Examples of weak $L^{p}$

A function $f \in L^{1,\infty}$

- divide up interval $[a,b]$ into $2^n$ sub-intervals of length $\frac{b-a}{2^n}$

Kolmogorov Theorem:

Require $\sigma$-finite space $(X,\mu)$ i.e. $X$ is the countable union of finite measure sets.

Then let $(X,\mu)$ be a $\sigma$-finite measure space and consider $g: X \rightarrow \mathbb{C}$ $\mu$-measurable.

Then for each $p \in (0,\infty)$ the following seminorm holds: $(0 < p)$

$g \in L^{p,\infty}(X,\mu) \Leftrightarrow \forall \varepsilon > 0, \exists C_{\varepsilon,p} > 0$ s.t.

$\forall E \mu$-measurable

$$\int_{E} |g(x)|^p \, d\mu(x) \leq C_{\varepsilon,p} \mu(E)^{1-\frac{1}{p}}$$

Proof:

$(X,\mu)$ $\sigma$-finite $X = \bigcup_{n=1}^{\infty} B_n$ $\mu(B_n) < \infty$ and $B_n \subseteq B_{n+1}$

$\Rightarrow$ Fix $\lambda > 0$ and $n \in \mathbb{N}$

$E = E_{\lambda} \cap B_n = \{ x \in X : |g(x)| > \lambda \} \cap B_n$

$\mu$-measurable since $g$ is $\mu$-measurable

By hypothesis

$$\lambda^p \mu(E_{\lambda}) \leq \int_{E_{\lambda}} |g(x)|^p \, d\mu(x) \leq C_{\varepsilon,p} \mu(E_{\lambda})$$

Examples properly in weak $L^{p,\infty}$ functions which is $L^{p}$ on interval $(\varepsilon, \frac{2\varepsilon}{1-N})$

Kolmogorov inequality $X$ a $\sigma$-finite measure space then $g \in L^{p,\infty} \Leftrightarrow \forall \varepsilon > 0, \exists C_{\varepsilon,p}$ s.t.

that is weak $L^{p}$ iff $L^{p}(E)$ norm for all $p \in (0,\infty)$ controlled by $\mu(E)^{1-\frac{1}{p}}$
Dividing by $\mu(E_{\lambda^n})$ is OK as long as $E_{\lambda^n}$ is measurable.
So
$$\lambda^5 \leq C_{spg} \mu(E_{\lambda^n})$$
$$\lambda^5 \mu(E_{\lambda^n})^{\frac{5}{\rho}} \leq C_{spg} \quad \text{or} \quad \left(\lambda \mu(E_{\lambda^n})^{\frac{5}{\rho}}\right) \leq C_{spg}$$
$$\lambda \mu(E_{\lambda^n})^{\frac{5}{\rho}} \leq C_{spg}$$
And this holds for any $n \in \mathbb{N}$.

Thus, for all $\lambda \in [0,1]$, we have
$$\lim_{n \to \infty} \lambda \mu(E_{\lambda^n})^{\frac{5}{\rho}} = \lambda \mu(E_{\lambda})^{\frac{5}{\rho}} \leq C_{spg}$$
justified via monotone convergence since $\mu(E_{\lambda^n}) \to \mu(E_{\lambda})$ as the integral of a characteristic function.

Therefore
$$\|g\|_{L^{5/\rho},\infty} \leq C_{spg}$$

If there is no $n_0 \in \mathbb{N}$ such that $\mu(E_{\lambda^n} \cap B_{\eta^n}) \neq 0$, the above argument does not work, but we have
$$\mu(E_{\lambda}) = 0 \quad \text{and so} \quad \lambda \mu(E_{\lambda})^{\frac{5}{\rho}} = 0$$
This finishes the $\Leftarrow$ implication.

$\Rightarrow g : X \to C$ $\mu$-measurable $g \in L^{5/\rho}(X,\mu)$ for some $\rho \in (0,\infty)$.

