Linear Algebra 28 November 2012

Tridiagonal Hermitian - show all eigenvalues are simple

\[ A - \lambda I \]

\[ \text{rank}(A - \lambda I) \leq m - 1 \]

Two sided

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

via triangulation, given for zero households to repair

Simultaneous Iterations

\[ Q^{(0)} = I \]

\[ Z = AQ^{(k)} \]

\[ Z = Q^{(k+1)}R^{(k+1)} \]

\[ Z = Q^{(k+1)} \]

\[ \text{So } \begin{bmatrix}
R^{(1)} \\
R^{(2)} \\
\vdots \\
R^{(n)}
\end{bmatrix} = A^{(0)} \begin{bmatrix}
Q^{(0)} \\
Q^{(1)} \\
\vdots \\
Q^{(n)}
\end{bmatrix} \]

\[ A^{(k)} = Q^{(k)} A Q^{(k)} \]

\[ A^{(k)} \rightarrow \Lambda \]
**Pure QR Algorithm**

\[
A^{(0)} := A \\
Q^{(k)} R^{(k)} := A^{(k-1)} \\
A^{(k)} := R^{(k)} Q^{(k)}
\]

\[\text{L factorization}\]

\[R^{(k)} = Q^{(k)} A^{(k-1)} \]

\[
A^{(k)} = R^{(k)} Q^{(k)} = Q^{(k)} A^{(k-1)} Q^{(k-1)} = Q^{(k)} A^{(k-1)} Q^{(k-1)}
\]

\[
Q^{(k)} = Q^{(1)} Q^{(2)} \ldots Q^{(k)}
\]

\[\text{Need to show we get the same factorization}\]

\[A = A^{(0)} = Q^{(1)} R^{(1)} = Q^{(1)} Q^{(1)}\]

\[A^k = A A^{k-1} = A Q^{(k-1)} R^{(k-1)}\]

\[
A^k = Q^{(k-1)} A^{(k-1)} Q^{(k-1)}
\]

\[
\text{So}\}
\]

\[
A^{(k)} Q^{(k)} R^{(k)} = Q^{(k)} Q^{(k)} R^{(k)} R^{(k)} = Q^{(k)} R^{(k)}
\]

So this method produces the same factorization as the simultaneous iterations.
So why QR and not simultaneous iterations?

1. Do not start $A^{(0)} = A$ but with $Q^T A Q$
   - Hessenberg or upper Hessenberg

2. As soon as $|q^{(1)}| < \varepsilon$ we can decouple into two blocks which calculate much faster

3. Use QR with shifts $\mu^{(k-1)}$

   $Q^{(k)} R^{(k)} = (A^{(k-1)} - \mu^{(k-1)} I)$
   
   $A^{(k)} = R^{(k)} Q^{(k)} + \mu^{(k-1)} I$

   With appropriate (Rayleigh quotient) shifts, can get cubic convergence

The QR method is like inverse power method (not just power shift/deflation)

$q^{(k)} \rightarrow Q = [q_1, ..., q_m]$

This is simultaneous iteration i.e. power method /deflation

$Q^{(0)} = I$

Instead, think of as simultaneous inverse iteration (power method) with $Q^{(0)} = P_\lambda|A^{-1}|$ on $A^{-1}$

$A^{-k} P \rightarrow q_m$

A something - got a bit confused, frankly.

Note $A^{(k)} = (A - \mu^{(k-1)} I)^{-1} \cdots (A - \mu I) = Q^{(k)} R^{(k)}$
The shifts are the Rayleigh quotients, as on the diagonals of $A^{(k)}$, cycled through to catch all of them.

N.B. End of chapter Wilkinson shifts or even better.

The algorithm is backward stable.

Week 8

Presentations Monday

Oral final exam

Friday 14 December 08 as arranged

30 minutes each approximate

Want us to prepare a topic for the first 5 minutes

Schedule to be determined next week.