Chapter 1

\[ P_1 = (P_1)^2 = (-P_2)^2 = P_2^2 = P_2 \]
\[ P_1 + P_2 = 2P_2 = 0 \]

HW help eps
\[ \| (1 + \text{eps}) - 1 \| \]

Notice
\[ \frac{|x - \bar{x}|}{|x|} < C E_{\text{mach}} = O(E_{\text{mach}}) \]

Test will cover up to CH 17

Asymptotics
\[ f(x) = O(|x - x_0|^2) \]
means \[ \exists C \text{ s.t.} \]
\[ |f(x)| < C |x - x_0|^k \quad \text{as} \quad x \to x_0 \]

\[ \psi(t, s) = O(t^h) \]
\[ |\psi(t, s)| \leq c t^h \]
\[ |x - \tilde{x}| \leq O(\varepsilon_{\text{mach}}, |x|) \]

**Stability - Backward, Forward, and Accuracy**

A problem \( f : X \rightarrow Y \)

the problem has conditioning concerns

An algorithm for the problem

\( \tilde{f} : X \rightarrow Y \)

Is \( \tilde{f} \) accurate? (i.e. \( \tilde{f} \) is close to \( f \))

Is \( \tilde{f} \) forward stable?

Is \( \tilde{f} \) backward stable?

We say that \( \tilde{f} \) is **backward accurate** if \( \frac{\|f - \tilde{f}\|}{\|f\|} = O(\varepsilon_{\text{mach}}) \)

An algorithm \( \tilde{f} \) is stable if \( \frac{\|\tilde{f}(x) - \tilde{f}(x')\|}{\|f(x)\|} = O(\varepsilon_{\text{mach}}) \)

\( \forall x \frac{\|x - \tilde{x}\|}{\|x\|} = O(\varepsilon_{\text{mach}}) \)

So \( \tilde{f}(x) \) is close to \( f(\tilde{x}) \) for \( x \) close to \( \tilde{x} \)

Backward stability strengthens this to equality

\( \tilde{f}(x) = f(\tilde{x}) \) for some \( \tilde{x} \in B_p(x) \) where \( \frac{\|x - \tilde{x}\|}{\|x\|} = O(\varepsilon_{\text{mach}}) \)

so the algorithm gives the exact solution of a nearby problem
Notice

Backward stable is for some $\tilde{x}$
Stable is for all $\tilde{x}$

Thm Backward stable $\&$ well conditioned $\implies$ accurate

Let $\tilde{f}$ be an algorithm for $f: X \rightarrow Y$
when $\tilde{f}$ is backward stable and $f$ is well-conditioned $K(x)$

Then

$$\frac{\| \tilde{f}(x) - f(x) \|}{\| f(x) \|} = O(K(x) \cdot \epsilon_{\text{mach}})$$

Proof

Backward stable, so $\exists \tilde{x}$

$\tilde{f}(x) = f(\tilde{x})$

$\frac{\| x - \tilde{x} \|}{\| x \|} = O(\epsilon_{\text{mach}})$

$f(x)$ has condition number $K(x)$

$$\limsup_{\delta \rightarrow 0} \frac{\| f(x) - f(x + \delta x) \|}{\| f(x) \|} \frac{\| \delta x \|}{\| x \|} = K(x)$$

i.e.

$$\frac{\| f(x) - f(\tilde{x} + \delta) \|}{\| f(x) \|} \leq K(x) \frac{\| \delta \|}{\| x - \tilde{x} \|} \frac{\| x - \tilde{x} \|}{\| x \|} + \epsilon$$
So substituting \( f(x) = f(x) \)

\[
\lim_{\|x\|} \frac{\|f(x) - f(x)\|}{\|f(x)\|} \leq K(x) \lim_{\|x\|} \frac{\|x - x\|}{\|x\|} = K(x) \mathcal{O}(\varepsilon_{\text{mach}})
\]

Consider

\( x_1, x_2 \). Then

\[
f(x_1) = x_1(1 + \varepsilon_1)
f(x_2) = x_2(1 + \varepsilon_2)
\]

\[
f(x_1) \pm f(x_2) = x_1(1 + \varepsilon_1) \pm x_2(1 + \varepsilon_2)
\]

So

\[
\frac{f(x_1) - f(x_2)}{f(x_1) - f(x_2)} = \left( \frac{x_1(1 + \varepsilon_1) - x_2(1 + \varepsilon_2)}{x_1(1 + \varepsilon_1) - x_2(1 + \varepsilon_2)} \right) (1 + \varepsilon_3)
\]

\[
= \left( \frac{x_1 - x_2 + x_1 \varepsilon_1 - x_2 \varepsilon_2}{x_1 - x_2} \right) (1 + \varepsilon_3)
\]

\[
= \frac{x_1 - x_2 + x_1 \varepsilon_1 - x_2 \varepsilon_2 + (x_1 - x_2) \varepsilon_3 + x_1 \varepsilon_3 \varepsilon_3 - x_2 \varepsilon_1 \varepsilon_3}{x_1 - x_2}
\]

\[
= x_1(1 + \varepsilon_4) - x_2(1 + \varepsilon_5)
\]

The claim is that all of the basic operations are backward stable.

\( +, -, /, \cdot, \text{inner product} \)

As an example of a nonbackward stable operation -- out product

\[
XY^\ast
\]

The algorithm is stable however
Unstable

A square matrix

compute eigenvalues via finding the roots of the characteristic polynomial

$$\text{det} \left| A - zI \right|$$

Read into §16.