Let $s > 0$ and $E \mu$-measurable $\subseteq X$.

$$\int_{E} |g(x)|^5 \mu(dx) = \int_{0}^{\infty} s \lambda^5 \mu(E \{ |g(x)| > s \}) \, d\lambda \quad \text{(previous line)}$$

In $\Leftarrow$ direction, manipulation gives good bound.

In $\Rightarrow$ direction, integrate over distribution function.
\[
\int_0^a s^{s-1} \mu_1(x) dx \leq \mu(E) a^s
\]

hence \( g \) is large.

Split integral into \( \lambda < a \) and \( \lambda > a \). For \( \lambda < a \), take upper bound.

For \( \lambda > a \), \( g \) is large but then the measure of the set \( \{ g \geq \lambda \} \) is small.

Pick a good \( \alpha \) so both quantities are small, get

\[
C_{s,p} g = 1 + \frac{s}{p-s} \| g \|_{L^p}^p
\]
Def. Let \((X, \mu)\) and \((Y, \nu)\) be measure spaces, \(0 < p_0, p_1 \leq +\infty\), and assume that \(T: L^{p_0}(X, \mu) \rightarrow L^{p_1, \infty}(Y, \nu)\) via measurable functions on \(Y\) is called of **weak type** \((p_0, p_1)\), provided that in fact

\[
\|Tf\|_{L^{p_1, \infty}(Y, \nu)} \leq C\|f\|_{L^{p_0}(X, \mu)}
\]

i.e., \(\exists C > 0\) \(\|Tf\|_{L^{p_1, \infty}(Y, \nu)} \leq C\|f\|_{L^{p_0}(X, \mu)}\) is a bounded function.

**T** is called **strong type** \((p_0, p_1)\) provided that

\[
T: L^{p_0}(X, \mu) \rightarrow L^{p_1, \infty}(Y, \nu)
\]

is a bounded function on \(Y\).

**Remark:**

\[
T: L^{p}(X, \mu) \rightarrow L^{p_1, \infty}(Y, \nu)
\]

is of weak type \((p_0, p_1)\) if \(\forall \lambda > 0\)

\[
\lambda \nu\{y \in Y: |TF(y)| > \lambda\}^{1/p_1} \leq C\|f\|_{L^{p_0}(X, \mu)}
\]

\[
\Leftrightarrow \exists C > 0\) \(\forall \lambda > 0\)

\[
\nu\{y \in Y: |TF(y)| > \lambda\} \leq \left(\frac{C\|f\|_{L^{p_0}(X, \mu)}}{\lambda}\right)^{p_1/p_0}
\]

**Remark:**

**Strong** \((p_0, p_1)\) \(\Rightarrow\) **Weak** \((p_0, p_1)\)

since \(T\) is a continuous linear mapping \(L^{p_0}(X, \mu) \rightarrow L^{p_1, \infty}(Y, \nu)\)

\[
\|Tf\|_{L^{p_1, \infty}(Y, \nu)} \leq C\|f\|_{L^{p_0}(X, \mu)}
\]

**T** on \(L^{p_0}(X, \mu)\) is of **weak type** \((p_0, p_1)\) if \(T\) maps \(L^{p_0}(X, \mu)\) boundedly into weak \(L^{p_1, \infty}(Y, \nu)\) and **strong** if \(T\) is bounded from \(L^{p_0}(X, \mu)\) to \(L^{p_1, \infty}(Y, \nu)\).
Theorem (Marcinkiewicz Interpolation Theorem)

Let \((X, \mu)\) and \((Y, \nu)\) be measure spaces

Fix \(1 \leq p_0 < p_1 \leq +\infty\)

Assume that

\[ T: L^{p_0}(X, \mu) + L^{p_1}(X, \mu) \rightarrow \nu\text{-measurable functions} \]

