Harmonic Analysis
17 January 2012

Possible time changes

Mon/Weds
10:00/11:20
11 - 12:20
12:00/13:20
13:00/14:20

No Schedule Changes

Concepts in this class from
harmonic analysis
functional analysis
PDEs
Geometric measure theory

No notes

Introduction to Harmonic Analysis methods for study of PDEs
Harmonic analysis
Functional analysis
PDEs
Geometric measure theory

No text - much of material still under development

Course will be self-contained but for a few fundamental results
will refer to the mathematical literature.

Discussion of open problems
No collected homework

Semester will culminate in a project
- basic topic part, e.g. review of background topics
- more advanced part. Teams of 3,
e.g. a paper in mind on K^2 boundedness of Calderon integral operators
on Lipschitz curves. Sparse integrable functions \to sparse integrable

Office Hours Tues/Thurs after class

Will draw together topics from harmonic analysis, functional analysis, PDEs and geometric measure theory.

No text - notes based. Will put into current & open areas of research

Grade will be based on a two part project - a background part & an advanced part
Harmonic Analysis

Problems

Calderon - Zygmund theory
non-tangential maximal functions
function spaces
- BMO bounded mean oscillation
- VMO vanishing mean oscillation
- Atomic Hardy spaces

Functional Analysis
- Fredholm theory
- Function spaces
  - Sobolev spaces
  - Besov spaces
    - interpolation between Lebesgue & Sobolev spaces

PDEs
- Laplacian (Partial differential operators)
- Fundamental solutions
- Distributions
- Integration by parts / Green's formulas

Geometric Measure Theory
- Classes of domains
  - boundary value problems in real and nice domains,
  - domains of locally finite perimeter / largest space for integration by parts

Singular Integral Operators

There is a long list of topics, but in particular, maximal functions, spaces of functions
and how operators relate different spaces, along with analysis of more difficult domains
Example
Boundary value
Laplace problem

Data in one space implies solution in another space

(BVP) \( \Omega \subseteq \mathbb{R}^n \) domain

\[
\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}
\]

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega \\
\frac{\partial u}{\partial n} = f & \text{on } \partial \Omega
\end{cases}
\]

\( f \) in certain function spaces on \( \partial \Omega \)

\( u \) in some function space on \( \Omega \)

Wish to study relationship between function space for \( f \) on \( \partial \Omega \) and space for solution \( u \) in \( \Omega \).

Example: Upper Half Space

Upper half space:

\( \Omega = \mathbb{R}^n_+ \) i.e. \( \Omega = \{ (x, t) : x \in \mathbb{R}^{n-1}, t > 0 \} \)

\( \partial \Omega = \{ (x, 0) : x \in \mathbb{R}^{n-1}, t = 0 \} \)

\[
\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^n_+ \\
\frac{\partial u}{\partial n} = f & \text{in } \mathbb{R}^{n-1} \text{ given}
\end{cases}
\]

When \( \frac{\partial u}{\partial n}(x) = \lim_{t \to 0^+} u(x, t) \)

For example, Laplace's equation \( \{ \Delta u = 0 \} \) on the upper half space \( \{ (x, t) : t > 0 \} \)

can be solved in the sense that \( \frac{\partial u}{\partial n} = \lim_{t \to 0^+} u(x, t) = f(x) \)
\[ P(x) = \frac{c_n}{(1|x|^2 + 1)^{n/2}} \]

\[ P_t(x) = t^{-(n-1)} P\left( \frac{x}{t} \right) \]

\[ P_t(x) = c_n t \left( \frac{1}{1 + \left(\frac{|x|}{t}\right)^2} \right)^{n/2} \]

\[ \int_{\mathbb{R}^n} P(x) \, dx = 1 \quad \text{(in terms of } P(x)\text{)} \]

Naively:

\[ \lim_{t \to 0^+} P_t(x) = \lim_{t \to 0^+} c_n t \left( \frac{1}{1 + \left(\frac{|x|}{t}\right)^2} \right)^{n/2} = 0 \quad \text{if } x \neq 0 \]

\[ = +\infty \quad \text{if } x = 0 \quad \text{and } n > 1 \]

thus approaches Dirac \( \delta \) function

Rigorously:

\[ \lim_{t \to 0^+} (P_t \ast f)(x) = f(x) \]

for all "reasonable" \( f \).

