head squares approximation

Given \( \{x_i, y_i\} \quad i = 0, 1, 2, \ldots, m \)

This is equivalent to the normal equation

\[
\begin{bmatrix}
\sum_{i=0}^{m} x_i^2 & \sum_{i=0}^{m} x_i \\
\sum_{i=0}^{m} x_i & m
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1
\end{bmatrix}
= \begin{bmatrix}
\sum_{i=0}^{m} y_i \\
\sum_{i=0}^{m} x_i y_i
\end{bmatrix}
\]

The linear least squares approximation is then \( y = a_0 + a_1 x \)

Fourier Series

functions on unit circle

truncated series

\( L^2 \) norm optimization

Fourier Coefficients

Approximation by Trigonometric Functions

Suppose that \( f \in L^2[0, \pi] \). We will approximate \( f \) using "trigonometric polynomials," that is

\[
S_N(x) = \frac{1}{2} a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx
\]

and we wish to minimize the error in the \( L^2 \) norm.

\[
a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) \, dx \quad b_1 = \frac{1}{\pi} \int_0^{\pi} f(x) \cos x \, dx \quad \cdots
\]

From the normal equation when the coefficient matrix is diagonal, we have

\[
a_k = \frac{1}{\pi} \int_0^{\pi} f(x) \cos kx \, dx \\
b_k = \frac{1}{\pi} \int_0^{\pi} f(x) \sin kx \, dx
\]
\( a_k \) and \( b_k \) are the Fourier coefficients.

The Fourier series is
\[
\frac{a_0}{2} + \sum_{k=1}^{\infty} \left( a_k \cos kx + b_k \sin kx \right)
\]

If \( f \) is piecewise continuous on \([0, 2\pi]\) then the Fourier series converges to \( f \) uniformly (n.b. \( f \) periodic \( 2\pi \)).

Let \( f_\chi \) be a complex valued periodic function on \([0, 2\pi]\).

Given equally spaced points \( x_j = \frac{2\pi j}{N} \) \((j = 0, 1, \ldots, N-1)\).

Let \( f_j := f(x_j) \). The set of functions
\[
\{1, e^{ix}, e^{2ix}, \ldots\}
\]
are orthonormal in \([0, 2\pi]\).

\( \phi_j := (1, e^{ix}, e^{2ix}, \ldots, e^{i \frac{2\pi (N-1)}{N}}) \)
and in the discrete analog these are orthonormal as well.

\[
\langle \phi_j, \phi_k \rangle = \sum_{k=0}^{N-1} e^{i \frac{2\pi j k}{N}} e^{-i \frac{2\pi k j}{N}} = \begin{cases} \frac{N}{N} & j = k \\ 0 & j \neq k \end{cases}
\]

The best approximation in \(L^2\) is then
\[
S(x) = \sum_{k=0}^{N-1} C_k e^{ikx}
\]
where \( C_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-i \frac{2\pi k j}{N}} \).

If \( N = \infty \) then \( S(x) \) becomes an interpolation of \( f(x) \) on \([0, 2\pi]\).

This is known as the discrete Fourier transform.
FFT

Motivation: As written, the DFT requires $N^2$ multiplications

$$\frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-\frac{2\pi i}{N} k}$$

The basic idea

$$ab + ac + ad = a(b + c + d)$$

3 products 1 product

3 sums 3 sums

Notice $e^{-\frac{2\pi i}{N} k}$

$N \times N$ products, but only $O(N)$ many different values, since

$$e^{-\frac{2\pi i}{N} k} = e^{-i \frac{2\pi}{N} (k \cdot j \mod N)}$$

The remainder mod $N$ is the only relevant value

$$f_k = e^{-i \frac{2\pi i}{N} k}.$$