(when \(L^{p_0} + L^{p_1}\) being the vector space generated by \(L^{p_0} + L^{p_1}\))

in these two spaces

with conditions

1. sublinear: \( |T(f_1 + f_2)| \leq |Tf_1| + |Tf_2| \) (up to \(\nu\) a.e.) for every \(f_1, f_2 \in L^{p_0} + L^{p_1}\)

2. \(T\) is of weak type \((p_0, p_0)\) and weak type \((p, p_1)\)

Then \(T\) is of strong type \((p, p)\) for all \(p \in (p_0, p_1)\)

Notice what does it mean here to be strong \(L^{p_0} \rightarrow L^{p_1}\).

Claim \(\forall p \in (p_0, p_1), \ L^{p}(X, \mu) \subseteq L^{p_0}(X, \mu) + L^{p_1}(X, \mu)\)

Take \(f \in L^{p}\) and consider constant \(C > 0 \) and \(\lambda > 0\)

\[ Ef^{\lambda} := \{x \in X : |f(x)| > \lambda \} \]

From this, Marcinkiewicz interpolation if \(T\) takes \(L^{p_0} + L^{p_1}\) to measurable functions, is sublinear, is weak \((p_0, p_0)\) and weak \((p, p_1)\)

then \(T\) is strong \((p, p)\) for \(p_0 < p < p_1\)

i.e. \(L^{p} \subseteq L^{p_0} + L^{p_1}\) and \(T\) maps \(L^{p}\) to \(L^{p}(Y)\)
Now write
\[ f = f_\lambda + f_{\lambda}^* \]
when \( f_\lambda = f \chi_{E_\lambda} \)
\[ f_{\lambda}^* = f \chi_{X \setminus E_\lambda} \]

Then
\[ \|f_\lambda\|_{L^p(X, \mu)} = \int_{E_\lambda} |f_\lambda|^p \, d\mu = \int_X |f_\lambda|^p \, d\mu = \int_{E_\lambda} |f|^p \, d\mu \]

Caution: negative exponent, so \( f_\lambda \) is this set so ok

\[ \|f\|_{L^p(X, \mu)} ^p \leq (c\lambda)^{p-1} \int_X |f|^p \, d\mu = (c\lambda)^{p-1} \|f\|_{L^p(X, \mu)} ^p \]

Thus \( f_\lambda \in L^p(X, \mu) \)

Split \( f = f \chi_{E_\lambda} + f \chi_{X \setminus E_\lambda} \) where \( E_\lambda = \{ |f| > \lambda \} \)

Then \( f \chi_{E_\lambda} \in L^p(X) \) since \( \|f_\lambda\|_{L^p} = (c\lambda)^{p-1} \|f\|_{L^p} \)
Now \( \|f_\lambda\|_{L^P(X, \mu)}^{p'} = \int_X |f_\lambda|^{p'} d\mu = \int_{X \setminus E_\lambda^C} |f_\lambda|^{p'} d\mu \)

\[ = \int_{X \setminus E_\lambda^C} |f_\lambda|^{p'} |f_\lambda|^{p-p'} d\mu \]

positive power \( c \) if \( |f_\lambda| < c\lambda \) on \( X \setminus E_\lambda^C \), so:

\[ = \int_{X \setminus E_\lambda^C} (c\lambda)^{p-p'} |f_\lambda|^p d\mu \leq (c\lambda)^{p-p'} \|f\|_{L^p(X, \mu)}^p \]

Therefore \( f \in L^p(X, \mu) \) is the sum of an \( L^{p_0}(X, \mu) \) and an \( L^{p'}(X, \mu) \) function.

Then when \( f_\lambda \) is nice if \( 1 < c\lambda \) \[ \|f_\lambda\|_{L^p}^{p'} \leq (c\lambda)^{p-p'} \|f\|_{L^p} \]

So \( f \in L^p(X, \mu) \) if also \( f \in L^{p_0} \cup L^{p'} \)

Next time \( f \in L^p(Y) \)
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