Numerics  23 March 2012

Error analysis of forward & Backward Euler's method.

Forward Euler result:

\[ (\text{global error})_n = (1 + hf_y(t_n, \tilde{y}_n)) (\text{global error})_n + (\text{local error})_n \]

Condition for stability is:

\[ |1 + hf_y(t_n, \tilde{y}_n)| < 1 \iff -2 < hf_y(t_n, \tilde{y}_n) < 0 \]

Backward Euler Error

\[ y(t_{n+1}) - y_{n+1} = y(t_n) - y_n + h \left[ f(t_{n+1}, y(t_{n+1})) - f(t_n, y_n) \right] \frac{\partial y}{\partial t}(t_n) \]

\[ (1 - hf_y(t_{n+1}, \tilde{y}_{n+1})) (\text{global error})_{n+1} = (\text{global error})_n + (\text{local error})_n \]

Condition for stability is:

\[ |1 - hf_y(t_{n+1}, \tilde{y}_{n+1})| < 1 \]

\[ hf_y(t_{n+1}, \tilde{y}_{n+1}) < 0 \text{ or } hf_y(t_{n+1}, \tilde{y}_{n+1}) > 0 \]

Systems of ODES

\[ \dot{y}(t) = f(t, y(t)) \in \mathbb{R}^n \]

Define the Jacobian matrix

\[ J(t) = \frac{\partial f(t, y(t))}{\partial y} \]
\[
\begin{align*}
\mathbf{y}'(t) &= \begin{bmatrix}
2y_1(t) + 3y_2^2(t) \\
-6t + 7y_2(t)
\end{bmatrix} \\
\text{so} \quad \mathbf{J}(t) &= \begin{bmatrix}
2 & 6y_2(t) \\
0 & 7
\end{bmatrix}
\end{align*}
\]

The system of order is stable at \((t_0, y(t_0))\) if and only if the real parts of all eigenvalues of the Jacobian matrix are negative.

Consider \(\mathbf{y}'(t) = A_{\text{diag}} \mathbf{y}(t)\). Then \(\mathbf{y}(t) = A_{\text{diag}} \mathbf{y}(t_0)\).

Assuming that \(A\) is diagonalizable \(A = V \Lambda V^{-1}\). Then \(\mathbf{y}(t) = V \Lambda(t) V^{-1} \mathbf{y}(t_0)\).

\[\begin{align*}
V^{-1} \mathbf{y}'(t) &= \Lambda \mathbf{y}(t) \\
V^{-1} \mathbf{z}'(t) &= \Lambda \mathbf{z}(t)
\end{align*}\]

Thus \(\mathbf{z}(t) = \mathbf{z} \exp(\Lambda t) = (e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t})\).

So for this to be stable, the real part of all \(\lambda_i\) must be negative.

For an ODE system, the stability condition of forward Euler's method is that \(I + hJ(t)y(t_0)\) has spectral radius bounded by \(1\), i.e., all eigenvalues are inside the unit circle.

The stability condition for backward Euler method is that all eigenvalues of \((I - hJ(t)y(t_0))^{-1}\) are inside the unit circle.

Now, we have shown that local error is on the order of \(h^2\), hence we say global error is on the order of \(h\), i.e., is first order accurate.

An improvement of Euler's method:

\[
\begin{align*}
y_{n+1} &= y_n + h f(t_n, y_n) \\
y_n &= y_{n-1} - h f(t_{n-1}, y_{n-1}) \\
y_{n+1} &= y_n + \frac{h}{2} (f(t_n, y_n) + f(t_{n+1}, y_{n+1}))
\end{align*}
\]

So better to use trapezoidal rule for the integral estimation instead of Euler.