Linear Algebra
26 September 2012

\[
\begin{align*}
\|X\|_P^2 &= \sum |x_j| P_i \leq \left( \sum |x_j| P_i \right) P_i P_j (\Sigma 1) \left( P_i P_j - 1 \right) P_j P_i \tilde{p}_j \\
\text{Project onto } \mathbb{R}(Y) \text{ along } N(X)^\perp \\
\tilde{p}_j &= Y (X^* Y)^{-1} X^* \quad \text{projects onto } y \text{ along the hyperplane normal to } x
\end{align*}
\]

Recall \( P^2 = P \text{ then } \|P\| \geq 1 \)

\[ \forall v \in \mathbb{R}(P) \quad P v = v \quad \Rightarrow \quad \|P\| \geq 1 \]

Now for \( vv^* \), \( \|P\| \leq \|v\| \|v^*\| = 1 \quad (v \text{ norm 1}) \)

For an oblique projection rank 1, \( v = \frac{y}{\|y\|} \quad w = \frac{x}{\|x\|} \)

\[ P = v (w^* v)^{-1} w^* = \frac{y}{\|y\|} \left( \frac{y^* x}{\|y\| \|x\|} \right)^{-1} \frac{x}{\|x\|} = y (x^* y)^{-1} x^* \]

So for general projections, we can find \( v^* v = I \), \( \mathbb{R}(v) = \mathbb{R}(y) \)

\[ w^* w = I \quad N(w) = N(v) \]

So \[ P = y (x^* y)^{-1} x^* = \sqrt{v^* v} w^* \]

And finally the special case \( P = vv^* \)
Thus the question – how to find an orthogonal / unitary matrix with the same range as a given matrix.

**QR Decomposition**

\[ A = QR \]

where upper triangular

**Ch 9 & Exercises** will convince you that not all QR algorithms are equally good

\[ R(Q) = R(A) \]
\[ R(Qa_1) = R(a_1) \]
\[ R[a_1 a_2] = R[a_1 q_2] \]

So \( q_1 = \frac{a_1}{\|a_1\|} \) (or neglect this, so let us demand the diagonals positive)

\[ q_2 = \frac{a_2 - r_{12} q_1}{r_{22}} \]

where

\[ r_{22} = \|a_2 - r_{12} q_1\| \]

\[ \langle q_1 | q_2 \rangle = \langle q_1 | q_2 \rangle - r_{12} \langle q_1 | q_1 \rangle = 0 \]

\[ \langle q_1 | a_2 \rangle = r_{12} \]

So

\[ q_1 = \frac{a_1}{\|a_1\|} \]

\[ q_2 = \frac{a_2 - (a_2^T q_1) q_1}{\|a_2 - (a_2^T q_1) q_1\|} \]

\[ q_3 = \frac{a_3 - P_1 a_3 - P_2 a_3}{\|a_3 - P_1 a_3 - P_2 a_3\|} \]

**i.e. Gram-Schmidt process**
Notice that

\[ Ax = b \]
\[ QRx = b \]
\[ Rx = Q^*b \]

which is easy to solve.

For a rectangular, can use this to solve the least squares approximation

\[ \| b - Ax \| = \| Q^*b - Rx \| \text{ not quite right} \]

Problem:

Using this algorithm

\[ A \approx QR \]
\[ \| Q^*Q - I \| > 0 \]

\[ q_j = \frac{(I - \hat{Q}_j) a_j}{\| (I - \hat{Q}_j) a_j \|} \text{ is our current procedure} \]

\[ \hat{Q}_j = Q_j Q_j^* \]

\[ P_j = I - \hat{Q}_{j-1} = P_{q_1} \cdots P_{q_{j-1}} P_{q_j} \]

\[ = (I - q_{j-1} q_{j-1}^*) \cdots (I - q_2 q_2^*) (I - q_1 q_1^*) \]

\[ (I - q_2 q_2^*) (I - q_1 q_1^*) a_3 \]

So if \( q_j \) actually orthogonal, this is the same, but if errors present, better.

This is "Modified Gram Schmidt."
GS

\[ j = 1 \ldots m \]
\[ v_j = a_j \]
\[ \text{for } i = 1 \ldots j-1 \]
\[ r_{ij} = q_i^* a_j \]
\[ v_j = v_j - r_{ij} q_i \]
\[ r_{ij}^2 = \|v_j\|^2 \]
\[ q_j = \frac{1}{r_{ij}} v_j \]

MG S

\[ \text{for } j = 1 \ldots m \]
\[ b_j = a_j \]
\[ \text{for } j = 1 \ldots n \]
\[ r_j = \|q_j\| \]
\[ q_j = v_j / r_j \]
\[ \text{for } i = j+1 \ldots n \]
\[ q_{ij} \]
\[ r_j = q_i^* v_j \]
\[ v_{ij} = v_i - r_{ij} v_j \]