Linear 1 October 2017

\[ V_1, V_2, W \text{ all norm 1, coplanar} \]
\[ V_1 \perp V_2 \]
\[ \| Av_1 \| = \| Aw \| = 6 \quad ? \quad \| Av_2 \| = 6 \]
\[ \| Av_2 \| \leq 6 \]

Exercise from previous:
\[ \sigma_1 = \| A \|_2 \]
\[ \| Av_1 \| = 6 \quad \| v_1 \| = 1 \]

Want to show that if
\[ \exists w \neq av_i \text{ s.t. } \| Aw \| = 6 \| w \| \text{ then} \]

\[ \exists v_2 \text{ s.t. } v_2 = av_i + bw \quad v_2 \perp v_1 \quad \Rightarrow \quad \| Av_2 \| = \sigma \| v_2 \| \]

\[ \text{and } Av_1 \perp Av_2 \]

So. Exercise as stated had a counterexample.
Consult T&H to see related lemma on SVD

\[ P^2 = P \text{ is a projector} \]

If inner product space, have orthogonality, norm, induced norms

**Def** \( P \) is an orthogonal projector \( \iff \)
- \( R(P) \perp N(P) \)
- \( R(P) \perp R(I-P) \)
- \( P = P^* \)
- \( \| P \| = 1 \quad P \neq 0, \ P \neq I \)
In a sense, we do for QR

\[
AR_1R_2\ldots R_n = Q
\]

\[
\overbrace{R^{-1}}^{R}
\]

We did Gram-Schmidt & Modified Gram-Schmidt

another way of thinking about it — Householder

\[
Q_iA = [\quad]
\]

\[
\vdots
\]

\[
Q_n\ldots Q_1A = R
\]

\[
Q^* = Q^*
\]

\[
A = QR
\]

HW Monday Ex 10.1 10.2 10.3 9.9 ?

Want

\[
Qa_i = \alpha e_i
\]

and choose \( \alpha = \|a_i\| \)

In \( \mathbb{R}^2 \) rotation

\[
\cos \theta = \frac{\langle x^*e_i \rangle}{\|x\|} = \frac{x_i}{\|x\|}
\]

\[
[ \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} ]
\]

So rotate last z coordinates, then next z, etc. "Givens Rotations"
Noting when using QR to solve problems, it is done "in place" without actually explicitly computing $Q$, etc.

i.e.

$$QRx = b$$

$$Rx = Q^*b$$

so as we compute $Rx$, compute $Q^*b$ as well.

Another method - Householder Reflections

$a_i$, $e_i$ lie in some plane - project/reflect in a sense

$$V = a_i - \|a_i\|e_i$$

$V$ is difference $a_i - \text{Proj}_{Q}(a_i)$

$$I - \frac{VV^*}{VV^*}$$

is orthogonal projection along $V$

so

$$R = I - 2\frac{VV^*}{VV^*}$$

Note $R^* = R$

$$R^2 = R^*R = (I - 2\frac{\langle VV^* \rangle}{\langle VV^* \rangle})^2 = I - 4\frac{\langle VV^* \rangle}{\langle VV^* \rangle} + 4\frac{\langle VV^* \rangle}{\langle VV^* \rangle}$$

$$= I.$$

Hence $R$ is orthogonal.
So we have
\[ H = I - 2\langle v | v \rangle \quad \text{or} \quad I - 2v v^T \]
\[ v \equiv \alpha_i - \| \alpha_i \| e_i \quad \text{or} \quad v \equiv \alpha_i + \| \alpha_i \| e_i \]

For stability concerns, suggest choosing \( v \) such that
\( v \) is large.

So for
\[ A_1 = Q_1 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \end{bmatrix} \]
\[ Q_2 = \begin{bmatrix} 1 \\ F \end{bmatrix} \quad F \text{ is a Householder reflection in the smaller space} \]

Gram-Schmidt
Modified Gram-Schmidt
Givens rotations
Householder reflections
all can find \( R \) in \( QR \)
If \( Q \) is actually wanted, use MSG
but Householder has fewer operations
$k = 1, \ldots, n$

$$x = A(k:n, k) \text{ i.e. rows } k \text{ to } n \text{ column } n$$

$$V = \text{sign}(x) \|x\|_2 e_1 + x$$

$$V = \frac{V}{\|V\|}$$

$$A(k:n, k:n) = A(k:n, k:n) - 2V(V^* A(k:n, k:n))$$

$$b(k:n) = b - 2V(V^* b(k:n))$$

Ex 10.1

Eigenvalues etc of Householder reflections

Try computing $Q$ from reflections in several ways. (Apply to $I$, multiply and reflect, etc.)

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Review

$A \in m \times n$, $m > n$ \hspace{1cm} A = \boxed{\vphantom{11}}$

Ex 6.3 $A^*A$ is nonsingular \hspace{1cm} (i.e. $AA^*$ must be singular)

If $A$ is full rank

Proof via SVD $A = U \Sigma V^*$

$A^*A = V \Sigma U^* U \Sigma V^* = V \Sigma^2 V^*$ is full rank if $\Sigma$ is

(Both $A^*A$, $AA^*$, $A$ all have same rank.)
A full rank \( \implies \begin{bmatrix} \end{bmatrix} = 0 \)

Take \( x \in \mathcal{N}(A^*A) \)

\[ A^*A x = 0 \]

WTS \( x = 0 \)

\[ x^*A^*A x = 0 \]

\[ (Ax)^*(Ax) = \|Ax\|^2 = 0 \]

\[ \implies Ax = 0 \]

but \( \mathcal{N}(A) = \{0\} \)

\[ A^*A \text{ is nonsingular} \]

Reverse

Converse

\[ A^*A \text{ nonsingular} \implies A \text{ full rank} \]

Assume no \( v \in \mathcal{N}(A) \setminus \{0\} \)

\[ A^*A v = 0 \] contradiction.