Forward analysis - distance between true and computed solutions

Backward analysis - distance between true and computed problem

For a numerical algorithm, makes more sense, since when we have an ill-conditioned problem, it may be that we can not get a good solution, but can get a solution to a closely perturbed problem.

Example

\[
\begin{bmatrix}
  1 & 1 \\
  1 & 1
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
= 
\begin{bmatrix}
  1 \\
  0
\end{bmatrix},
\]

Assume that \( S < \frac{\varepsilon}{2} \)

\[
\delta x_1 + \delta x_2 = 1
\]

\[
x_2 = -\delta x_1
\]

Angle is not too small, so small perturbations in the lines move the solution/intersection just a little bit.

Consider Gaussian elimination

\[
\begin{bmatrix}
  1 & 1/2 \\
  1 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
  1 & 1/2 \\
  0 & -1/2
\end{bmatrix} = \begin{bmatrix}
  1 \\
  -1/2
\end{bmatrix}
\]

So, \( x_2 = 1 \)

\( x_1 = 0 \)

\( \delta x_2 \) is huge since \( S < \frac{\varepsilon}{2} \)

So Gaussian elimination is not backward stable (without pivoting)

Better approach - Gaussian elimination with pivoting.

We give an example of Gaussian elimination where catastrophic cancellation occurs, even though the problem itself is well conditioned.
For a well-conditioned problem, the solution should be close. For an ill-conditioned problem, the solution might be wrong, but the back-solution will be close.

**Second Example**

For a well-conditioned problem, the solution should be close. For an ill-conditioned problem, the solution might be wrong, but the back-solution will be close.

**Second Example**

$$\begin{bmatrix}
0.661 & 0.991 \\
0.500 & 0.793
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
0.330 \\
0.250
\end{bmatrix}$$

Gaussian with Pivoting

In Gaussian, we try to clear out columns

With partial pivoting - interchange rows so that largest magnitude entry in each column is 1 or on the diagonal

$$\begin{bmatrix}
8 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}
\rightarrow 
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1.8 & 1
\end{bmatrix}$$

To prove that Gaussian with pivoting is backward stable exists,

But experience shows that it work well

**Second Example**

$$\begin{bmatrix}
0.661 & 0.991 \\
0.500 & 0.793
\end{bmatrix}
\begin{bmatrix}
x \\
x_0
\end{bmatrix}
= 
\begin{bmatrix}
0.330 \\
0.250
\end{bmatrix}$$

So $x_0$ and $x$ are not close. However $x$ residual (backward error)

$$r = b - Ax_0 = 
\begin{bmatrix}
-0.000507 \\
-0.000250
\end{bmatrix}$$

Notice relative error is $rac{\|r\|}{\|b\|} \approx \frac{1}{100}$.}

For badly conditioned problems, the forward error cannot be expected to be small but

backwards error may (hopefully) be small.

Partial pivoting is backward stable

Eg. for ill-conditioned, forward error is large, but the relative backward error

(even in terms of residual) is as small as possible.
Sensitivity Analysis

\( \frac{dx}{dt} \)

The sensitivity of \( x \) to \( t \) is \( \frac{dx}{dt} \), i.e., how much does a change after a small change in \( t \)?

Eq: \( x^2 + 6x + c = 0 \)

Use implicit differentiation to compute \( \frac{dx}{db} \):

\[ \frac{dx}{db} + 1x + b \frac{dx}{db} + 0 = 0 \]

\[ (ax + b) \frac{dx}{db} = -x \]

\[ \frac{dx}{db} = \frac{-x}{ax + b} \]

\( x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4c}}{2a} \)

Consider \( x_2 \):

\[ \frac{dx_2}{db} = \frac{1}{a} \left( -1 + \frac{ab}{a \sqrt{b^2 - 4c}} \right) \]

Check: \( \frac{-X_2}{ax + b} = \frac{-\frac{1}{2} \left( -b + \sqrt{b^2 - 4c} \right)}{\sqrt{b^2 - 4c}} \)

What is the condition under which the roots are extremely sensitive to small changes in \( b \)?

\( b^2 - 4c < 0 \)

i.e., \( x_1 = x_2 \)

This derivative is \( \infty \), and one of the roots will split real/complex!

Rank-deficient matrices have fewer columns which are nearly linearly dependent.

We can estimate the sensitivity of a problem via the derivative of solutions with respect to the parameters.