I missed this class. These are a reconstruction of the material from notes borrowed from Meredith Hegg.

1. Interpolation

1.1. Lagrangian Interpolation. If we have two points \((x_k, y_k)\) and \((x_{k+1}, y_{k+1})\) we can do linear interpolation as

\[
L_1(x) = y_k + \frac{y_{k+1} - y_k}{x_{k+1} - x_k} (x - x_k) = \frac{x_{k+1} - x}{x_{k+1} - x_k} y_k + \frac{x - x_k}{x_{k+1} - x_k} y_{k+1}
\]

\[
= L_k(x)y_k + L_{k+1}(x)y_{k+1}
\]

We call \(L_k, L_{k+1}\) basis functions for interpolation.

Notice,

\[
L_k(x_k) = 1
\]
\[
L_k(x_{k+1}) = 0
\]
\[
L_{k+1}(x_k) = 0
\]
\[
L_{k+1}(x_{k+1}) = 1
\]

1.1.2. Polynomial Interpolation. For quadratic interpolation (3 data points) we have

\[
L_2(x) = y_k \cdot \frac{(x - x_k)(x - x_{k+1})}{(x_{k+1} - x_k)(x_{k-1} - x_k)} + y_{k+1} \frac{(x - x_{k-1})(x - x_{k+1})}{(x_{k+1} - x_{k-1})(x_{k+1} - x_k)} + y_{k+1} \frac{(x - x_{k-1})(x - x_k)}{(x_{k+1} - x_{k-1})(x_{k+1} - x_k)}
\]

\[
= L_{k-1}(x)y_{k-1} + L_k(x)y_k + L_{k+1}(x)y_{k+1}
\]

So we have a similar form but different basis functions for quadratic interpolation.

For a given set of nodes, \((n + 1)\) many, with \(x_0 < x_1 < \ldots < x_{n-1} < x_n\) we have that the interpolation polynomial is

\[
L_n(x) = \sum_{k=0}^{n} y_k L_k(x)
\]
where our basis functions are
\[ L_k(x) = \prod_{i=0}^{n} \frac{(x-x_i)}{(x_k-x_i)} \]

Therefore we have a polynomial which is a linear combination of \( n + 1 \) many basis polynomials, producing a polynomial of degree \( n \).

As a warning, the points must be independent in some sense (to be elaborated later), so it is more appropriate to say that it is a polynomial of degree at most \( n \).

1.1.3. Uniqueness.

**Theorem 1** (Uniqueness). Given \( x_0 < x_1 < \ldots < x_{n-1} < x_n \) and \( y_0, \ldots, y_n \) there exists a unique interpolation polynomial of order no larger than \( n \) such that \( L_n(x_i) = y_i \) for all \( i = 0, \ldots, n \).

**Proof.** Suppose some other polynomial \( P_n \) satisfies the requirements. Then \( P_n - L_n \) is of degree no more than \( n \) but has \( n + 1 \) distinct zeros, therefore \( P_n - L_n \equiv 0 \).

**Theorem 2** (Remainder). Suppose \( f^{(n)}(x) \) is continuous in \([a, b]\) and \( f^{(n+1)}(x) \) exists in \((a, b)\). Suppose \( a \leq x_0 < x_1 < \ldots < x_{n-1} < x_n \leq b \). Let \( y_i := f(x_i) \) and \( L_n(x) \) be the corresponding Lagrangian interpolation polynomial. Then for any \( x \in [a, b] \),
\[ \exists \xi \in (a, b) \text{ s.t. } R_n(x) = f(x) - L_n(x) = \frac{f^{(n+1)}(\xi)}{(n + 1)!} \prod_{i=0}^{n} (x-x_i) \]

**Proof.** The proof uses Taylor series and is straightforward.

2. Newton Interpolation

Newton interpolation is a different formulation of the interpolating polynomial (by the uniqueness result above they must be equivalent polynomials), but is convenient for adding additional points.

We define the divided differences orders 1, 2, \ldots, \( k \) of the function.

\[ [x_i, x_j]^f = \frac{f(x_i) - f(x_j)}{x_i - x_j} \]
\[ [x_i, x_j, x_k]^f = \frac{[x_i, x_j]^f - [x_i, x_k]^f}{x_j - x_k} \]
\[ \ldots \]
\[ [x_0, x_1, \ldots, x_k]^f = \frac{[x_0, \ldots, x_{k-2}, x_k]^f - [x_0, \ldots, x_{k-1}]^f}{x_0 - x_{k-1}} \]

\(^1\)Maybe it would be better to use the notation \( L_k^n(x) \) for the basis?