j = y_j(h) \quad j = 0, 1, \ldots, k-1. \]

Then the method converges to the true solution of the IVP on a fixed \([0, T]\) if and only if the method satisfies the root condition

\[ y_i - y_0 = O(h^r) \quad ? \]

Test problem: \( y' = k y \)
For one-step methods we have in general \( y_{n+1} = R(z) y_n \) where \( z = h \).

For example, forward Euler:
\[ R(z) = 1 + z \]

Backward Euler:
\[ R(z) = (1 - z)^{-1} \]

Heun:
\[ R(z) = \frac{1 + z}{1 - z} \]

RK4:
\[ \begin{align*}
R(z) &= 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} \\
\end{align*} \]

If \( |R(z)| \leq 1 \), the method is absolutely stable for that value of \( \beta \).

The corresponding region of \( z \) is called the stability region.
\[ y' = \lambda y \]

**Figure 7.1.** Stability regions for (a) Euler, (b) backward Euler, (c) trapezoidal, and (d) midpoint (a segment on imaginary axis).

**Figure 10.4:** Explicit Runge-Kutta Stability Regions
Figure 7.2. Stability regions for some Adams–Bashforth methods. The shaded region just to the left of the origin is the region of absolute stability. See Section 7.6.1 for a discussion of the other loops seen in figures (c) and (d).

Figure 7.3. Stability regions for some Adams–Moulton methods.
The pink region shows the stability region of the BDF methods