Divided Differences

first order

second order

nth order

get higher order differences from lower ones

Newton interpolation polynomial

\[ f(x) = f(x_0) + (x-x_0)[x_0, x_1]f + (x-x_0)(x-x_1)[x_0, x_1, x_2]f + \ldots + (x-x_0)(x-x_1)(x-x_2)\ldots(x-x_{n-1})[x_0, x_1, \ldots, x_n]f \]

Call \( \left( \prod_{i=0}^{n-1} (x-x_i) \right)^{-1} [x_0, x_1, \ldots, x_n]f = P_n(x) \)

And the rest of the section \( N_n(x) \)
Interpolation condition

Satisfies the interpolation condition

\[ N_n(x) = f(x) \]
\[ N_n(x) = f(x_0) + (x - x_0) \frac{f(x_0) - f(x_1)}{x_0 - x_1} \]
\[ = f(x_0) \times (x - x_0) \]
\[ = f(x) \]

So \[ N_n(x) = f(x_i) \] for \[ i = 0, 1, \ldots, n \]

This polynomial is equivalent to the Lagrange polynomial, since there is only one polynomial of degree \( n+1 \) through \( n+1 \) points. But this formulation is easier to state.

\[
\begin{array}{c|c|c|c|c|c}
  x_i & f(x_i) & \Delta f_i & \Delta^2 f_i & \Delta^3 f_i & \Delta^4 f_i \\
  x_0 & f(x_0) & & & & \\
  x_1 & f(x_1) & \Delta f_1 & & & \\
  x_2 & f(x_2) & \Delta f_2 & \Delta^2 f_2 & & \\
  x_3 & f(x_3) & \Delta f_3 & \Delta^2 f_3 & \Delta^3 f_3 & \\
\end{array}
\]

Special Case

\[ x_i - x_{i-1} = h \]

Forward difference \( \Delta f_k \)

\[ \Delta f_k = f(x_{k+1}) - f(x_k) \]

Backward difference \( \nabla f_k \)

\[ \nabla f_k = f(x_k) - f(x_{k-1}) \]

\[ \Delta^2 f_k = \Delta f_{k+1} - \Delta f_k = f(x_{k+2}) - 2f(x_{k+1}) + f(x_k) \]

\[ \nabla^2 f_k = \nabla f_{k+1} - \nabla f_k = f(x_{k+2}) - f(x_k) \]
High order differences

\[ \Delta^m f_k = \Delta^{m-1} f_{k+1} - \Delta^{m-1} f_k \]

\[ \Delta^m f_k = \Delta \Delta^{m-1} f_k - \Delta^{m-1} f_{k-1} \]

Formulas from values

\[ \Delta^m f_k = \frac{\sum_{i=0}^{m} (-1)^i \binom{m}{i} f(x_{m+k-i})}{n!} \]

\[ \Delta^m f_k = \frac{\sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} f(x_{k+i-m})}{n!} \]

So forward & backward difference a linear combination of function values

With the special case

\[ \left[ x_k \ x_{k+1} \right] f = \frac{f(x_{k+1}) - f(x_k)}{x_{k+1} - x_k} = \frac{\Delta f_k}{h} \]

\[ \left[ x_k \ x_{k+1} \ x_{k+2} \right] = \frac{\Delta^2 f_k}{2h^2} \]

\[ \left[ x_k \ x_{k+1} \ \ldots \ x_{k+n} \right] = \frac{\Delta^n f_k}{n! h^n} \]

\[ \left[ x_k \ x_{k+1} \ \ldots \ x_{k+n-1} \right] = \frac{\Delta^n f_k}{n! h^n} \]

Newton's Interpolation for Equidistant Points

\[ N_n(x_0 + th) = f(x_0) + t \Delta f_0 + \frac{t(t+1)}{2!} \Delta^2 f_0 + \ldots + \frac{t(t+1) \ldots (t+n-1)}{n!} \Delta^n f_0 \]

\[ N_n(x_k + th) = f(x_k) + t \Delta f_k + \ldots + \frac{t(t+1) \ldots (t+n-1)}{n!} \Delta^n f_k \]