Composite rules increase accuracy by performing lower-order quadrature on increasingly many intervals.

Trapezoidal rule error is proportional to \( h^2 \) and \( f''(x) \) (and length of interval).

Simpson's composite proportional to \( h^4 \) and \( f^{(4)}(x) \) (and length of interval).

Romberg approximation is even finer for increasing accuracy.

**Numerics** 16 March 2012

**Composite Trapezoidal Rule**

\[
T_n = \frac{h}{2} \left[ f(a) + 2 \sum_{k=1}^{n-1} f(x_k) + f(b) \right]
\]

\[
R_n = I - T_n = \sum_{k=0}^{n-1} \left[ \frac{h^3}{12} f^{(2)}(\eta_k) \right] \quad \text{where } \eta_k \in (x_k, x_{k+1})
\]

**Composite Simpson's Rule**

\[
\begin{align*}
\text{1st } x_{k+\frac{1}{2}} &= x_k + \frac{1}{2} h \\
S_n &= \frac{h}{3} \sum_{k=0}^{n-1} \left[ f(x_k) + 4 f(x_{k+\frac{1}{2}}) + f(x_{k+1}) \right] \\
&= \frac{h}{6} \left[ f(a) + 4 \sum_{k=0}^{n-1} f(x_{k+\frac{1}{2}}) + 2 \sum_{k=0}^{n-1} f(x_k) + f(b) \right]
\end{align*}
\]

\[
R_n = -\frac{(b-a)}{180} \left( \frac{h}{2} \right)^4 f^{(4)}(\eta) \quad \text{where } \eta \in (a, b)
\]

**Romberg**

\[
I - T_n = -\frac{b-a}{12} \frac{h^2}{12} f''(\eta)
\]

\[
I - T_{2n} = -\frac{b-a}{12} \frac{h^4}{12} f''(\eta)
\]

Assume that \( f''(\eta) \approx f''(\eta) \)

\[
\frac{I}{I - T_n} \approx \frac{1}{4}
\]

So \( 3I \approx 4T_n - T_2n \)

Assume \( I \approx 4T_{2n} - T_n \)
Romberg's method gives Simpson's rule.

\[ T_0 = \frac{h}{4} \left[ f(a) + 2f(\frac{a+b}{2}) + f(b) \right] \]

\[ T_k = \frac{h}{4} \left[ f(a) + 2f(\frac{a+b}{2}) + f(b) \right] \]

\[ Q_k = T_k - T_{k-1} = \frac{h}{2} \left[ f(a) + 2f(\frac{a+b}{2}) + f(b) \right] - \frac{h}{2} \left[ f(a) + f(b) \right] \]

\[ T_k = \frac{h}{2} \left[ f(a) + 2f(\frac{a+b}{2}) + f(b) \right] \]

which is Simpson's rule.

Likewise for Simpson's rule:

\[ I - S_n = -\frac{b-a}{16} \left( \frac{h}{2} \right)^4 f^{(4)}(\eta) \]

\[ \frac{I - S_{2n}}{I - S_n} \approx \frac{1}{16} \]

\[ I \approx \frac{1}{15} \left( 16 S_{2n} - S_n \right) \]

which will be equivalent to Coates' rule, \( C_n \).

"\( n \)th order accurate" means exact for polynomials of degree \( n \).

Similarly, a linear combination of \( C_n \) and \( C_{2n} \):

\[ I \approx \frac{1}{63} \left( 64 C_{2n} - C_n \right) \]

Romberg's rule (reliability?)

**Gauss Quadrature**

The idea is to space nodes non-uniformly so that the quadrature is exact for polynomials of degree \( n \) or less.

These nodes will be the zeros of some special family of polynomials.