Numerical Analysis, 2012-02-13

Assume \( f \in C^4[a,b] \) and \( S(x) \) is a cubic spline interpolation of \( f(x) \) which satisfies the first or second kind of boundary condition. Let \( h_i = x_i - x_{i-1} \) \( i = 0, 1, \ldots, n-1 \) and \( h = \max h_i \).

Then
\[
\max_{a \leq x \leq b} |f^{(k)}(x) - S^{(k)}(x)| \leq C_k \max_{a \leq x \leq b} |f^{(4)}(x)| h^{4-k}
\]

where \( k = 0, 1, 2 \).

\[
C_0 = \frac{5}{384}, \quad C_1 = \frac{1}{24}, \quad C_2 = \frac{3}{8}
\]

When \( k > 2 \) was here, spoke on basic functional analysis.

Induced Norms etc.

Weierstrass Theorem

Assume \( f(x) \in C[a,b] \) continuous. Then for any \( \varepsilon > 0 \)

there exists a polynomial \( P(x) \) s.t.

\[
||f(x) - P(x)||_\infty < \varepsilon
\]

when \( f \) has \( 1 \)-norm in sup \( f(x) \) below, has a maximum.

Bernstein Polynomials

Uniform conv on \([0,1] \)

derivative converges

Bernstein Polynomial

\[
B_n(f,x) := \sum_{k=0}^{n} f^{(k)}(x) \frac{n!}{k!} (x)^k (1-x)^{n-k}
\]

Then \( B_n(f, x) \longrightarrow f(x) \) uniformly on \([0,1] \).

If \( f(x) \in C^n[0,1] \), the limit \( \lim_{n \to \infty} B_n(f,x) = f^{(n)}(x) \)

Error term for cubic spline \( S(x) \) of \( f \in C^4[a,b] \) bounded by \( \max |f^{(4)}(x)| h^4 \).

Weierstrass theorem \( f \in C[a,b] \) can be approximated \( L^0 \) by polynomials.

Example: Bernstein polynomials apply Weierstrass theorem to \( (x f(-x))^n \).
\[ \sum |P_R(x)| = \sum P_R(x) = (x+1-x)^n = 1 \] 

**Basics of Inner Product Spaces**

**Def.** Assume that \( S \) is a norm space with \( \langle \cdot, \cdot \rangle : S \times S \to \mathbb{C} \) such that \( \forall u, v \in S \)

- Conjugate symmetric
- Homogeneous
- Linear
- Nonnegative

Cauchy Schwarz

\[ |\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle \]

**NS. Cauchy-Schwarz** for any \( u, v \in S \)

Proof left as an exercise.

Homework to be penalized 20% per day.

**Graham Matrix**

Let \( u_1, u_2, \ldots, u_n \in S \). Then the matrix \( G = [g_{ij}] = [\langle u_i, u_j \rangle] \)

is nonsingular iff \( u_1, \ldots, u_n \) are linearly independent.

Inner product spaces.

Cauchy-Schwarz inequality

\[ |\langle u, v \rangle|^2 \leq \langle u, u \rangle \langle v, v \rangle \]

The graham matrix \( [g_{ij}] = [\langle u_i, u_j \rangle] \) for \( \{u_1, \ldots, u_n\} \subset S \) is nonsingular iff \( \{u_1, \ldots, u_n\} \) linearly independent.
Most common inner product is the $L^2$ product

$$\langle u, v \rangle = \overline{v}^H u = \sum_{i=0}^{n} \overline{v}_i u_i \quad \text{for } u, v \in \mathbb{C}^n$$

We also have an inner product with weighted coefficients

$$\langle u, v \rangle = \sum_{i=0}^{n} \omega_i \overline{v}_i u_i \quad u, v \in \mathbb{R}^n \cap \mathbb{C}^n$$

**Weighted $\langle f, g \rangle$**

Assume that a nonnegative function $\rho(x)$ on $[a,b]$ satisfies the following conditions:

1) $\int_a^b x^k \rho(x) dx < \infty \quad \text{for } k = 0, 1, \ldots$

2) For all nonnegative $q(x)$ if $\int_a^b q(x) \rho(x) dx = 0 \Rightarrow q(x) \equiv 0$

Then $\rho(x)$ is called a weight function on $[a,b]$ with $\rho(x)$, we can define $\langle f, g \rangle = \int_a^b \rho(x) f(x) g(x) dx$

The weighted inner product various kinds $\langle u, v \rangle_\rho = \sum_{i=0}^{n} \rho_i \overline{v}_i u_i$

In $L^2[a,b]$ $\langle f, g \rangle_\rho = \int_a^b \rho(x) f(x) g(x) dx$ where $\rho \geq 0$ and $\int_a^b \rho(x) dx < \infty$, $\int g_p = 0 \Leftrightarrow g_p \equiv 0$