To solve, we use the Poisson Kernel. A Naive check suggests that \( \lim_{t \to 0^+} P_t(x) = \delta(x) \), which can be confirmed by checking that

\[ \lim_{t \to 0^+} (P_t \ast f)(x) = f(x) \]
$$u := P_t \ast f$$

$$u(x,t) = \int_{\mathbb{R}^{n-1}} P_t(x-y)f(y) \, dy$$

Boundary condition

$$\lim_{t \to 0^+} u(x,t) = f(x)$$

$$u(x,0)$$

so we can check $$u|_{\partial \Omega} = f|_{\partial \Omega}$$

Recall

$$\Delta = \Delta_x + \Delta_t$$

$$\Delta u = \Delta_x u (\Delta_x + \Delta_t) u(x,t)$$

Assuming all goes well, can differentiate under integral, only $$P_t(x-y)$$ needs differentiated — explicit computation shows $$\Delta u = 0$$

$$u$$ is called the Poisson extension of $$f \in \mathbb{R}^n_+$$

**Sample Theorem**

$$\forall x \in \mathbb{R}^{n-1} \gamma(x) := x + x$$

i.e. translation of cone to vertex of $$x$$

$$u : \mathbb{R}^n_+ \to \mathbb{R}$$

$$N(u)(x) := \sup$$

"non-tangential maximal function of $$u$$ applied at $$x$$"

$$N(u)(x) := \sup \{ |u(x+y)| : (y,t) \in \gamma(x) \} \quad \text{for } x \in \mathbb{R}^{n-1}$$

$$u := P_t \ast f$$ is the Poisson extension of $$f$$ to $$\mathbb{R}^n_+$$. It needs the boundary condition $$u|_{\partial \Omega} = f$$ and via differentiation under the integral, meets the $$\Delta u = 0$$ condition.

The non-tangential maximal function of $$u$$ at $$x$$, sup of $$u$$ in cone above $$x$$

$$N(u)(x) = \sup \{ u(x+y) : (y,t) \in \gamma(x) \}$$

Maximal value "not tangent to surface"
Simple Theorem

If given function on $\mathbb{R}^{n-1}$, reasonable,

$$u(x,t) = \mathcal{P}_t f(x) \quad x \in \mathbb{R}^{n-1}, \quad t > 0$$

i) if $1 < p < \infty$ then

$$f \in L^p(\mathbb{R}^{n-1}) \iff \mathcal{N}u \in L^p(\mathbb{R}^{n-1})$$

ii) $1 < p < \infty$ then

$$f \in L^p(\mathbb{R}^{n-1}) \iff \mathcal{N}(\mathcal{V}u) \in L^p(\mathbb{R}^{n-1})$$

if $\mathcal{V}$ is integrable and gradient is as well a Sobolev space in

iii) if $1 < p < \infty$, $s \in (0,1)$

$$f \in B^{p,s}_s(\mathbb{R}^{n-1}) \iff \int_0^\infty \int_{\mathbb{R}^{n-1}} \left| t^{(1-s)-1} \mathcal{V}u(x,t) \right|^p dx dt < \infty$$

"diagonal" Besov space

Question

$\mathcal{N}(\mathcal{V}u)$ is maximum of $\mathcal{N}(U_{x_k})$ max of each component non-tangential.

$B^{p,s}_s \to$ regular integrability

$B^{p,s}_s \to$ regular smoothness

These $B^{p,s}_s$ fall in between $L^p$ and $L^1$.

$L^p(\mathbb{R}^{n-1}) = \{ f \in L^p(\mathbb{R}^{n-1}) : \mathcal{V}f \in L^p(\mathbb{R}^{n-1}) \text{ in sense of distributions} \}$

Besov spaces are very good in sense that functions in Sobolev spaces have

traces in Besov spaces.

$L^p$ Sobolev space, both function & gradient $L^p$

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Recall \( 0 < \alpha < 1 \)

\[ C^\alpha(\mathbb{R}^{n-1}) \quad \text{Hölder space} \]

\[
\sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{1 + \alpha}} < +\infty
\]

iv) if \( \alpha \in (0,1) \)

\[ f \in C^\alpha(\mathbb{R}^{n-1}) \iff t^{-1-\alpha} |u(t, \cdot)| \in L^\infty(\mathbb{R}^{n-1}) \]

Heuristically, multiplying by distance to boundary is almost like integrating.

Spend some time on the upper half plane

Office Hours

Thurs after class

Likewise, if \( f \) in Homogeneous Hölder space \( t^{1-\alpha} |u| \) essentially bounded

Notice often in this subject multiplying by \( |x| \) distance to the boundary is like integration, in a heuristic sense.